CARTESIAN COMPOSITIONS IN FOUR DIMENSIONAL SPACE WITH AFFINE CONNECTIONS, WITHOUT TORSION AND ADDITIONS

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ABSTRACT. The affine connection space $A_4$, product spaces, product affinor $(a^a, b^a, c^a)$ with symmetrical and additional connections (asymmetric $P^a_{ab}$) where affine of structures continue to be transformed in parallel way along the lines in space; see [16].

1. Introduction

Let us take affine product on the four-dimensional space all along with symmetrical connections and addition $A_4$ which have been studied, see [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 16]. Let us take $A_4$, the affine symmetric space. In $A_4$ there have already been defined the products $X_3 \times X_2$, $Y_2 \times Y_2$, $Z_2 \times Z_2$, and $X_3 \times X_1$, (addition) in such a way that each of them has a multiple base on $A_4$, and they have been analyzed in [1, 4, 6, 8, 14]. We have already discussed the space $A_4$ with the additional structure on the space of independent vectors in [4, 11, 12, 13, 14, 15].

2. Preliminaries

Let $A_4$ be the space with affine symmetric connection. This will be presented with the formula $\Gamma_{\alpha\beta}^\gamma$ where the connection coefficients will be denoted $(\alpha, \beta, \gamma = 1, 2, 3, 4)$. In $A_4$ we consider the product $X_n \times X_m$ where $(n + m = 4)$. Both multipliers have differential bases. Let us take two transformation positions of $P(X_n)$ and $P(X_m)$, or $(P(X_n)$ and $P(X_n))$ of the multipliers at any point $A_4$, see [4, 6, 7, 9]. It is known that the product

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is in general defined based on affinors fields:

\[
\begin{align*}
\alpha^b = V^b_1 V^1_\alpha + V^b_2 V^2_\alpha - V^b_3 V^3_\alpha - V^b_4 V^4_\alpha \\
\beta^b = V^b_1 V^1_\alpha + V^b_3 V^3_\alpha + V^b_4 V^4_\alpha.
\end{align*}
\]

(2.1)

Also

\[
\begin{align*}
P^b_\alpha = V^b_1 V^1_\alpha - V^b_3 V^3_\alpha
\end{align*}
\]

(2.2)

and with affinor (2.2) we present the additional structure. Affinors (2.1) and affine (2.2) are called product affinors. We take them as affinors connected with the space \( A_4 \) in an integral structure of product. According to [1, 2, 3, 4, 9] and [8] the integral condition of structure is characterized with the equation:

\[
\begin{align*}
\nabla_\sigma a^\beta_\alpha = 0.
\end{align*}
\]

(2.3)

Using [16] there is differential of the equation (2.3) for the field of vectors \( V = (V^1_1, V^1_2, V^3_3, V^4_4) \) and the result is:

\[
\begin{align*}
\nabla_\sigma V^\beta_\alpha = T^\sigma_\alpha V^\beta, \quad \nabla_\sigma V^\beta_\beta = -T^\sigma_\sigma V^\beta_\beta.
\end{align*}
\]

(2.4)

With \( \{V_\alpha\} \) we mark the net of vectors. Let us take an independent vector in their field \( V^\beta_\alpha \). If we take \( \{V_\alpha\} \) the widen net of the coordinates where we have affine, projected affinors \( a^\beta_\alpha \) and \( a^\alpha_\alpha \) are defined by equation:

\[
\begin{align*}
a^\beta_\alpha = \frac{1}{2}(\delta^\beta_\alpha + a^\beta_\alpha), \quad a^\alpha_\alpha = \frac{1}{2}(\delta^\alpha_\alpha - a^\alpha_\alpha).
\end{align*}
\]

This equations meets the conditions:

\[
\begin{align*}
a^\beta_\alpha a^\alpha_\alpha = a^\beta_\beta, \quad a^\alpha_\alpha a^\beta_\beta = a^\beta_\alpha, \\
\alpha^\beta_\alpha a^\beta_\alpha = \delta^\beta_\alpha, \quad \alpha^\beta_\alpha a^\alpha_\beta = a^\alpha_\beta,
\end{align*}
\]

see [1, 2, 5, 6, 8].

