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A NOTE ON CESÀRO OF ORDER K SUMMABILITY OF DILUTED P-FABER SERIES

NAZLIM DENIZ ARAL[1] AND TUNCAY TUNÇ

ABSTRACT. In this study, some relationships between the overconvergence of p-Faber series and the existence of an elongation of the sequence of the partial sums of the p-Faber series, whose Cesàro means of order k is convergent, are investigated.

1. INTRODUCTION

Let $G \subset \mathbb{C}$ be a Jordan domain, that is, its boundary $\partial G := L$ is a Jordan curve, and Φ be the conformal mapping of the domain $\Omega := \mathbb{C}_\infty \setminus \overline{G}$ where $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$, onto $\Delta := \{w : \ |w| > 1\}$ with the usual normalization at infinity:

$$w = \Phi(z) = \alpha z + a_0 + \frac{a_1}{z} + \cdots, \qquad \alpha > 0, \quad z \in \Omega. \tag{1.1}$$

Let $\Psi := \Phi^{-1} : \Delta \to \Omega$ denote the inverse conformal map. Then,

$$z = \Psi(w) = \beta w + b_0 + \frac{b_1}{w} + \cdots, \qquad |w| > 1,$$

where $\beta = 1/\alpha$ gives the capacity $cap(L)$ of L.

Let $0 < p < \infty$. The p-Faber polynomials $F_{n,p}$ associated the domain G are defined as the polynomial part of the Laurent expansion of

$$\Phi^n(z) \left(\Phi'(z)\right)^{1/p}, \qquad n = 0, 1, 2, \cdots,$$

in a neighborhood of the infinity. Therefore, from (1.1), we have the p-Faber polynomial of degree n

$$F_{n,p}(z) := \alpha^{n+1/p} z^n + \cdots$$

Also, the p-Faber polynomials associated with G can be defined by the generating function

$$g(w) := \frac{\left(\Psi'(w)\right)^{1-1/p}}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_{n,p}(z)}{w^{n+1}}, \qquad z \in G, \quad |w| > 1.$$

If p tends to infinity the p-Faber polynomials coincide with the usual Faber polynomials F_n.

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[1] *corresponding author*

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For every $R > 1$, the level curve $L_R := \{z : |\Phi(z)| = R\}$ is a Jordan curve. We denote the interior of L_R by G_R, and the exterior of L_R by Ω_R. A p-Faber series

$$f(z) = \sum_{n=0}^{\infty} a_n F_{n,p}(z) \qquad \text{with} \qquad \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{R}, \text{ and } R > 1 \tag{1.2}$$

is uniformly convergent to the holomorphic function f on each compact subset of G_R and divergent for all $z \in \Omega_R$. On the other hand, given a function f which is holomorphic in G_R with $R > 1$, then the representation (1.2) holds with the p-Faber coefficients

$$a_n = \frac{1}{2\pi i} \int_{\Gamma_s} \frac{f(\Psi(w))(\Psi'(w))^{1/p}}{w^{n+1}} dw, \quad 1 < s < R.$$

Let us denote the partial sums of (1.2) by

$$S_n(z) = \sum_{k=0}^{n} a_k F_{k,p}(z). \tag{1.3}$$

From the above results, it is easily seen that the sequence $\{S_n\}_{n=0}^{\infty}$ converges compactly in G_R, and for every $z \in \Omega_R$, we have

$$\limsup_{n\to\infty} |S_n(z)|^{\frac{1}{n}} = \frac{|\Phi(z)|}{R} > 1. \tag{1.4}$$

For more details on p-Faber polynomials, we refer the reader to ([4], [12] and [13]).

It is known that it can be constructed such a Faber series with the property that a certain subsequence of $\{S_n\}$ converges to f on the sets in Ω_R, where the function f is regular. This is the phenomenon of overconvergence. A Faber series in (1.2) is called *compactly overconvergent* if there exists an open set $U \subset \Omega_R$ and a monotone increasing sequence of positive integers $\{n_k\}$ such that $\{S_{n_k}\}$ converges on every compact subset of U. Examples of overconvergent Faber series are given in ([5], [8], [14]).

