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# FROM EUCLID TO ARTIFICIAL INTELLIGENCE

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ABSTRACT. We give 19 proofs of the famous Angle Bisector Theorem from Euclid's Elements. The first proof is the Euclid's original proof, the remaining proofs use the methods of Euclidean Geometry, Trigonometry, Analytic Geometry, Complex Numbers, and Gröbner Bases. The Gröbner Bases proof is in the area of Automatic Proving and Artificial Intelligence, so that the proofs in a way symbolise the development of mathematics from 300 BC (the Euclid's time) to modern days. We discuss what the proofs illustrate and why they are important for Math Education. All the proofs, except the first one, are original.

## 1. INTRODUCTION

This article, we hope, will have certain pedagogical value for Math Education. According to [3], curriculum changes of the mathematics education in secondary schools (since nineteen seventies on) have added to the curriculum: elementary set theory, a wider use of various algebraic notions, earlier introduction of basic concepts of calculus. At the same time, the topics that were excluded are mainly from the area of traditional Euclidean Geometry. The process of emphasizing algebra at the expense of geometry is even more widespread in university teaching. We would like to argue by this article that traditional Euclidean Geometry should

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not be excluded from the curriculum. One cannot deny the efficiency and practicality of algebra in solving various problems, however purely algebraic solutions are often mechanical, without any insight and without any intuitive justification as why would the fact in question hold. Descartes invented Analytic Geometry with the idea to reduce geometry to algebra, however, we show in our article that the Analytic Geometry solution of the problem which we are considering is more complicated than several traditional Euclidean geometry solutions and, even more, that the Analytic Geometry solution has to use elements of the traditional Euclidean Geometry in some form. Further, we illustrate in our article that the best approach is combining the traditional Euclidean Geometry with modern tools, like Analytic Geometry, Trigonometry, Complex Numbers, Gröbner Bases. Thus there has to be place for both approaches, modern and traditional, side by side.

The article offers 19 proofs of the well-known Angle Bisector Theorem, which is (one direction of) Proposition 3 in the Book 6 of Euclid's Elements (please read articles [4,5]). Euclid himself states the theorem in the following way (see [2]):

*"If an angle of a triangle is cut in half, and the straight- line cutting the angle also cuts the base, then the segments of the base will have the same ratio as the remaining sides of the triangle."*

At the beginning of his proof he gives another formulation, namely:

*"Let ABC be a triangle. And let the angle BAC have been cut in half by the straight-line AD. I say that as BD is to CD, so BA [is] to AC."*

Our formulation of the theorem is basically this second Euclid's formulation. We will use the notation from our formulation, and from Figure 0 below, in the rest of the paper.

**The Angle Bisector Theorem.** *Let $\triangle ABC$ be a triangle. Let $D$ be a point in the segment $BC$ such that $\angle BAD = \angle DAC$. Then*

$$\overline{BD} : \overline{CD} = \overline{AB} : \overline{AC}.$$

If we denote $b = \overline{AC}, c = \overline{AB}, m = \overline{BD}, n = \overline{CD}$, the proportion has the form

$$m : n = c : b.$$

(See Figure 0.)

We would like to say that we did not collect the proofs from the literature (except for the first proof, which is the proof from the Elements), but we invented all of them.

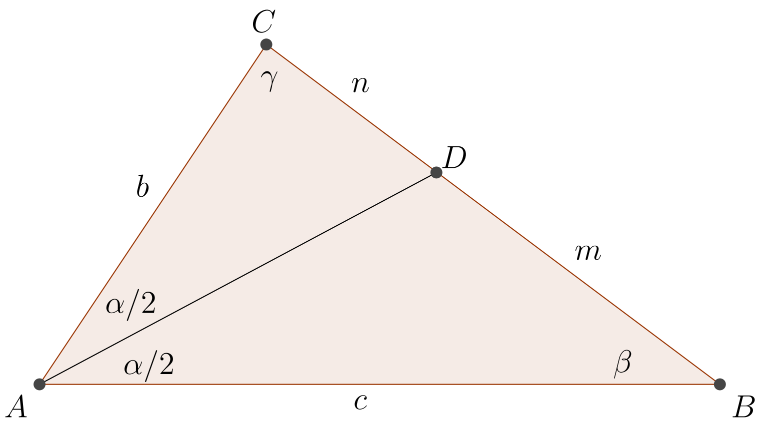


Figure 0

Let us give some historical comments.

The proofs 1-11 are in the spirit of Euclid's Elements' Book 6 "Similar Figures" (in which the theory of proportions is applied to plane geometry). Proofs 12 and 13 involve respectively the formula for the area of triangle and addition of proportions, things that Euclid did not use in his Elements, but other Greek mathematicians immediately after Euclid's time did (see [6]). The proofs 14, 15 and 16 involve trigonometric functions (the level of Greek mathematics of the third century BC). The proof 17 uses the method of coordinates, invented by René Descartes (latinized name Cartesius) in the 17$^{\text{th}}$ century (see [6]). The proof 18 uses complex numbers (in combination with coordinates). (The theory of complex numbers was for the first time developed in the 16$^{\text{th}}$ century by Rafael Bombelli (see [6]). Finally, the proof 19 uses Gröbner bases and computer algebra. It belongs to the area of automatic proving of geometric theorems (whose development started in late 1980's), which is one of the subjects in Artificial Intelligence. We constructed the proof using the technique explained in [6]. The proof can be verified using any computer algebra system, like, for example, Macaulay 2.

One can say that in a way all aspects of plane geometry are covered by these proofs. In particular, the proofs illustrate:

(a) the beauty of pure euclidean geometry;

(b) the usefulness of introducing the numbers to geometry and associating them with the geometric figures;

(c) the usefulness of trigonometry in plane geometry;

(d) the practicality and efficiency of the coordinate method;

(e) the usefulness of complex numbers in plane geometry;

(f) the power of the automatic proving method (and Artificial Intelligence) when applied to plane geometry.

In the pure euclidean geometry proofs 1-11, as well as in the proofs 12 and 13, one has to introduce some new objects, not initially present in the problem, and then, by analyzing the relations between those objects, to get the wanted conclusion. Introducing the new objects is the creative part of the solution. In the remaining proofs we do not stay in the realms of pure eculidean geometry and we use the trigonometry, the coordinates and the complex numbers. The proof 17, which uses the coordinates, is, as expected, efficient and does not contain any surprises. However, at one moment it uses the proportionality of the lengths of the sides of similar triangles, in an obvious situation though. Nevertheless, it illustrates that one can not ignore Euclidean Geometry and use "pure Analytic Geometry." In fact, the truth is that the method of coordinates is a wonderful new tool, which in plane geometry is most efficient when we combine it with the traditional methods, including construction of auxiliary triangles. And becuase of that we need to keep in our schools serious courses of classical Euclidean Geometry. To further emphasize this conclusion, we note that in the proof 19 (which belongs to the area of automatic proving of geometric theorems) one first needs certain Euclidean Geometry reasonnings in order to get the problem ready for the automatic proving. And, as always, there is an ethical issue if one is willing to accept the confirmation by computer that certain polynomial belongs to the certain ideal in a polynomial ring, the fact that cannot be verified without computer and which does not offer any intuitive hint as whether it is true or not.

## 2. Proofs

**First proof.**

*Method*: Introducing new objects, namely the line through $B$ parallel with $AD$ and a new triangle $\triangle CEB$. Then using similarity of triangles.

*Facts used*: the lengths of the sides of similar triangles are proportional.