For each vector \( V^\alpha \in A_4 \), in \((X_2 \times \overline{X}_2), (Y_2 \times \overline{Y}_2), (Z_2 \times \overline{Z}_2), (X_2 \times Y_2), (X_2 \times Z_2), (Y_2 \times \overline{X}_2), (Z_2 \times \overline{Z}_2)\), and \((X_2 \times X_1)\), we have:

\[
V^\alpha = a^\beta_\alpha V^\beta + a^\alpha_\beta V^\beta = V^\alpha + V^\beta.
\]

and the following equations hold:

\[
\begin{align*}
\nabla V^\alpha = a^\beta_\alpha V^\beta \in P(X_2), \quad V^\alpha = a^\alpha_\beta V^\beta \in P(\overline{X}_2).
\end{align*}
\]

These products have been studied in [1, 2, 3, 6, 5, 8].
The product \((C, C)\) (Cartesian, Cartesian) is called of type Cartesian if the positions of \(P(X_2)\) and \(P(X_3)\) are put parallel along the lines of \(A_4\) and are characterized with \((2.3)\). Let us see the vectors:

\[
W^\alpha_i = V^\alpha_i \\
W^\alpha_i = \frac{1}{\sqrt{2}} \left( V^\alpha_{i-4} + V^\alpha_i \right).
\]

From \((2.5)\) and the condition

\[
\overset{\alpha}{W_\beta} V^\sigma = \delta^\alpha_\beta \iff \overset{\sigma}{W_\alpha} W^\alpha = \delta^\alpha_\beta,
\]

we have that

\[
W^\alpha = V^\alpha - \sqrt{2} V^\alpha_i, \quad W^\alpha_i = \sqrt{2} V^\alpha_i,
\]

where

\[
\alpha, \beta, \sigma = (1, 2, 3, 4), \quad i = 1, 2, \quad \bar{i} = 3, 4.
\]

Let us see affine

\[
a^\beta_\alpha = W^\beta_i \overset{\alpha}{W^i} - \overset{\bar{i}}{W^i} \overset{\alpha}{W^\bar{i}}.
\]

From \((2.6)\) and \((2.8)\) we have \(a^\beta_\alpha a^\alpha_\delta = \delta^\beta_\delta\) and we say that affine \(a^\beta_\alpha\) satisfies the condition of production.

**Theorem 2.1.** The product \(X_2 \times X_2\) is of the type \((C, C)\) (Cartesian, Cartesian) if it satisfies the condition \(\nabla_\sigma \overset{\bar{i}}{a} = 0\).

**Proof.** Let us consider the condition

\[
\nabla_\sigma a^\beta_\alpha = 0, \quad \nabla_\sigma \delta^\beta_\alpha = 0.
\]

Based on \((2.3)\) and \((2.9)\) the condition for the product \(X_2 \times X_2\) is satisfied, and the product is of the type \((C, C)\). Further, based on the relations \((2.7)\) and \((2.8)\) we have that:

\[
\nabla_\sigma a^\beta_\alpha = 0, \quad \nabla_\sigma \overset{\bar{i}}{a} = 0,
\]

where

\[
d^\beta_\alpha = V^\beta_i \overset{\alpha}{V^i}, \quad d^\beta_\alpha = V^\beta_i \overset{\bar{i}}{V^\alpha}.
\]

Affine \(d^\beta_\alpha\) and \(d^\beta_\alpha\) are nilpotent because

\[
d^\beta_\alpha d^\sigma_\beta = 0 \quad \text{and} \quad d^\beta_\alpha d^\sigma_\alpha = 0.
\]

Finally, according to \((2.11)\) and \((2.10)\) even the products \((Y_2 \times Y_2), (Z_2 \times Z_2)\) are of the type \((C, C)\) by using relation \((2.1)\). So, according to \((2.9)\), \((2.10)\) and \((2.11)\) it holds \(\nabla_\sigma d^\beta_\alpha = 0\).

**Theorem 2.2.** If the products \(X_2 \times X_2, X_2 \times Y_2, X_2 \times Z_2, Y_2 \times X_2, \text{ and } Z_2 \times X_2,\) are of the type \((C, C)\), then the space \(A_4\) is affine.
Proof. According to the theorem 2.1 the products \( X_2 \times X_2, X_2 \times Y_2, X_2 \times Z_2, Y_2 \times X_2, \) and \( Z_2 \times X_2 \) are of the type \((C, C)\) if the condition (2.9) hold. Based on equation (2.8) and (2.11), equation (2.9) will be as the following:

\[
\nabla_\alpha \left( V^i_\beta \dot{V}_\alpha - V^j_\beta \dot{V}_\alpha \right) = 0
\]

\[
\nabla_\alpha \left( V^n_\beta \dot{V}_\alpha \right) = 0,
\]

\( i, j = 1, 2, \quad i, j = 3, 4, \quad n = 2. \)

