VISIBILITY IN PROXIMAL DELAUNAY MESHES AND STRONGLY NEAR WALLMAN PROXIMITY

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Dedicated to the Memory of Som Nainpally

ABSTRACT. This paper introduces a visibility relation \( v \), leading to the strongly visible relation \( \tilde{v} \) on proximal Delaunay meshes. Two main results in this paper are that the visibility relation \( v \) is equivalent to Wallman proximity and the strongly near proximity \( \tilde{v} \) is a Wallman proximity. In addition, a Delaunay triangulation region endowed with the visibility relation \( v \) has a local Leader uniform topology.

1. INTRODUCTION

Delaunay triangulations, introduced by B.N Delone [Delaunay] [3], represent pieces of a continuous space. A triangulation is a collection of triangles, which includes the edges and vertices of the triangles in the collection.

A 2D Delaunay triangulation of a set of sites (generators) \( S \subseteq \mathbb{R}^2 \) is a triangulation of the points in \( S \). The set of vertices (called sites) in a Delaunay triangulation define a Delaunay mesh. A Delaunay mesh endowed with a nonempty set of proximity relations is a proximal Delaunay mesh. A proximal Delaunay mesh is an example of a proximal relator space [20], which is an extension of a Szász relator space [21, 22, 23].

Let \( S \subseteq \mathbb{R}^2 \) be a set of distinguished points called sites (mesh generating points), \( p, q \in S \), \( pq \) a straight line segment in the Euclidean plane. A site \( p \) in a straight line segment \( pq \) is visible to another site \( q \) in the same straight line segment, provided there is no other site between \( p \) and \( q \). New forms of proximity are found via the geometry of visibility.

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Example 1.1. Visible Points.
Let \( p, q, r, s, t \in S \), a set of sites. A pair of Delaunay triangles \( \triangle(pqr), \triangle(rst) \) are shown in Fig. 1. Points \( r, q \) are visible from \( p \), since \( \overline{pT} \) and \( \overline{pr} \) are straight line segments with no other sites in between the endpoints. However, in the straight line segment \( \overline{ps}, s \) is not visible from \( p \), since site \( r \) is blocking \( p \)'s view of \( s \). Similarly, points \( p, r \) are visible from \( q \) but in the straight line segment \( \overline{qt}, t \) is not visible from \( q \). From \( r \), points \( p, q, s, t \) are visible. For more about visibility, see [7].

A straight edge connecting sites \( p \) and \( q \) is a Delaunay edge if and only if the Voronoi region of \( p \) [6, 18] and Voronoi region of \( q \) intersect along a common line segment [5, §1.1, p. 3]. For example, in Fig. 2, the intersection of Voronoi regions \( V_p, V_q \) is a common edge, i.e., \( V_p \cap V_q = \overline{pq} \), and \( p \) and \( q \) are connected by the straight edge \( \overline{pq} \). Hence, \( \overline{pq} \) is a Delaunay edge in Fig. 2.

\[ \triangle(pqr) \]

A triangle with vertices \( p, q, r \in S \) is a Delaunay triangle (denoted \( \triangle(pqr) \) in Fig. 2), provided the edges in the triangle are Delaunay edges. This paper introduces proximal Delaunay triangulation regions derived from the sites of Voronoi regions [18], which are named after the Ukrainian mathematician Georgy Voronoi [25].

A nonempty set \( A \) of a space \( X \) is a convex set, provided \( aA + (1 - a)A \subset A \) for each \( \alpha \in [0, 1] \) [1, §1.1, p. 4] (see, also, [10]). A simple convex set is a closed half plane (all points on or on one side of a line in \( R^2 \) [6]) The edges in a Delaunay mesh are examples of convex sets. A closed set \( S \) in the Euclidean space \( E^n \) is convex if and only if to each point in \( E^n \) there corresponds a unique nearest point in \( S \). In this paper, \( E \) denotes a normed linear space and the set of sites \( S \) is a subset of \( E \). For \( z \in S \), a closed set in \( R^n \),

\[ S_z = \left\{ x \in E : ||x - z|| = \inf_{y \in S} ||x - y|| \right\} \]

which is a convex cone [24].

Lemma 1.1. [6, §2.1, p. 9] The intersection of convex sets is convex.