Let $m = \{m_n\}_{n=0}^{\infty}$ be an arbitrary sequence of positive integers. It is called that a sequence $\{s_n\}_{n=0}^{\infty}$ is being *elongated* with respect to the sequence m if for each n the term s_n is listed m_n-times, i.e. if it is written by the following way:

$$(\underbrace{s_0, s_0, ..., s_0}_{m_0-\text{times}}, \underbrace{s_1, s_1, ..., s_1}_{m_1-\text{times}}, ..., \underbrace{s_n, s_n, ..., s_n}_{m_n-\text{times}}, ...). \tag{1.5}$$

The sequence (1.5) is called $m-$elongation of $\{s_n\}$. It is obvious that the sequence $\{s_n\}$ is convergent if and only if any $m-$elongation of $\{s_n\}$ is convergent with the same limit.

Let $A = (a_{n,k})(n, k = 0, 1, 2, 3, ...)$ be an infinite matrix of real (or complex) numbers. A sequence $\{s_n\}$ of real (or complex) numbers is said to be *summable* to a number S by the method $A = (a_{n,k})$, shortly *A-summable* to S, if the limit relation

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} S_k = S$$

holds, and it is written as $A - \lim_{n\to\infty} S_n = S$. The matrix $A = (a_{n,k})$ is called regular if it transforms convergent sequences to convergent sequences with the same limit. It is well known that the matrix $A = (a_{n,k})$ is regular if and only if it satisfies the following Silverman-Toeplitz conditions (see [7, p. 142], also [9]):

(i) There exists a constant $M > 0$ such that $\sum\limits_{k=1}^{\infty} |a_{n,k}| \leq M$, for each $n = 1, 2, 3, ...$;

(ii) For each positive integer k,

$$\lim_{n\to\infty} a_{n,k} = 0; \tag{1.6}$$

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1.$

Here, we mainly deal with a special regular method, namely with the Cesàro means (C, k) of order $k \in \mathbb{N}$ which transforms the given sequence $\{s_n\}$ into the sequence

$$\sigma_n^{(k)} := \frac{1}{\binom{n+k}{n}} \sum_{i=0}^{n} \binom{n-i+k-1}{n-i} s_i.$$

For $k = 1$ we especially obtain the arithmetical means

$$\sigma_n := \sigma_n^{(1)} = \frac{1}{n+1} \sum_{i=0}^{n} s_i.$$

If $\left\{\sigma_n^{(k)}\right\}_{n=0}^{\infty}$, $k \in \mathbb{N}$, converges, then $\left\{\frac{s_n}{n^k}\right\}$ tends to 0, as $n \to \infty$ [3].

In this paper, our aim is to investigate the equivalence between the overconvergence of the p-Faber series (1.2) and the existence of an elongation of $\{S_n\}$ the sequence of the partial sums of (1.2), whose (C, k) transformation converges. The obtained results can be used in the theory of universal series.

2. Main Results

Let f be a p-Faber series as in (1.2) with partial sums $\{S_n\}$ as in (1.3), and let $A = (a_{n,k})$ be any regular matrix transformation. The matrix A transforms the sequence $\{S_n(z)\}$ defined by (1.3) into the sequence $\{A_n(z)\}$, where

$$A_n(z) = \sum_{k=0}^{\infty} a_{n,k} S_k(z).$$

By the regularity of the matrix A, the sequence $\{A_n\}$ converges compactly to f on G_R. Especially, for the matrix transformation (C, k), $k \in \mathbb{N}$, of the sequence of the partial sums $\{S_n\}$,

$$\sigma_n^{(k)}(z) := \frac{1}{\binom{n+k}{n}} \sum_{i=0}^{n} \binom{n-i+k-1}{n-i} S_i(z),$$

it is well known that $\{\sigma_n^{(k)}\}$ converges to f on every compact subsets of G_R and diverges for all $z \in \Omega_R$. But, it is also known that, there may be convergent subsequences of $\{\sigma_n^{(k)}(z)\}$ in Ω_R (see e.g. [2], [10]).