*Proof.* Consider the line $\ell_1$ through $B$ parallel with $AD$. Let the line $\ell_2$ through $A$ and $C$ intersect $\ell_1$ at $E$. (See Figure 1.) Since $BE \parallel AD$, we have $\angle BEA = \alpha/2$ and $\angle ABE = \alpha/2$. So $\triangle AEB$ is equilateral and $\overline{AE} = c$. Now since $\triangle CEB \sim \triangle CAD$, we have $(m+n) : n = (c+b) : b$, hence $m : n = c : b$. □

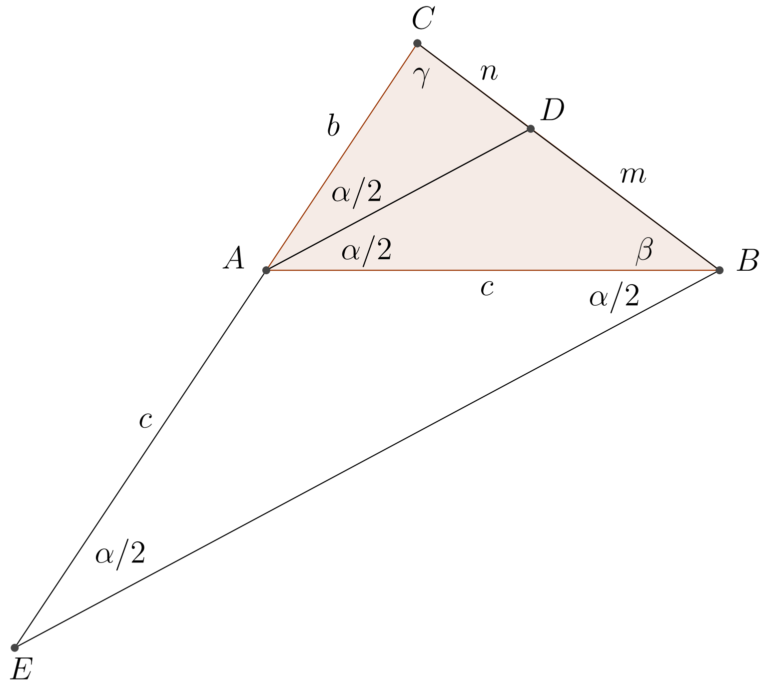


Figure 1

**Second proof.**

*Method*: Introducing new objects, namely the line through $B$ parallel with $AC$ and a new triangle $\triangle ABE$. Then using similarity of triangles.

*Facts used*: the lengths of the sides of similar triangles are proportional.

*Proof.* Consider the line $\ell_1$ through $B$ parallel with $AC$. Let the line $\ell_2$ through $A$ and $D$ intersect $\ell_1$ at $E$. (See Figure 2.) Since $BE \parallel AC$, we have $\angle DBE = \gamma$. Since $\angle ADC = \angle BDE$, it follows (by considering $\triangle ABC$ and $\triangle ABE$) that $\angle AEB = \alpha/2$. Hence $\triangle ABE$ is equilateral and $c = \overline{AB} = \overline{BE}$. Now since $\triangle ADC \sim BDE$, we have $m : n = c : b$. $\square$

**Third proof.**

*Method*: Introducing new objects, namely the line through $D$ parallel with $AC$ and a new triangle $\triangle EBD$. Then using similarity of triangles.

*Facts used*: the lengths of the sides of similar triangles are proportional.

*Proof.* Let $E$ be the point on $AB$ such that $DE \parallel AC$. Then $\angle BED = \alpha/2$. Hence $\angle AED = \beta + \gamma$ and (considering the sum of the angles in $\triangle AED$) $\angle ADE = \alpha/2$. Hence $\triangle AED$ is equilateral. (See Figure 3.) Let $p = \overline{AE} = \overline{DE}$. Since $\triangle ABC \sim \triangle EBD$, we have $c : (c - p) = b : p = (m + n) : m$. From these proportions we get $p = \frac{mb}{m+n}$ and $p = \frac{bc}{b+c}$. Hence $\frac{m+n}{mb} = \frac{b+c}{bc}$, which implies $m : n = c : b$. $\square$

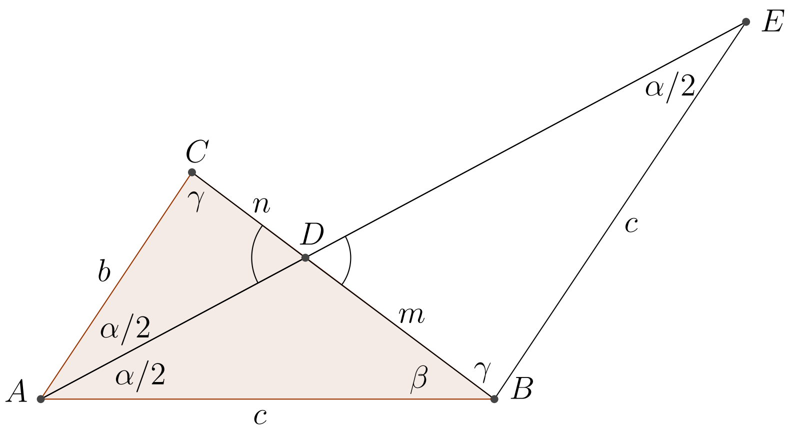


Figure 2

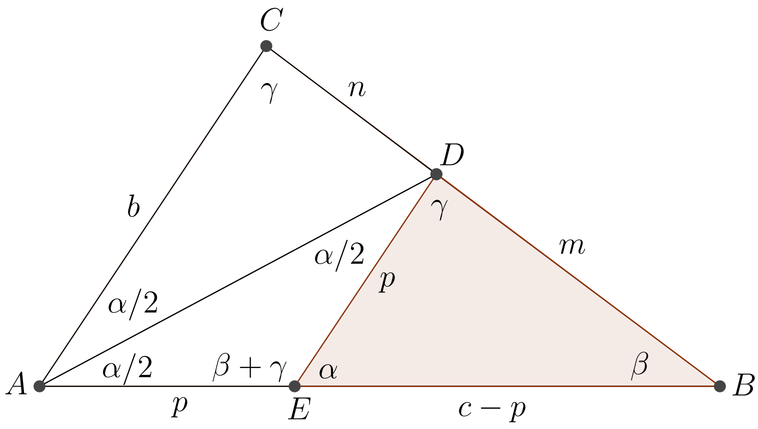


Figure 3

**Fourth proof.**

*Method*: Introducing new objects, namely the line through $B$ making the angle $\alpha/2$ with $BC$. Then using similarity of triangles.

*Facts used*: the lengths of the sides of similar triangles are proportional.

*Proof.* Let $\ell$ be the line through $B$ making the angle $\alpha/2$ with $BC$ (see Figure 4). Since $\angle BDE = \angle ADC + \alpha/2 + \beta$, we have $\angle BED = \gamma$. From the similarity of $\triangle ADC$ and $\triangle BDE$ we have $b : \overline{BE} = \overline{AD} : m$. From the similarity of $\triangle ABE$ and $\triangle ADC$ we have $c : \overline{AD} = \overline{BE} : n$. Hence $bm = cn$, i.e., $m : n = c : b$. $\square$

**Fifth proof.**

*Method*: Introducing new objects, namely the circumscribed circle of $\triangle ABC$ and the quadrilateral $ABEC$. Then using similarity of triangles.

*Facts used*: (a) the angles inscribed in the same arc are equal; (b) the lengths of the sides of similar triangles are proportional.