Proof. Let \( A, B \subset R^2 \) be convex sets and let \( K = A \cap B \). For every pair of points \( x, y \in K \), the line segment \( \overline{xy} \) connecting \( x \) and \( y \) belongs to \( K \), since this property holds for all points in \( A \) and \( B \). Hence, \( K \) is convex. \( \Box \)
2. Preliminaries

Delaunay triangles are defined on a finite-dimensional normed linear space \( E \) that is topological. For simplicity, \( E \) is the Euclidean space \( \mathbb{R}^2 \). The closure of \( A \subset E \) (denoted \( \text{cl}(A) \)) is defined by

\[
\text{cl}(A) = \{ x \in X : D(x, A) = 0 \}, \text{ where } D(x, A) = \inf \{ ||x - a|| : a \in A \},
\]

i.e., \( \text{cl}(A) \) is the set of all points \( x \) in \( X \) that are close to \( A \) (\( D(x, A) \) is the Hausdorff distance [10, §22, p. 128] between \( x \) and the set \( A \) and \( ||x - a|| \) is the Euclidean distance between \( x \) and \( a \)).

Let \( A^c \) denote the complement of \( A \) (all points of \( E \) not in \( A \)). The boundary of \( A \) (denoted \( \text{bdy}A \)) is the set of all points that are near \( A \) and near \( A^c \) [14, §2.7, p. 62]. An important structure is the interior of \( A \) (denoted \( \text{int}A \)), defined by \( \text{int}A = \text{cl}A - \text{bdy}A \). For example, the interior of a Delaunay edge \( \overline{pq} \) are all of the points in the segment, except the endpoints \( p \) and \( q \). Notice that the interior of a Delaunay triangle is empty.

In general, a relator is a nonvoid family of relations \( \mathcal{R} \) on a nonempty set \( X \). The pair \((X, \mathcal{R})\) is called a relator space. Let \( E \) be endowed with the relator \( \mathcal{R}_d \) defined by

\[
\mathcal{R}_d = \left\{ \delta, \delta^*, \delta^\#, \delta^\gamma \right\},
\]

called a proximal relator (cf. [20]), containing the the proximities \( \delta, \delta^*, \delta^\#, \delta^\gamma \). The Delaunay tesselated space \( E \) endowed with the proximal relator \( \mathcal{R}_d \) (briefly, \( \mathcal{R} \)) is a Delaunay proximal relator space.

The proximity relations \( \delta \) (near), \( \delta^* \) (strongly near) and their counterparts \( \delta^\# \) (far) and \( \delta^\gamma \) (strongly far) facilitate the description of properties of Delaunay edges, triangles, triangulations and regions. The strongly near proximity \( \delta^\# \) was introduced in [17].

Remark 2.1. The notation \( \delta^\gamma \) for the strongly far proximity was suggested by C. Guadagni [9]. For the use of \( \delta^\# \) in local proximity spaces, see [8, §2.2, p. 7]. The notation for the far proximity \( \delta^\# \) is commonly used (see, e.g., [4, 12]). The variant notation \( \delta^\# \) for the far proximity is also used [14]. For various forms of proximity, see [4, 8, 15, 12, 13, 14, 16].

Let \( A, B \subset E \). The set \( A \) is near \( B \) (denoted \( A \delta B \)), provided \( \text{cl}A \cap \text{cl}B \neq \emptyset \) [4] (closure axiom). The Wallman proximity \( \delta \) (named after H. Wallman [26]) satisfies the closure axiom as well as the four Čech proximity axioms [2, §2.5, p. 439] and is central in near set theory [14, 15]. Sets \( A, B \) are far apart (denoted \( A \delta^\# B \)), provided \( \text{cl}A \cap \text{cl}B = \emptyset \). For example, Delaunay edges \( \overline{pq} \delta \overline{qr} \) are near, since the edges have a common point, i.e., \( q \in \overline{pq} \cap \overline{qr} \) (see, e.g., \( pq \delta \overline{qr} \) in Fig. 2). By contrast, edges \( \overline{pr}, \overline{pq} \) have no points in common in Fig. 2, i.e., \( \overline{pr} \delta^\# \overline{pq} \).
Voronoi regions $V_p, V_q$ are strongly near (denoted $V_p \lessdot V_q$) if and only if the regions have a common edge.

