A NOTE ON WEAK ODD EDGE-COLORINGS OF GRAPHS

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ABSTRACT. An edge-coloring of a graph $G$ is said to be a weak-odd edge-coloring if each non-isolated vertex of $G$ uses at least one color odd number of times on its incident edges. The weak-odd chromatic index $\chi_{\text{wod}}(G)$ is the minimum number of colors needed for a weak-odd edge-coloring of $G$. In this paper, we prove that any graph without trivial nonempty components admits a weak-odd edge-coloring, and characterize connected graphs according to the value of their weak-odd chromatic index.

1. Introduction

Throughout the article we mainly follow terminology and notation used in [2]. A graph $G$ is always regarded as being finite (i.e. having finite nonempty set of vertices $V(G)$, and finite set of edges $E(G)$) with loops and multiple edges allowed. A loopless graph without multiple edges is referred to as a simple graph. The parameters $n(G) = |V(G)|$ and $m(G) = |E(G)|$ are called order and size of $G$, respectively. Whenever $n(G) = 1$ we say $G$ is trivial, and whenever $m(G) = 0$ we say $G$ is empty. For $X \subseteq V(G) \cup E(G)$, the subgraph of $G$ obtained by removing the vertices and edges of $X$ is denoted by $G - X$. If $X = \{x\}$ is a singleton, we write $G - x$ rather than $G - \{x\}$. Given a cut vertex $v$ of $G$, let $V_1, \ldots, V_k$ be the vertex sets of the components of $G - v$, and $H_i = G[V_i \cup \{v\}]$, for $i = 1, \ldots, k$. Each such $H_i$ is called a $v$-lobe of $G$.

We refer to each vertex $v$ of even (resp. odd) degree $d_G(v)$ as an even (resp. odd) vertex of $G$. In particular, a vertex of degree equal to 0 (resp. 1) is an isolated (resp. pendant) vertex. A graph is called even (resp. odd) whenever all its vertices are even (resp. odd).

Given a (not necessarily proper) edge-coloring $\varphi$ of a graph $G$ and a color $c$, we denote the fiber $\varphi^{-1}(c)$ by $E_c$, and the spanning subgraph of $G$ with edge set $E_c$ by $G_c$. For a vertex $v \in V(G)$, we say that $c$ appears at $v$ if $d_{G_c}(v) > 0$. Moreover, we say that $c$ is odd at $v$ whenever $d_{G_c}(v)$ is odd. The edge-coloring $\varphi$ is weak-odd at $v$ whenever at least one color is odd at $v$. And if this holds for every non-isolated vertex $v$ of $G$, then we speak of a weak-odd edge-coloring of $G$. Thus, we say that $\varphi$ is a weak-odd edge-coloring of $G$ if each non-isolated vertex appears as an odd vertex in at least one of the subgraphs induced by the different color classes.
A weak-odd edge-coloring of $G$ which uses at most $k$ colors is referred to as a **weak-odd $k$-edge-coloring**, and we then say that $G$ is **weak-odd $k$-edge-colorable**. Whenever $G$ admits a weak-odd edge-coloring, the **weak-odd chromatic index** $\chi'_\text{wo}(G)$ is defined to be the least integer $k$ for which $G$ is weak-odd $k$-edge-colorable.

Since each loop at a vertex $v$ colored with $c$ contributes 2 to the count of appearances of $c$ at $v$, it is obvious that a necessary and sufficient condition for the existence of a weak-odd edge-coloring of $G$ is the absence of vertices incident only to loops. Apart from this, the presence of loops does not influence the existence nor changes the value of the index $\chi'_\text{wo}(G)$.

A similar notion of **odd edge-coloring** of a graph $G$ was introduced by Pyber in his survey on graph coverings [7] as an edge decomposition of $G$ into (edge disjoint) odd subgraphs. Equivalently, it is an edge-coloring of $G$ such that at each non-isolated vertex every appearing color is odd. In his work, Pyber considered simple graphs and proved the following result.

**Theorem 1.1** (Pyber, 1991). *Every simple graph admits an odd edge-coloring with at most 4 colors.*

In [6], the authors considered the same notion for loopless graphs and proved an analogous result.

**Theorem 1.2** (Lužar et al., 2013). *Every loopless graph admits an odd edge-coloring with at most 6 colors.*

Furthermore, a characterization of the loopless graphs needing the maximum 6 colors is given in [6].

In the next section, we consider the related notion of weak-odd edge-coloring of graphs in more detail. We provide a tight upper bound for the weak-odd chromatic index and characterize graphs $G$ according to the value of $\chi'_\text{wo}(G)$.

## 2. Weak-odd edge-coloring

We have already determined which graphs $G$ are weak-odd edge-colorable. It is a simple matter to characterize those having $\chi'_\text{wo}(G) \leq 1$. Namely, $\chi'_\text{wo}(G) = 0$ holds exclusively for the empty graphs $G$, while $\chi'_\text{wo}(G) = 1$ if and only if $G$ is nonempty and the subgraph induced by the non-isolated vertices is odd.