*Proof.* Let $k$ be the circumscribed circle of $\triangle ABC$. Let $AD$ be extended to meet $k$ at $E$. Since the angles inscribed in the same arc are equal, we have $\angle AEB = \gamma$, $\angle AEC = \beta$ and $\angle CBE = \angle BCE = \alpha/2$. (See Figure 5.) Hence $\triangle BCE$ is equilateral. Let $p = \overline{BE} = \overline{CE}$. Since $\triangle ABD \sim \triangle DEC$, we have $c : m = p : \overline{DE}$. Since $\triangle ADC \sim \triangle BED$, we have $b : n = p : \overline{DE}$. Hence $c : m = b : n$, i.e., $m : n = c : b$. $\square$

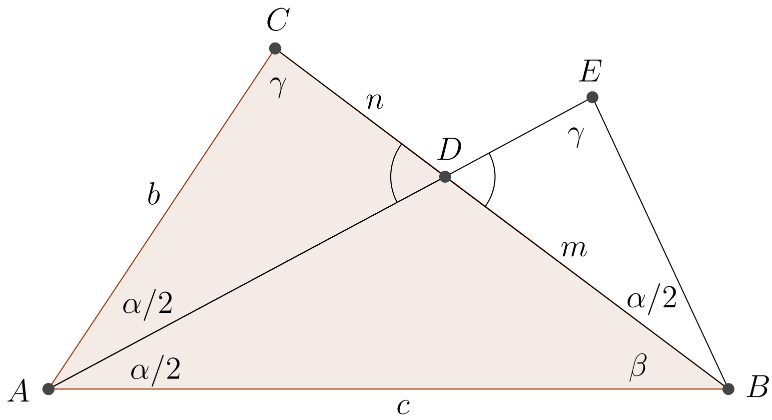


Figure 4

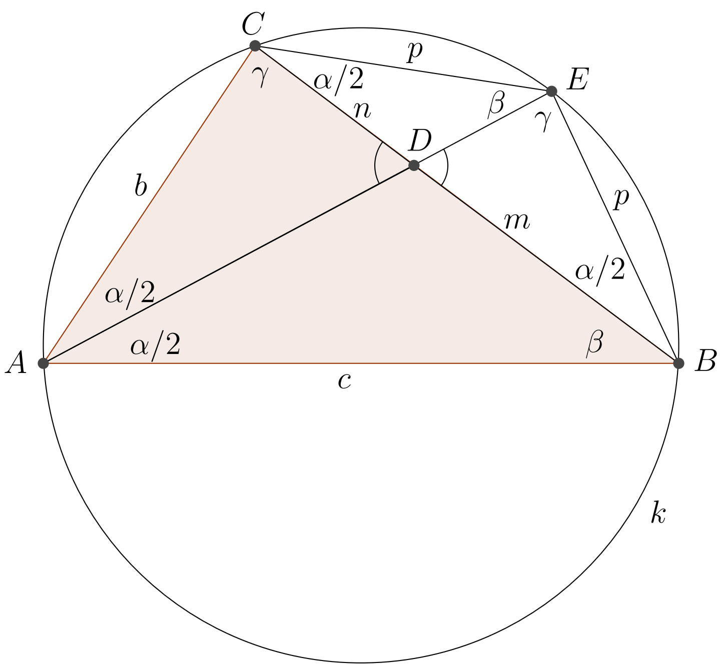


Figure 5

**Sixth proof.**

*Method*: Introducing new objects, namely the circumscribed circle of $\triangle ADC$ and (in one of the cases) the quadrilateral $AEDC$. Then using similarity of triangles.

*Facts used*: (a) the angles inscribed in the same arc are equal; (b) the lengths of the sides of similar triangles are proportional; (c) the central angle subtended by two points on a circle is twice the inscribed angle subtended by those points.

*Proof.* Let $k$ be the circumscribed circle of $\triangle ADC$. Let $\ell$ be the line thorugh $A$ and $B$. We will first consider the case when the point $E$ of intersection of $k$ and $\ell$ is inside the segment $AB$. (See figure 6.) Since the angles inscribed in the same arc are equal we have that $\angle ECD = \angle EAD = \alpha/2$. Hence $\triangle CED$ is equilateral and $\overline{DE} = n$. Also $\angle AEC = \angle ADC = \alpha/2 + \beta$. Hence $\angle DEC = \gamma$ and $\angle EDB = \alpha$. Thus $\triangle ABC \sim \triangle EDB$. Hence $c : m = b : n$, i.e., $m : n = c : b$.

The case when the point $E$ of intersection of $k$ and $\ell$ is outside of the segment $AB$ is similar. In the remaining case suppose that $A$ is the only point of intersection of $k$ and $\ell$. Let $S$ be the center of $k$. (See Figure 6a.) Since the central angle subtended by two points on a circle is twice the inscribed angle subtended by those points, we have $\angle SAD = 90° - \gamma$ (since $\triangle ASD$ is equilateral) and also $\angle CAS = 90° - \beta - \alpha/2$. Hence $\alpha/2 = 90° - \gamma + 90° - \beta - \alpha/2$ and so $\gamma = \alpha/2$. Hence $\triangle ADC$ is equilateral and $\overline{AD} = n$. Also $\angle ADC = \alpha$. Hence $\triangle ABC \sim \triangle ABD$. It follows that $c : m = b : n$, i.e., $m : n = c : b$. □

**Seventh proof.**

*Method*: Introducing new objects, namely two segments starting at $D$ and parallel with $AB$ and $AC$ respectively. Then using similarity of triangles.

*Facts used*: the lengths of the sides of similar triangles are proportional.

*Proof.* Let $\ell_1$ be the line through $D$ parallel with $AC$ and $\ell_2$ the line through $D$ parallel with $AB$. Let $E$ be the intersection of $\ell_1$ and $AB$ and $F$ the intersection of $\ell_2$ and $AC$. (See Figure 7.) Then $AEDF$ is a rhombus. Denote $p = \overline{AE} = \overline{ED} = \overline{DF} = \overline{FA}$. Since $\triangle ABC \sim \triangle EBD$, we have $c : (c - p) = (m + n) : m$. Hence $p = \frac{cn}{m+n}$. Since $\triangle FDC \sim \triangle EBD$, we have $n : (b - p) = m : p$. Hence $p = \frac{bm}{m+n}$. Thus $\frac{cn}{m+n} = \frac{bm}{m+n}$ and so $m : n = c : b$. □

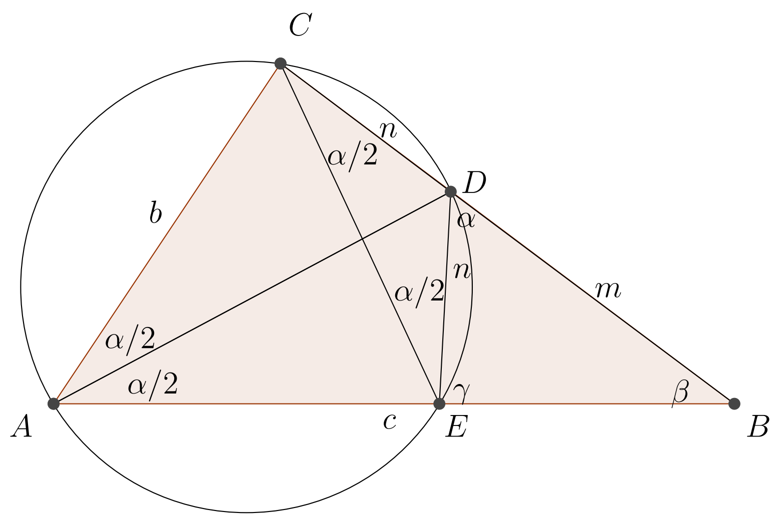


Figure 6

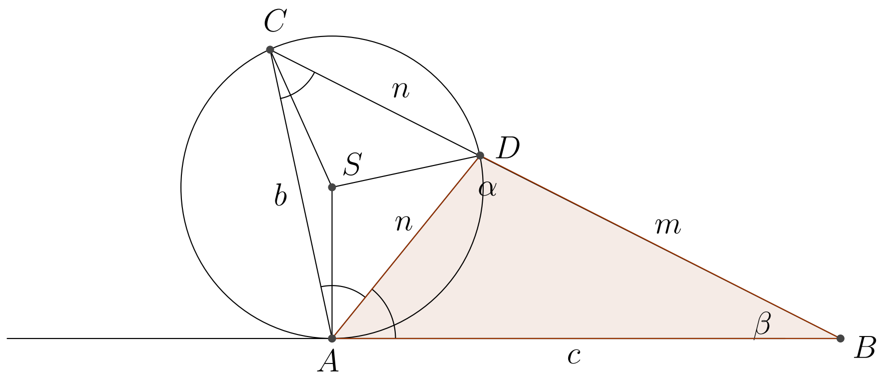


Figure 6a

**Eighth proof.**

*Method*: Introducing new objects, namely two segments starting at $D$ and making the angle $\alpha/2$ with $DB$ and $DC$ respectively. Then using similarity of triangles.