This has been studied in [3, 4, 6, 7, 8]. From equation (2.4) we have:

\[
\begin{align*}
\frac{v^\beta}{T_\beta} u_i v_\alpha - \frac{v^\beta}{T_\beta} u_i v_\alpha - \frac{v^i}{T_\beta} u^\beta v_\alpha + \frac{v^i}{T_\beta} u^\beta v_\alpha &= 0 \\
\frac{v^\beta}{T_\beta} n_{i+i} - \frac{n^i}{T_\beta} \beta v_\alpha &= 0.
\end{align*}
\]

(2.12)

From the equation (2.12) we have the following:

\[
\begin{align*}
\frac{v^\beta}{T_\beta} - \frac{v^\beta}{T_\beta} &= 0, \\
\frac{v^\beta}{T_\beta} - \frac{v^\beta}{T_\beta} &= 0, \\
\frac{v^\beta}{T_\beta} - \frac{v^\beta}{T_\beta} &= 0.
\end{align*}
\]

If we work with independent vectors \( \left\{ V^\beta_\alpha \right\} \) we will get the equation:

\[
\begin{align*}
\frac{v^\beta}{T_\beta} &= 0, \\
\frac{i^\beta}{T_\beta} &= 0, \\
\frac{i^\beta}{T_\beta} - \frac{v^\beta}{T_\beta} &= 0.
\end{align*}
\]

(2.13)

If we use the net \( \left\{ V_\alpha \right\} \) of coordinate \( \left( V^\alpha_1, V^\alpha_2, V^\alpha_3, V^\alpha_4 \right) \) then the equation (2.13) would appear like the following:

\[
\begin{align*}
\Gamma_{\alpha i}^{\beta} &= 0, \\
\Gamma_{\alpha i}^{\beta} &= 0, \\
\Gamma_{\alpha i}^{\beta} - \Gamma_{\alpha i}^{2+i} &= 0.
\end{align*}
\]

Then \( \Gamma_{i\alpha}^{\beta} = 0 \) and we have that \( A_4 \) is affine. \( \square \)

3. Cartesian products with additional structure

Let \( P_\alpha^\beta \) be the affine in the relation (2.2). Then it is called paracontact affine and it holds:

\[
\begin{align*}
P_\alpha^\beta &= V^\beta_\alpha - V^i_\beta \dot{V}_\alpha.
\end{align*}
\]

We know that:

\[
\begin{align*}
V^\alpha_1 i &\delta_{\alpha}^\beta \\ \Rightarrow \quad DSP_\alpha^\beta &= \delta_{\sigma}^\alpha
\end{align*}
\]
From (3.1) and (3.2) we get that \( P^\beta_\alpha = \delta^\beta_\alpha - V^\beta_\alpha \). The affine (3.1) defines the paracontact structure in the space \( A_4 \), see [12, 13, 14, 16]. Using (3.2) and equations
\[ V^\alpha(1, 0, 0, 0), V^\alpha(0, 1, 0, 0), V^\alpha(0, 0, 1, 0), V^\alpha(0, 0, 0, 4), \]
\[ V^\alpha(1, 0, 0, 0), V^\alpha(0, 1, 0, 0), V^\alpha(0, 0, 1, 0), V^\alpha(0, 0, 0, 1), \]
with parameters of coordinated net \( \{ V_\alpha \} \) the matrix \( P^\beta_\alpha \) would look like the following:
\[
P^\beta_\alpha = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

**Theorem 3.1.** The equality \( \nabla_\sigma P^\beta_\alpha = 0 \) is fulfilled if and only if it holds
\[
(3.3) \quad \begin{array}{ccc}
T^1_i = T^3_3 = T^4_3 = 0.
\end{array}
\]

**Proof.** From relations (2.4) and (3.1) we can write the equation
\[
(3.4) \quad \nabla_\sigma P^\beta_\alpha = 0
\]
like:
\[
(3.5) \quad T^i_\sigma V^\beta_\sigma - T^i_\sigma V^\beta_\sigma - T^3_\sigma V^3_\sigma + T^3_\sigma V^3_\sigma = 0.
\]

Using simple operation the equation (3.5) with \( V^\alpha \) and \( V^\alpha \), and reading independence of vector fields \( V^\beta_\alpha \) we get that the equation (3.3) and (3.4) are equivalent, proving the theorem. \( \square \)

Next, using theorem 3.1 and equation
\[
\Gamma^\sigma_\alpha_\beta = T^\sigma_\alpha_\beta,
\]
we can write the tensor of the curve \( R^\alpha_\beta_\gamma_\sigma \) in the space \( A_4 \) like the following:
\[
(3.6) \quad R^\alpha_\beta_\gamma_\sigma = \partial_\alpha \Gamma^\alpha_\beta_\gamma - \partial_\beta \Gamma^\alpha_\gamma_\sigma + \Gamma^\alpha_\gamma_\delta \Gamma^\delta_\beta_\sigma - \Gamma^\alpha_\delta_\gamma \Gamma^\delta_\beta_\sigma.
\]