**Example 2.1. Near and Strongly Near Sets.**

In the Delaunay mesh in Fig. 3, let $C, G, H$ be mesh triangles. $H \lessdot C$, since each these pairs of triangles have a common vertex. $H \lessdot G$ and $G \lessdot C$ (the triangles are strongly near), since these triangles have a common edge. Similarly, in Fig. 2, $V_p \lessdot V_q$. In general, strongly near Delaunay triangles have a common edge. Delaunay triangles $\triangle(pqr)$ and $\triangle(qrt)$ are strongly near in Fig. 4, since edge $\overline{qr}$ is common to both triangles. In that case, we write $\triangle(pqr) \lessdot \triangle(qrt)$.

Nonempty sets $A \not\lessdot C$ are strongly far apart (denoted $A, C$), provided $C \subset \text{int}(\text{cl}B)$ and $A \not\lessdot B$.

**Example 2.2. Far and Strongly Far Sets.**

In the Delaunay mesh in Fig. 3, sets $A$ and $B$ have no points in common. Hence, $A \not\lessdot B$ (A is far from B). Also in Fig. 3, let $C = \{\triangle(pqu)\}$. Consequently, $C \subset \text{int}(\text{cl}B)$, such that triangle $\triangle(pqu)$ lies in the interior of the closure of $B$. Hence, $A \not\lessdot C$.

**Figure 4. Strongly Visible Sets $A \lessdot^* B$**

Let $A, B$ be subsets in a Delaunay mesh, $\triangle(pqr) \in B, \triangle(qrt) \in A$. Subsets $A, B$ in a Delaunay mesh are visible to each other (denoted $A \lor B$), provided at least one triangle vertex $x \in \text{cl}A \cap \text{cl}B$. That is, if there is at least one site in $A$ visible to at least one site in $B$, then $A \lor B$. $A, B$ are strongly visible to each other (denoted $A \lor^* B$), provided at least one triangle edge is common to $A$ and $B$.

**Example 2.3. Visibility in Delaunay Meshes.**

In the Delaunay mesh in Fig. 3, $A \lor D$, since $A$ and $D$ have one triangle vertex is common, namely, vertex $r$. Sets $B$ and $D$ in Fig. 3 are strongly visible (i.e., $B \lor^* D$), since edge $\overline{BD}$ is common to $B$ and $D$. In Fig. 3, let $C = \{\triangle(pqu)\}$. Then $C \lor B$, since $C \subset B$. In Fig. 4, edge $\overline{qr}$ is common to $A$ and $B$. $\overline{qr}$ is visible from $p \in B$ and from $t \in A$. Hence, $A \lor^* B$.

Subsets $A, B$ in a Delaunay mesh are invisible to each other (denoted $A \not\lor B$), provided $\text{cl}A \cap \text{cl}B = \emptyset$, i.e., $A$ and $B$ have no triangle vertices in common. $A, B$ are strongly
invisible to each other (denoted $A \nLeftarrows B$), provided $C \not\ni A$ for all sets of mesh triangles $C \subset B$.

Example 2.4. Invisible and Strongly Invisible Subsets in a Delaunay Mesh.
In the Delaunay mesh in Fig. 3, $A$ and $B$ are not visible to each other, since $clA \cap clB = \emptyset$, i.e., $A$ and $B$ have no triangle vertices in common. In Fig. 3, let $C = \{ \triangle(pqu) \}$. Then $A \nLeftarrows B$ ($A$ and $B$ are strongly invisible to each other), since $C \not\ni A$ for all sets of mesh triangles $C \subset B$.

3. MAIN RESULTS

The Delaunay visibility relation $\nu$ is equivalent to the proximity $\delta$.

Lemma 3.1. Let $A, B$ be subsets in a Delaunay mesh. $A \Leftarrow B$ if and only if $A \nu B$.

Proof. $A \Leftarrow B \iff clA \cap clB \neq \emptyset \iff A$ and $B$ have a triangle vertex in common if and only if $A \nu B$. $\Box$

Theorem 3.1. The visibility relation $\nu$ is a Wallman proximity.

Proof. Immediate from Lemma 3.1. $\Box$

Lemma 3.2. Let $A, B$ be subsets in a Delaunay mesh. $A \nLeftarrows B$ if and only if $A \nu B$.