*Facts used*: (a) the lengths of the sides of similar triangles are proportional; (b) properties of inscribed quadrilaterals.

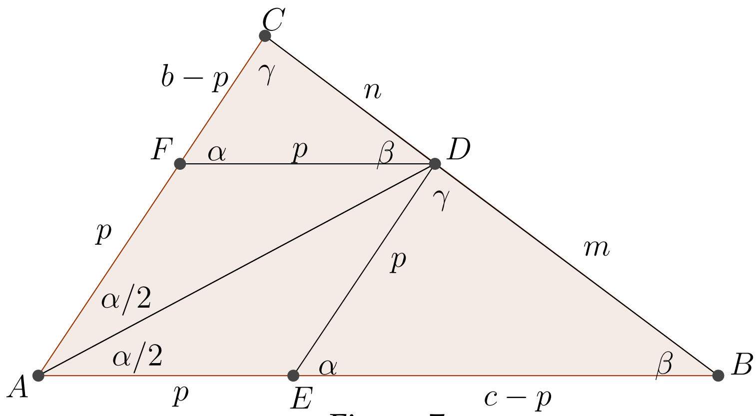


Figure 7

*Proof.* Let $E$ be the point on $AB$ such that $\angle BDE = \alpha/2$ and let $F$ be the point on $AC$ such that $\angle CDF = \alpha/2$. Then $\angle ADE = \gamma$ and $\angle ADF = \beta$. Note that $AEDF$ is an inscribed quadrilateral since $\angle FAE + \angle EDF = 180°$. Hence $\overline{ED} = \overline{DF}$ since these chords subtend equal angles. Since $\triangle ADC \sim \triangle AED$, we have $b : n = \overline{AD} : \overline{DE}$. Also $\triangle ABD \sim \triangle ADF$, so $c : m = \overline{AD} : \overline{DF} = \overline{AD} : \overline{DE}$. Hence $b : n = c : m$, i.e., $m : n = c : b$. □

**Nineth proof.**

*Method*: Introducing new objects, namely two segment $CE$ such that $\angle ACE = \beta$ and the triangles resulting from that.

*Facts used*: the lengths of the sides of similar triangles are proportional.

*Proof.* If $\beta = \gamma$, then $b = c$ and $m = n$, so the statement is true. Without loss of generality we can assume that $\gamma > \beta$. Let $E$ be the point on $AB$ such that $\angle ACE = \beta$. Let $S$ be the intersection of $AD$ and $CE$. (See Figure 9.) We have $\angle CSD = \angle CDS$, hence $\overline{CS} = n$. Now from $\triangle ASC \sim \triangle ABD$ we have $b : n = c : m$. Hence $m : n = c : b$. □

**Tenth proof.**

*Method*: Introducing new objects, namely two segment $AE$ of length $b$, the segment $EF$ parallel with $AD$ and the triangles resulting from that.

*Facts used*: the lengths of the sides of similar triangles are proportional.

*Proof.* If $b = c$, then $m = n$, so the statement is true. Without loss of generality we can assume that $c > b$. Let $E$ be the point on $AB$ such that $\overline{AE} = b$. Since $\triangle AED \cong \triangle ADC$, $\overline{DE} = n$ and $\angle AED = \gamma$. [Note that $\angle DEB = \alpha + \beta$ and since $\angle DEB > \angle EBD$, we have $m > n$.] Let $F$ be the point on $BD$ such that $EF \parallel AD$. (See Figure 10.) Then $\angle DEF = 180° - \gamma - \alpha/2 = \beta + \alpha/2$ and $\angle DFE = \angle CDA = \beta + \alpha/2$. Hence $\overline{DF} = n$ and $\overline{BF} = m - n$. Now from $\triangle ABD \sim \triangle EBF$ we have $(c - b) : c = (m - n) : m$, hence $m : n = c : b$. $\square$