Given a graph $G$, let $T$ be an even-sized subset of $V(G)$. Following [2], a spanning subgraph $H$ of $G$ is called a $T$-join if $d_H(v)$ is odd for every $v \in T$ and even for every $v \in V(G) \setminus T$. For example, if $P$ is an $xy$-path in $G$ (i.e. a path having endvertices $x$ and $y$), the spanning subgraph of $G$ with edge set $E(P)$ is an $\{x,y\}$-join. An obvious necessary condition for the existence of a $T$-join is that $T$ intersects every component of $G$ in an even number of vertices (possibly equal to 0). The following classical result about $T$-joins claims that this condition is also sufficient (see e.g. [8]).

**Lemma 2.1.** *Given a connected graph $G$, for every even-sized subset $T$ of $V(G)$ there exists a $T$-join in $G$.*

In particular, for a connected graph $G$ of even order, let $T$ denote the subset of even vertices. Since $T$ is even-sized, a $T$-join $H$ can be found in $G$. By setting $K = G - E(H)$ we obtain an odd factor of $G$, i.e. a spanning odd subgraph $K$. This proves the following result.
Lemma 2.2. Every connected graph $G$ of even order has an odd factor $K$.

While considering edge-colorings, it suffices to restrict attention to connected graphs. As an immediate consequence of the last lemma, we have the next proposition.

Proposition 2.1. If $G$ is a connected graph of even order, then $\chi_{wo}(G) \leq 2$.

Note that by leaving out the constraint about the order of $G$, we can do almost as good, as the following proposition demonstrates.

Proposition 2.2. Given a connected graph $G$ and a vertex $w \in V(G)$, there exists a 2-edge-coloring of $G$ that is weak-odd at each vertex distinct from $w$.

Proof. By Proposition 2.1, we may assume that $G$ is of odd order. Consider first the case when $w$ is a non-cut vertex of $G$. Lemma 2.2 implies that $G - w$ has an odd factor $K$. Thus, by coloring each member of $E(K)$ with 1 and each member of $E(G) \setminus E(K)$ with 2, we obtain a 2-edge-coloring of $G$ which is weak-odd at each vertex distinct from $w$.

Consider now the case when $w$ is a cut vertex of $G$. Denote by $V_1, \ldots, V_k$ the vertex sets of the components of $G - w$, and let $H_i = G[V_i \cup \{w\}], i = 1, \ldots, k$, be the $w$-lobes of $G$. For each $i$, since $w$ is a non-cut vertex of $H_i$, there exists an edge-coloring $\varphi_i$ of $H_i$ with the color set $\{1, 2\}$, which is weak-odd at each vertex distinct from $w$. Then, the union $\varphi_1 \cup \cdots \cup \varphi_k$ satisfies the same for $G$. \hfill $\Box$

Corollary 2.1. If a connected graph $G$ has at least one odd vertex, then $\chi'_{wo}(G) \leq 2$.

Proof. Let $w$ be an odd vertex of $G$. Clearly, any edge-coloring of $G$ is weak-odd at $w$. By the previous proposition, there exists a 2-edge-coloring $\varphi$ of $G$ which is weak-odd at every vertex distinct from $w$. Thus, $\varphi$ is a weak-odd 2-edge-coloring of $G$. \hfill $\Box$

Next, we show that three colors suffice for a weak-odd edge-coloring of any connected nontrivial graph.

Proposition 2.3. For every connected nontrivial graph $G$ it holds that $\chi'_{wo}(G) \leq 3$. Moreover, equality is attained if and only if $G$ is a nontrivial even graph of odd order.

Proof. We may restrict to even connected graphs of odd order at least 3. Let $G$ be such and take $v$ to be any non-cut vertex of $G$ (there are at least two such vertices). By Lemma 2.2, there exists an odd factor $K$ of $G - v$. Select an arbitrary edge $e$ incident to $v$. We obtain a weak-odd 3-edge-coloring of $G$ by coloring $E(K)$ with 1, the edge $e$ with 2, and each of the remaining non-colored edges with 3.

For the second part of the statement, suppose there exists a nontrivial connected even graph $G$ of odd order that is weak-odd 2-edge-colorable. For such an edge-coloring, each color class induces an odd factor of $G$. But this is clearly impossible, for it implies that any such odd factor is a graph with odd number of odd vertices. This completes the proof. \hfill $\Box$

Thus, we are able to characterize the connected graphs according to the value of their weak-odd chromatic index. Recall from the introduction that a connected graph admits weak-odd edge-colorings if and only if its edge set does not consist only of loops.
Theorem 2.1. Given a connected nontrivial graph $G$, it holds that

$$
\chi'_{wo}(G) = \begin{cases} 
1 & \text{if } G \text{ is odd,} \\
3 & \text{if } G \text{ is even and of odd order,} \\
2 & \text{otherwise.} 
\end{cases}
$$

3. Concluding remarks and further work

The notion of weak-odd edge-coloring of graphs naturally gives an analogous notion for digraphs. Namely, a (not necessarily proper) edge-coloring of a digraph $D$ is said to be weak-odd whenever for each vertex $v \in V(D)$ at least one color $c$ satisfies the following requirement: if $d^+(v) > 0$ then $c$ appears an odd number of times on the outgoing edges at $v$; and if $d^-(v) > 0$ then $c$ appears an odd number of times on the ingoing edges at $v$. We will address this matter in forthcoming works (for example, in [5]).

References


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