Proof. $A \nLeftarrows B \iff \overline{pq}$ for some triangle edge common to $A$ and $B \iff AuB$, since $\overline{pq}$ is visible from a vertex in $A$ and from a vertex in $B$ and $A$ and $B$ have vertices in common. $\Box$

Theorem 3.2. The strong visibility relation $\nLeftarrows$ is a Wallman proximity.

Proof. Immediate from Theorem 3.1 and Lemma 3.2. $\Box$

Theorem 3.3. $\Leftarrow$ is a Wallman proximity.

Proof. Immediate from Lemma 3.1, Lemma 3.2 and Theorem 3.2. $\Box$

Remark 3.1. From Theorem 3.3, $\nLeftarrows$ is a strongly near Wallman proximity.

Theorem 3.4. Let $A, B$ be subsets in a Delaunay mesh. Then

1° $A \nLeftarrows B$ implies $A \nu B$.

2° $A \nLeftarrows B$ if and only if $A \nLeftarrows B$.

Proof.

1°: Given $A \nLeftarrows B$, then $A$ and $B$ have no triangle vertices in common. Hence, $A \not\ni B$.

2°: $A \nLeftarrows B$ if and only if $A$ and $B$ have no triangles in common if and only if $A \nLeftarrows B$. $\Box$

Theorem 3.5 is an extension of Theorem 3.1 in [19], which results from Theorem 3.1.

Theorem 3.5. The following statements are equivalent.

1° $\triangle(pqr)$ is a Delaunay triangle.

2° Circumcircle $\circ(pqr)$ has center $u = clV_p \cap clV_q \cap clV_r$. 
$3^v V_p \overset{\hat{v}}{\supset} V_q \overset{\hat{v}}{\supset} V_r$.

$\Delta(pqr)$ is the union of convex sets.

Let $P$ be a polygon. Two points $p, q \in P$ are visible, provided the line segment $\overline{pq}$ is in $\text{int}P$ [7]. Let $p, q \in S, L$ a finite set of straight line segments and let $\overline{pq} \in L$. Points $p, q$ are visible from each other, which implies that $\overline{pq}$ contains no point of $S - \{p, q\}$ in its interior and $\overline{pq}$ shares no interior point with a constraining line segment in $L - \overline{pq}$.

That is, $\text{int}\overline{pq} \cap S = \emptyset$ and $\overline{pq} \cap \overline{xy} = \emptyset$ for all $\overline{xy} \in L [5, \S II, p. 32]$.

**Theorem 3.6.** If points in $\overline{pq}$ are visible from $p, q$, then $\text{int}\overline{pq} \supseteq S - \{p, q\}$ and $\overline{pq} \cap \overline{xy} \subseteq L - \overline{pq}$ for all $x, y \in S - \{p, q\}$.

**Proof.** Symmetric with the proof of Theorem 3.2 [19].

A Delaunay triangulation region $\mathcal{D}$ is a collection of Delaunay triangles such that every pair of triangles in the collection is strongly near. That is, every Delaunay triangulation region is a triangulation of a finite set of sites and the triangles in each region are pairwise strongly near. Proximal Delaunay triangulation regions have at least one vertex in common. From Lemma 1.1 and the definition of a Delaunay triangulation region, observe

**Lemma 3.3.** [19] A Delaunay triangulation region is a convex polygon.

**Theorem 3.7.** [19] Proximal Delaunay triangulation regions are convex polygons.

A local Leader uniform topology [11] on a set in the plane is determined by finding those sets that are close to each given set.

**Theorem 3.8.** [19] Every Delaunay triangulation region has a local Leader uniform topology (application of [11]).

**Theorem 3.9.** [19] A Delaunay triangulation region endowed with the visibility relation $\nu$ has a local Leader uniform topology.

**Proof.** Let $\mathcal{D}$ be a Delaunay triangulation region. From Theorem 3.1 and Theorem 3.8, determine all subsets of $\mathcal{D}$ that are visible from each given subset of $\mathcal{D}$. For each $A \subseteq \mathcal{D}$, this procedure determines a family of Delaunay triangles that are visible from (near) each $A$. By definition, this procedure induces a local Leader uniform topology on $\mathcal{D}$.

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