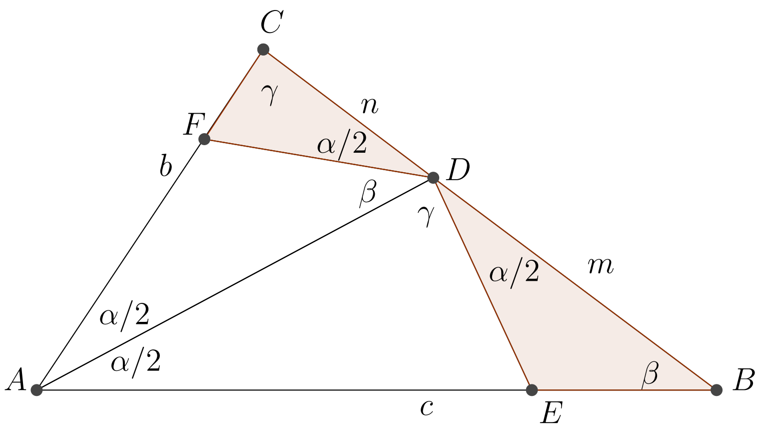


Figure 8

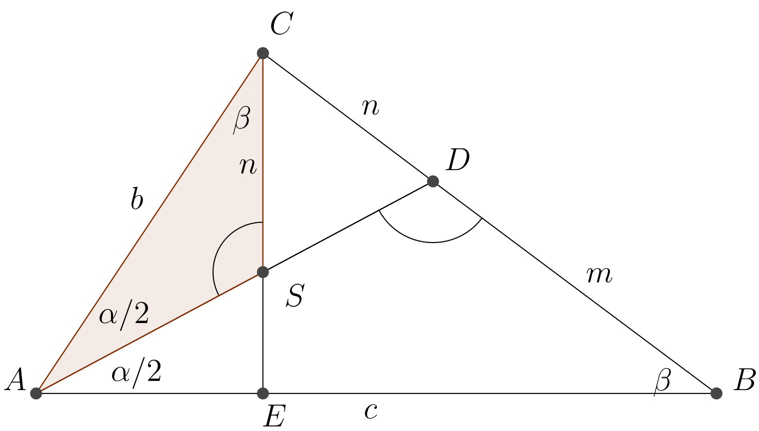


Figure 9

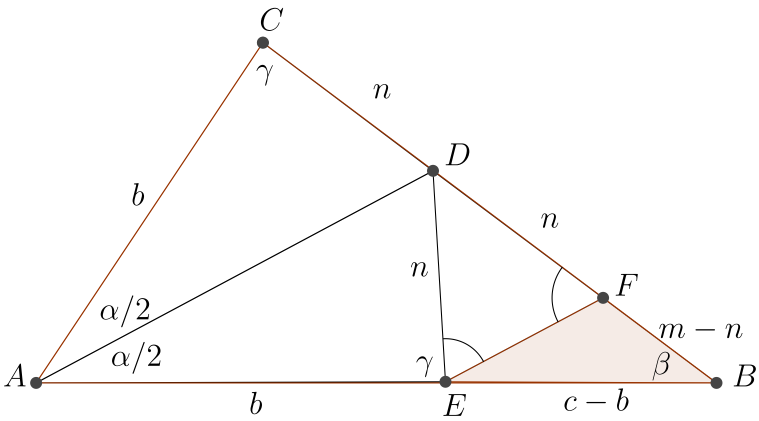


Figure 10

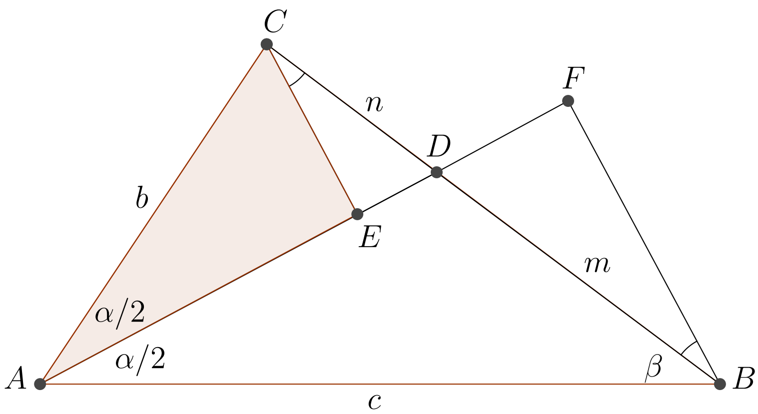


Figure 11

**Eleventh proof.**

*Method*: Introducing new objects, namely the segments $CE$ and $BF$ orthogonal to the line through $A$ and $D$, and triangles resulting from that.

*Facts used*: the lengths of the sides of similar triangles are proportional.

*Proof.* If $BC$ is orthogonal to $AD$, then $b = c$ and $m = n$, so the statement is true. Suppose that $BC$ and $AD$ are not orthogonal. Let $\ell$ be the line through $A$ and $D$. Let $E$ be the point on $\ell$ such that $CE$ is orthogonal on $\ell$ and let $F$ be the point on

$\ell$ such that $BF$ is orthogonal on $\ell$. (See Figure 11.) From $\triangle EDC \sim \triangle FDB$ we get $c : b = \overline{BF} : \overline{CE}$. From $\triangle AEC \sim \triangle AFB$ we get $m : n = \overline{BF} : \overline{CE}$. Hence $m : n = c : b$. $\square$

**Twelveth proof.**

*Method*: Introducing new objects, namely the center $S$ of the inscribed circle of $\triangle ABC$, and triangles resulting from that.

*Facts used*: (a) the bisectors intersect at a common point, which is the center of the inscribed circle; (b) the ratio of the areas of the triangles with the altitudes of the same length is the same as the ratio of the lengths of the sides corresponding to those altitudes.

*Proof.* Let $E$ be the point on $AC$ such that $BE$ is the bisector of $\beta$ and let $F$ be the point on $AB$ such that $CF$ is the bisector of $\gamma$. Let $S$ be the point of intersection of the three bisectors. Then $S$ is the center of the inscribed circle of $\triangle ABC$. (See Figure 12.) Let $r$ be the radius of that circle. Hence $r$ is the length of the altitude of $\triangle BSD$ corresponding to $BD$ and $r$ is the length of the altitude of $\triangle DSC$ corresponding to $DC$. Hence Area($\triangle BSD$) : Area($\triangle DSC$) $= m : n$. Since the triangles $\triangle ABD$ and $\triangle ACD$ have the same length altitudes corresponding to $BD$ and $CD$ respectively, we have Area($\triangle ABD$) : Area($\triangle ADC$) $= m : n$. It follows that [Area($\triangle ABD$) − Area($\triangle BSD$)] : [Area($\triangle ADC$) − Area($\triangle DSC$)] $= m : n$, i.e., Area($\triangle ASB$) : Area($\triangle ASC$) $= m : n$. However, the length of the altitude of $\triangle ASB$ corresponding to $AB$ is the length of the altitude of $\triangle ASC$ corresponding to $AC$ are both equal to $r$, so Area($\triangle ASB$) : Area($\triangle ASC$) $= c : b$. Thus $m : n = c : b$. $\square$

**Thirteenth proof.**

*Method*: Introducing new objects, namely the segment $AE$ of length $b$, the line through $E$ and $D$ and its intersection $F$ with the line through $A$ and $C$, the segment $BF$ and its intersection $G$ with the line through $A$ and $D$, and triangles resulting from all that.

*Facts used*: (a) the lengths of the sides of similar triangles are proportional; (b) Van Obel's theorem.

*Proof.* If $b = c$, then $m = n$, so the statement is true. Without loss of generality we can assume that $c > b$. Let $E$ be the point on $AB$ such that $\overline{AE} = b$. It is easy to see that the line through $E$ and $D$ is not parallel with $AC$. Let $F$ be the intersection of that line with the line through $A$ and $C$. Let $G$ be the intersection of the segment $BF$ with the line through $A$ and $D$. (See Figure 13.) We have $\angle AFE = \beta$ (since the other two angles of $\triangle AEF$ are $\alpha$ and $\gamma$). Hence $\triangle CDF \cong \triangle BDE$ and so $\overline{CF} = c - b$ and $\overline{AF} = c = \overline{AB}$. Hence $\overline{BG} = \overline{GF}$. Now by Van Obel's theorem (see Appendix) we have: $\frac{m}{n} = \frac{c-b}{b} + \frac{\overline{BG}}{\overline{GF}}$, i.e., $\frac{m}{n} = \frac{c-b}{b} + 1 = \frac{c}{b}$. □

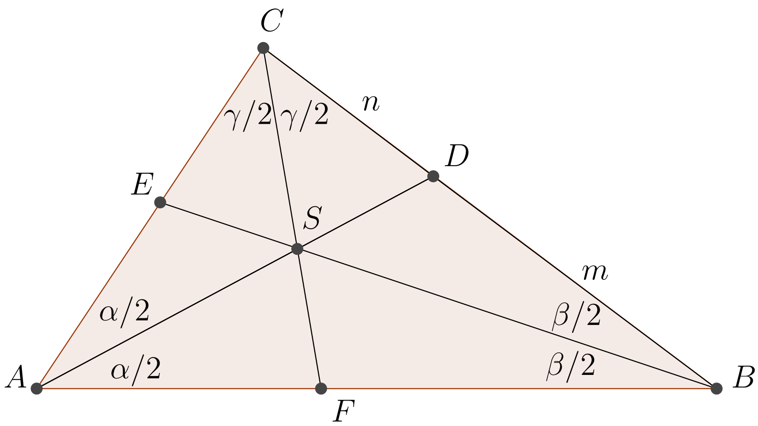


Figure 12

**Fourteenth proof.**

*Method*: Multiple application of a formula for the area of the triangle and some calculations with the results.

*Facts used*: the area of a triangle is the half of the product of the lengths of two sides and the sine of the angle between them.