**Corollary 3.1.** In parameters of coordinative net \( \{ V \} \) and equation (3.3) we get the following equation
\[
(3.7) \quad \Gamma^i_\sigma_\gamma = \Gamma^i_\gamma_\sigma = \Gamma^i_\sigma_\gamma = \Gamma^i_\gamma_\sigma = 0.
\]

Based on continuity and the relation (3.6), see [4, 13, 14] we have the following:

**Corollary 3.2.** If affine \( P^\beta_\alpha \) satisfies the condition \( \nabla_\sigma P^\beta_\alpha = 0 \), then the product \( X_2 \times X_3 \) and \( X_3 \times X_1 \) are of the type (C, C) (Cartesian, Cartesian).
Proof. If we take in the space $A_4$ with additional paracontact structure $P^\beta_\alpha$ with a new asymmetric connection, we will get

\[ (3.8) \quad 1^1 \Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} + 1^1 A^\mu_{\alpha\beta} \]

Where $1^1 A^\mu_{[\alpha,\beta]}$ is torsion tensor with a new connection, written with $1^1 \nabla$ and $1^1 R^\mu_{\alpha\beta\sigma}$ is the co-variation of derivation and the tensor of curve in relation to the $1^1 \Gamma^\mu_{\alpha\beta}$, see [7, 8, 9, 14].

**Theorem 3.2.** If $\nabla_\sigma P^\beta_\alpha = 0$ and $1^1 \nabla_\sigma P^\beta_\alpha = 0$ then the tensor $1^1 A^\mu_{\alpha\beta}$ satisfy the condition

\[ (3.9) \quad 1^1 A^\alpha_{\mu\beta} - 1^1 A^\beta_{\mu\alpha} = 1^1 A^\alpha_{\alpha\beta} = 1^1 A^\beta_{\alpha\alpha} = 0. \]

Also, in the contracting net $V_\alpha$ the parameters are replaced.

Proof. The equation $1^1 \nabla_\sigma P^\beta_\alpha = 0$, holds. Based on (3.4) and (3.8) the line of the curve is $1^1 \nabla_\sigma P^\beta_\alpha = L^\beta_\alpha$. Then

\[ (3.10) \quad L^\beta_\alpha = 1^1 A^\mu_{\alpha\beta} P^\mu_\alpha - 1^1 A^\mu_{\alpha\sigma} P^\mu_\sigma. \]

Now it follows that (3.4) and (3.10) are equivalent.

Next, let us take the net $V_\alpha$ as a single coordinate $L^\beta_\alpha$ which is changeable from zero. We introduce the following:

\[ (3.11) \quad L^\alpha_{\alpha\beta} = \eta \cdot 1^1 A^\alpha_{\beta\gamma}, \quad L^\alpha_{\gamma\beta} = \chi \cdot 1^1 A^\alpha_{\gamma\beta}, \]

and

\[ L^3_{\alpha\beta} = \pi \cdot 1^3 A^3_{\alpha\beta}, \quad L^3_{\alpha\gamma} = \mu \cdot 1^3 A^3_{\alpha\gamma} \quad \text{for} \quad \eta, \chi, \pi, \mu = \pm 1 \pm 2 \pm ... \]

Now, from (3.11) we have (3.9).

According to equations (3.7), (3.8) and (3.9) we have that:

\[ (3.12) \quad 1^1 \Gamma^i_{\alpha\beta} = 1^1 \Gamma^i_{\alpha\beta} = 0, \quad 1^1 \Gamma^i_{\gamma\beta} = 1^1 \Gamma^i_{\gamma\alpha} = 0, \quad 1^1 \Gamma^i_{\gamma\gamma} = 1^1 \Gamma^i_{\gamma\gamma} = 0. \]

Finally from equations (3.6), (3.9) and (3.12) we get the components of the tensor $R^\mu_{\alpha\beta\sigma}$ and $1^1 R^\mu_{\alpha\beta\sigma}$:

\[ 1^1 R^i_{\alpha\beta\sigma} = R^i_{\alpha\beta\sigma} = 1^1 R^i_{\alpha\beta\sigma} = 1^1 R^i_{\alpha\beta\sigma} = 1^1 R^i_{\alpha\beta\sigma} = 1^1 R^i_{\alpha\beta\sigma} = 1^1 R^i_{\alpha\beta\sigma} = 1^1 R^i_{\alpha\beta\sigma} = 0. \]

\[ \square \]

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