*Proof.* Using the notation on Figure 0 we have: Area$(\triangle ABD) = \frac{1}{2}c \cdot \overline{AD} \cdot \sin(\alpha/2) = \frac{1}{2}m \cdot \overline{AD} \cdot \sin(\angle ADB)$, hence $\frac{c}{m} = \frac{\sin(\angle ADB)}{\sin(\alpha/2)}$. Also Area$(\triangle ADC) = \frac{1}{2}b \cdot \overline{AD} \cdot \sin(\alpha/2) = \frac{1}{2}n \cdot \overline{AD} \cdot \sin(\angle ADC)$, hence $\frac{b}{n} = \frac{\sin(\angle ADC)}{\sin(\alpha/2)}$. Since $\angle ADC = 180° - \angle ADB$, we get $\frac{c}{m} = \frac{b}{n}$, i.e., $m : n = c : b$. □

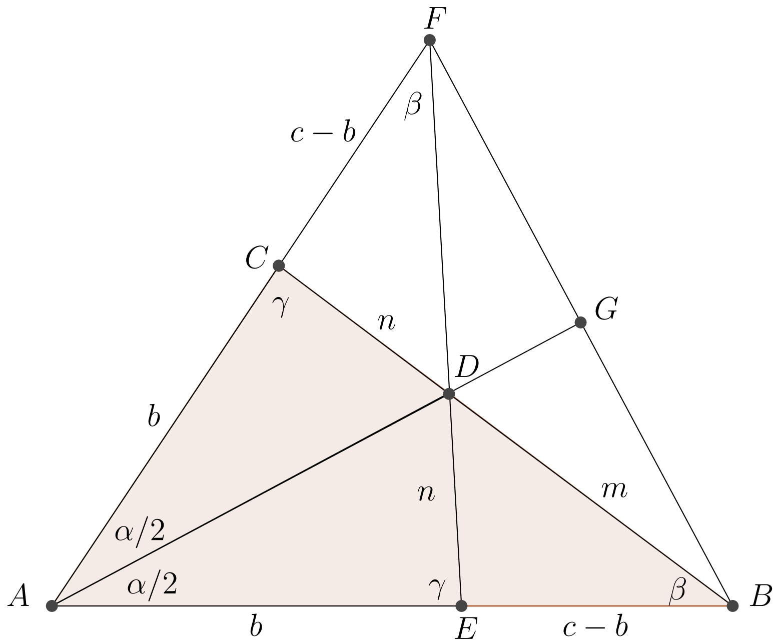


Figure 13

**Fifteenth proof.**

*Method*: Application of the Sine Theorem to two triangles and a calculation with the results.

*Facts used*: the Sine Theorem.

*Proof.* We will use the notation on Figure 0. Applying the Sine Theorem to $\triangle ABD$ we get $\frac{m}{\sin(\alpha/2)} = \frac{c}{\sin(\angle ADB)}$. Applying the same theorem to $\triangle ADC$ we get $\frac{n}{\sin(\alpha/2)} = \frac{b}{\sin(\angle ADC)}$. From these two relations we get $\frac{c}{m} = \frac{b}{n}$, i.e., $m : n = c : b$. $\square$

**Sixteenth proof.**

*Method*: Application of the Cosine Theorem to two triangles and a calculation with the results. In the calculation an auxiliary result about bisectors is used. To establish the auxiliary result, new objects are introduced, namely the circumscribed circle of $\triangle ABC$ and the quadrilateral $ABEC$. Then similarity of triangles is used.

*Facts used*: (a) the angles inscribed in the same arc are equal; (b) the lengths of the sides of similar triangles are proportional; (c) the Cosine Theorem.

*Proof.* Denote $d = \overline{DE}$. We will first show that $d^2 = bc - mn$ $(*)$. Let $k$ be the circumscribed circle of $\triangle ABC$. Let $AD$ be extended to meet $k$ at $E$. Denote $e = \overline{DE}$. Since the angles inscribed in the same arc are equal, we have $\angle BCE = \alpha/2$. (See Figure 5.) Hence $\triangle ABD \sim \triangle AEC$ and this implies $\frac{b}{d+e} = \frac{d}{c}$. Hence $bc = d^2 + ed$. Since $\triangle ADC \sim \triangle BDE$ implies $d : n = m : e$, i.e., $ed = mn$, we get $bc = d^2 + mn$, i.e., $(*)$ holds.

Now we apply the Cosine Theorem to $\triangle ADC$ and $\triangle ABD$. We get

$$\cos(\alpha/2) = \frac{d^2 + c^2 - m^2}{2dc} = \frac{d^2 + b^2 - n^2}{2db}.$$

Hence $b(d^2 + c^2 - m^2) = c(d^2 + b^2 - n^2)$. Hence (using $(*)$) $b(bc - mn + c^2 - m^2) = c(bc - mn + b^2 - n^2)$. After multiplying and cancelling, this implies $cn = bm$, i.e., $m : n = c : b$. □

**Seventeenth proof.**

*Method*: Using Analytic Geometry.

*Facts used*: (a) some trigonometric formulas; (b) the lengths of the sides of similar triangles are proportional.

*Proof.* Consider $\triangle ABC$ in the $xOy$ coordinate system, assuming that $A$ is at $(0, 0)$ and $B$ is on the $x$-axis, $B = (p, 0)$. Let $C = (r, s)$ and $D = (x_D, y_D)$. (See Figure 14.) Let $E = (r, 0)$ (respectively $F = (x_D, 0)$) be the projection of $C$ (respectively $D$) on the $x$-axis. We have $\cos\alpha = r/\sqrt{r^2 + s^2}$, hence $\tan\alpha/2 = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{s}{\sqrt{r^2+s^2}+r}$. The line through $A$ and $D$ has the equation $y = \tan\alpha/2 = \frac{s}{\sqrt{r^2+s^2}+r}\, x$. The line through $B$ and $C$ has the equation $y = \frac{s}{r-p}(x - p)$. Since $D$ is the intersection of these lines, we get

$$\frac{s}{\sqrt{r^2 + s^2} + r} x_D = \frac{s}{r - p}(x_D - p).$$

Hence

$$x_D = \frac{p(r + \sqrt{r^2 + s^2})}{p + \sqrt{r^2 + s^2}}.$$

We now have:

$$\begin{aligned} m : n &= \overline{BF} : \overline{FE} \\ &= (p - x_D) : (x_D - r) \\ &= (p - \frac{p(r + \sqrt{r^2 + s^2})}{p + \sqrt{r^2 + s^2}}) : (\frac{p(r + \sqrt{r^2 + s^2})}{p + \sqrt{r^2 + s^2}} - r) \\ &= p : \sqrt{r^2 + s^2} \\ &= \overline{AB} : \overline{AC} \\ &= c : b. \end{aligned}$$

□

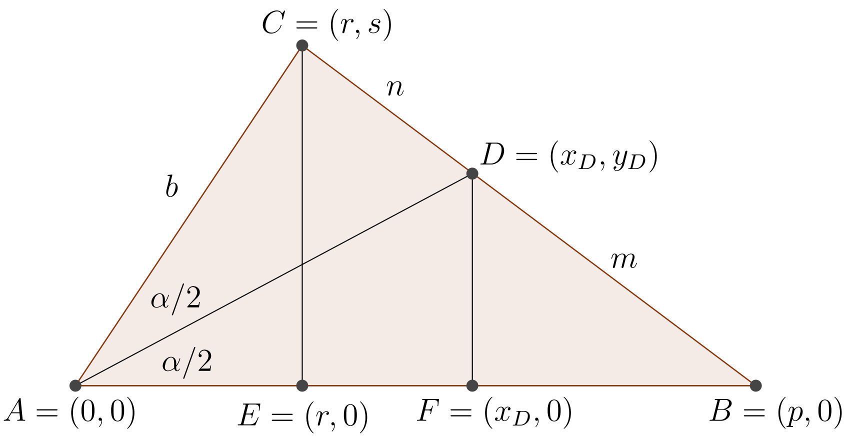


Figure 14

**Eighteenth proof.**

*Method*: Using Complex Numbers.

*Facts used*: properties of the complex numbers, in particular the complex number condition for a point to belong to the line through two given points.

*Proof.* We will consider all the points as points in the complex plane. We will denote the complex number corresponding to a point by the same letter which denotes the point itself. Without loss of generality we may assume that $A$, $B$,

$C$ belong to the unit circle. Moreover, without loss of generality we may also assume that $B$ and $C$ are conjugate numbers. (See Figure 15.) We will use the well-known fact that a complex number $Z$ belongs to the line through the complex numbers $W_1$ and $W_2$ if and only if $Z + W_1W_2\overline{Z} = W_1 + W_2$ $(*)$. Note that the bisector of $\angle BAC$ passes through the midpoint of the arc $BC$, i.e., through the point $1$. Hence by $(*)$ we have

$$D + A\overline{D} = A + 1. \tag{2.1}$$

Note also that

$$D + \overline{D} = B + \overline{B}. \tag{2.2}$$

We will show that $\frac{bm}{cn} = 1$, i.e., that $\left|\frac{(A-C)(B-D)}{(A-B)(C-D)}\right| = 1$, i.e., that $\left|\frac{(A-\overline{B})(B-D)}{(A-B)(\overline{B}-D)}\right| = 1$. We have:

$$\begin{aligned}
(A-\overline{B})(B-D) &= AB - \overline{B}B - AD + \overline{B}D \\
&= AB - 1 - AD + \overline{B}D \\
&= AB + A\overline{B} - A\overline{B} + \overline{B}D - 1 - AD \\
&= A(B+\overline{B}) - A\overline{B} + \overline{B}D - 1 - AD \\
&= A(D+\overline{D}) - A\overline{B} + \overline{B}D - 1 - AD && \text{by } (2) \\
&= A\overline{D} - A\overline{B} + \overline{B}D - 1 \\
&= A - D - A\overline{B} + \overline{B}D && \text{by } (1) \\
&= (A-D)(1-\overline{B}).
\end{aligned}$$

In an analogous way we get

$$(A-B)(\overline{B}-D) = (A-D)(1-B).$$

From these two relations we get $\left|\frac{(A-\overline{B})(B-D)}{(A-B)(\overline{B}-D)}\right| = \left|\frac{(A-D)(1-\overline{B})}{(A-D)(1-B)}\right| = \left|\frac{1-\overline{B}}{1-B}\right| = 1$. Thus $\frac{bm}{cn} = 1$, i.e., $m : n = c : b$. □

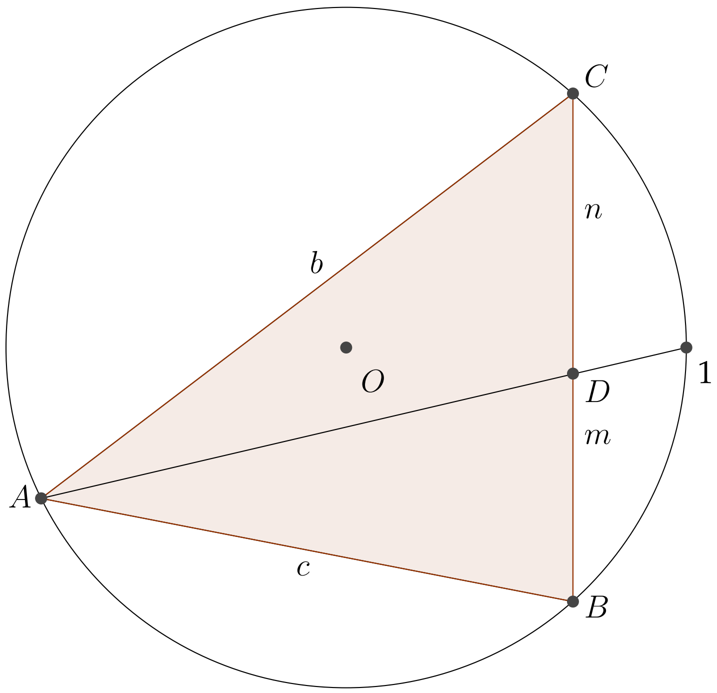


Figure 15

**Nineteenth proof.**

*Method*: Using Gröbner Bases.

*Facts used*: (Hilbert Nullstellensatz [1]) Let $K$ be a field, $L$ an algebraically closed extension field of $K$, and $f, g_1, \ldots, g_m \in K[X_1, \ldots, X_n]$. Then the following are equivalent:

(i) For all $\mathbf{z} \in L^n$, $g_1(\mathbf{z}) = \cdots = g_m(\mathbf{z}) = 0$ implies $f(\mathbf{z}) = 0$.

(ii) $f \in \operatorname{rad}(\operatorname{ideal}(g_1, \ldots, g_m))$.

We will use this theorem with $K = \mathbb{Q}$ and $L = \mathbb{C}$.

*Proof.* Consider $A, B, C$ as points in the coordinate plane with coordinates $A = (0, 0)$, $B = (b_1, 0)$, $C = (c_1, c_2)$. Let the coordinates of $D$ be $D = (d_1, d_2)$. Let $E$ be a point colinear with $A$ and $C$, such that $\overline{AB} = \overline{AE}$ and either $C$ is between $A$ and $E$, or $E$ is between $A$ and $C$ (including the possibility $E = C$). Let $E = (e_1, e_2)$ and let $F = (f_1, f_2)$ be the midpoint of the segment $BE$. (See Figure 16.) Consider the following conditions:

$$b_1^2 - e_1^2 - e_2^2 = 0, \tag{2.3}$$

$$c_1e_2 - c_2e_1 = 0, \tag{2.4}$$

$$2f_1 - e_1 - b_1 = 0, \tag{2.5}$$

$$2f_2 - e_2 = 0, \tag{2.6}$$

$$d_1f_2 - d_2f_1 = 0, \tag{2.7}$$

$$c_2(d_1 - b_1) - d_2(c_1 - b_1) = 0, \tag{2.8}$$

$$b_1 \neq 0, \tag{2.9}$$

$$c_2 \neq 0, \tag{2.10}$$

$$b_1^2((d_1 - c_1)^2 + (d_2 - c_2)^2) - (c_1^2 + c_2^2)((d_1 - b_1)^2 + d_2^2) = 0. \tag{2.11}$$

The condition (2.3) means that $\overline{AB} = \overline{AE}$; the condition (2.4) that $A, C, E$ are colinear; the conditions (2.5) and (2.6) together that $F$ is the midpoint of $BE$, which in turn means that $AD$ is the bisector of the angle $\angle BAC$; the condition (2.7) that $A, D, F$ are colinear; the condition (2.8) that $B, D, C$ are colinear; the conditions (2.9) and (2.10) that $A, B, C$ are not colinear; the condition (2.11) that $\dfrac{\overline{AB}}{\overline{AC}} = \dfrac{\overline{BD}}{\overline{CD}}$.

We will consider the polynomial ring $\mathbb{Q}[X_1, \ldots, X_9, Y_1, Y_9]$ and the following elements of it (corresponding to the above conditions in the order in which they are listed):

$$
\begin{aligned}
h_1 &= X_1^2 - X_6^2 - X_7^2,\\
h_2 &= X_2X_7 - X_3X_6,\\
h_3 &= 2X_8 - X_6 - X_1,\\
h_4 &= 2X_9 - X_7,\\
h_5 &= X_4X_9 - X_5X_8,\\
h_6 &= X_3(X_4 - X_1) - X_5(X_2 - X_1),\\
h_7 &= X_1Y_1 - 1,\\
h_8 &= X_3Y_2 - 1,\\
h_9 &= X_1^2((X_4 - X_2)^2 + (X_5 - X_3)^2) - (X_2^2 + X_3^2)((X_4 - X_1)^2 + X_5^2).
\end{aligned}
$$

Our goal is to prove that the conditions (2.3)-(2.10) imply the condition (2.11) for any positive real numbers $b_1, c_1, c_2, d_1, d_2, e_1, e_2, f_1, f_2$. Hence it is enough to prove that whenever $\mathbf{z} \in \mathbb{C}^{11}$ is a zero of the polynomials $h_1 - h_8$, it is a zero of the polynomial $h_9$. By the above Hilbert Nullstellensatz, that is equivalent to proving that $h_9$ belongs to the radical of the ideal generated by the polynomials $h_1 - h_8$. That can be easily verified using the computer algebra systems, like, for example, Macaulay 2. (Note that the variables $Y_1, Y_2$ were introduced to make sure that no common zero $\mathbf{z}$ of the polynomials $h_1 - h_8$ has the $X_1$ value nor the $X_3$ value equal to $0$, which secures the condition that $A, B, C$ are not colinear, which in turn secures that the ideal membership program in computer algebra systems works properly.) $\square$

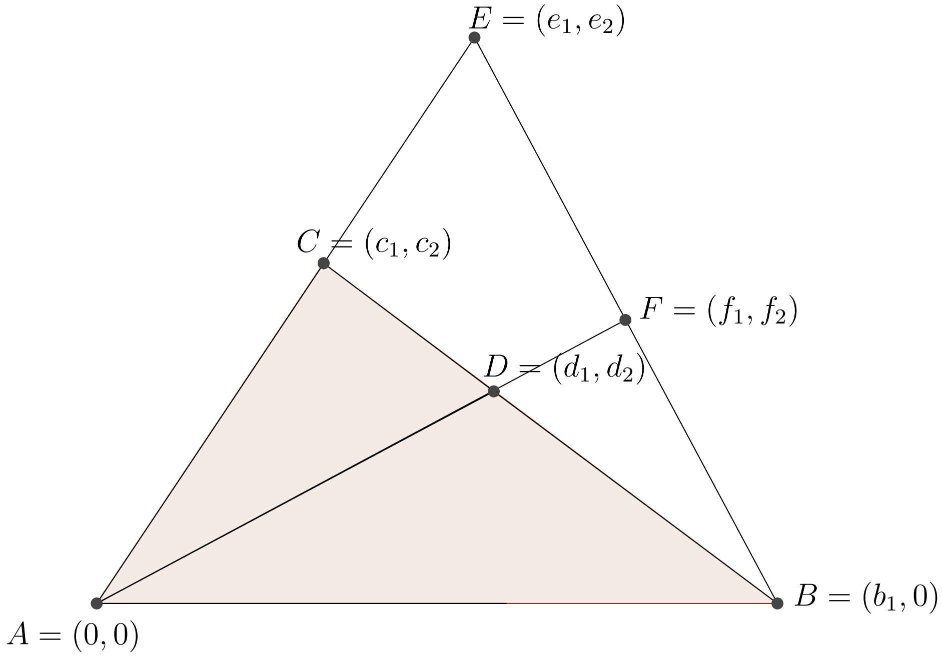


Figure 16

## 3. Appendix

**Van Obel's Theorem.** In a triangle $\triangle ABC$ let $D$, $E$, $F$ be points on $BC$, $CA$ and $AB$ respectively, such that the segments $AD$, $BE$ and $CF$ intersect at a common point $S$. Then

$$\frac{\overline{CS}}{\overline{SF}} = \frac{\overline{CE}}{\overline{EA}} + \frac{\overline{CD}}{\overline{DB}}.$$

*Proof.* Let $\ell$ be the line through $C$ parallel with $AB$. Let $G$ be the intersection of $\ell$ with the line through $A$ and $D$, and $H$ the intersection of $\ell$ with the line through $B$ and $E$. (See Figure 16.) Denote $c_1 = \overline{AF}$, $c_2 = \overline{FB}$, $b_1 = \overline{CE}$, $b_2 = \overline{EA}$, $d_1 = \overline{CS}$, $d_2 = \overline{SF}$. From $\triangle ASF \sim \triangle CSG$ we get $\frac{c_1}{d_2} = \frac{\overline{CG}}{d_1}$, hence $\overline{CG} = \frac{c_1 d_1}{d_2}$. Similarly from $\triangle BSF \sim \triangle CSH$ we get $\overline{CH} = \frac{c_2 d_1}{d_2}$. Hence $\overline{HG} = \frac{(c_1+c_2)d_1}{d_2}$ $(*)$. From $\triangle ABD \sim \triangle CDG$ we have $\frac{c_1+c_2}{a_2} = \frac{\overline{CG}}{a_1}$, hence $\overline{CG} = \frac{a_1(c_1+c_2)}{a_2}$. From $\triangle ABE \sim \triangle CEH$ we have $\frac{c_1+c_2}{b_2} = \frac{\overline{CH}}{b_1}$, hence $\overline{CH} = \frac{b_1(c_1+c_2)}{b_2}$. Hence $\overline{HG} = \overline{CG} + \overline{CH} = (\frac{a_1}{a_2} + \frac{b_1}{b_2})(c_1 + c_2)$ $(**)$. Now from $(*)$ and $(**)$ we get $\frac{d_1}{d_2} = \frac{a_1}{a_2} + \frac{b_1}{b_2}$. $\square$

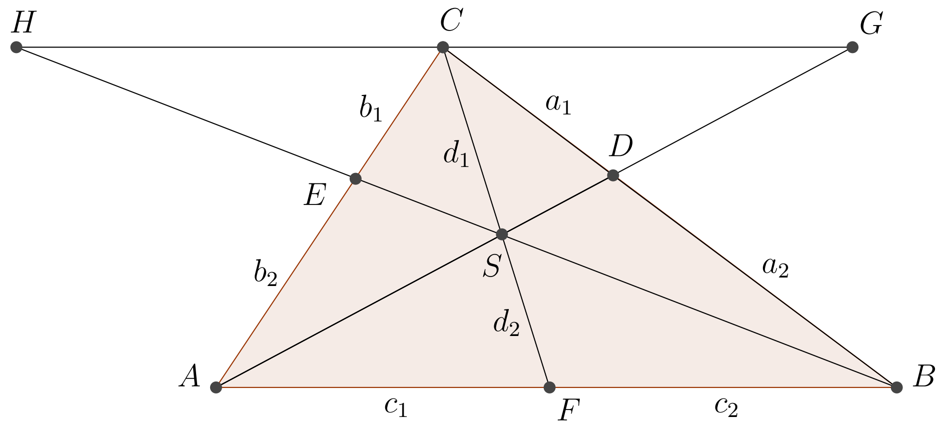


Figure 17

## References


[1] T. Becker, V. Weispfenning: *Gröbner Bases*, Springer-Verlag New York, Inc., 1993.

[2] R. Fitzpatrick: *Euclid's Elements of Geometr*, farside.ph.utexas.edu /euclid/Elements.pdf, 2008.

[3] R. Thom: *Modern" Mathematics: An Educational and Philosophic Error?*, p. 67-78, in *New Directions in the Philosophy of Mathematics*, edited by T. Tymoczko, Princeton University Press, 1998.

[4] Article *Euclid* on en.wikipedia.org.

[5] Article *Euclid's Elements* on en.wikipedia.org.

[6] D. Wang: *Gröbner Bases Applied to geometric Proving and Discovering*, p. 281-301 in *Gröbner Bases and Applications*, edited by B. Buchberger and F. Winkler, Cambridge University Press, 1998.



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