The following two theorems extend the results of Luh and Nieß in [6] on the overconvergence of Faber series.

Theorem 2.1. *Let $A = (a_{n,k})$ be any regular matrix transformation and $U \subset \Omega_R$ be an open set. If the p-Faber series (1.2) is compactly overconvergent to a limit function F in U, then there exists an elongation of the sequence (1.3) which is compactly $A-$summable on U to the function F.*

Proof. It is known that there exists an increasing sequence of compact sets $\{K_n\}$ with the property that if $K \subset U$ is a compact set, then there exists a positive integer $n_0 = n_0(K)$ such that $K \subset K_{n_0}$, since U is an open set in complex plane [11, p. 267].

Assume that there exists a monotone increasing sequence $\{n_k\}$ of positive integers such that $\{S_{n_k}\}$ is compactly convergent in U to a limit function F, that is,

$$\lim_{k\to\infty} \sup_{z\in K_m} |S_{n_k}(z) - F(z)| = 0 \qquad \text{for all} \quad m \in \mathbb{N}_0.$$

Let $\{m_k\}$ be a sequence of positive integers which will be determined later. We now elongate the sequence $\{S_n\}$ to the sequence $\{\widetilde{S}_n\}$ where the terms S_{n_k} for $k \geq 1$ are listed $m_k + 1$ times while the others remain unchanged, i.e.

$$\{\widetilde{S}_n\} = (S_0, S_1, ..., S_{n_0-1}, \underbrace{S_{n_0}, S_{n_0}, ..., S_{n_0}}_{m_0+1-\text{times}}, S_{n_0+1}, ..., S_{n_k-1}, \underbrace{S_{n_k}, S_{n_k}, ..., S_{n_k}}_{m_k+1-\text{times}}, ...).$$

If we denote the nth term of the transformation of the elongated sequence $\{\widetilde{S}_n\}$ under the matrix A by A_n, then

$$\begin{aligned} A_n(z) \quad &:= \quad \sum_{i=0}^{\infty} a_{n,i}\widetilde{S}_i(z) = \sum_{i=0}^{n_0} a_{n,i}S_i(z) + \sum_{i=n_0+1}^{n_0+m_0} a_{n,i}S_{n_0}(z) \\ &\quad + \sum_{i=n_0+1}^{n_1} a_{n,i+m_0}S_i(z) + \sum_{i=n_1+1}^{n_1+m_1} a_{n,i+m_0}S_{n_1}(z) + ... \\ &\quad + \sum_{i=n_{k-1}+1}^{n_k} a_{n,i+M_{k-1}}S_i(z) + \sum_{i=n_k+1}^{n_k+m_k} a_{n,i+M_{k-1}}S_{n_k}(z) + ... \\ &= \sum_{k=0}^{\infty} \sum_{i=n_{k-1}+1}^{n_k} a_{n,i+M_{k-1}} \left(S_i(z) - S_{n_k}(z)\right) + \sum_{i=0}^{\infty} a_{n,i}g_i(z), \end{aligned}$$

where $M_k = \sum\limits_{i=0}^{k} m_i$, $k \in \mathbb{N}$, $n_{-1} := M_{-1} := 0$ and $\{g_i\}$ is an elongation of $\{S_{n_i}\}$, that is

$$\{g_i\} = (\underbrace{S_{n_0}, S_{n_0}, ..., S_{n_0}}_{n_0+m_0-\text{times}}, \underbrace{S_{n_1}, S_{n_1}, ..., S_{n_1}}_{n_1-n_0+m_1-\text{times}}, ..., \underbrace{S_{n_k}, S_{n_k}, ..., S_{n_k}}_{n_k-n_{k-1}+m_k-\text{times}}, ...).$$

Since the subsequence $\{S_{n_k}\}$ of $\{S_n\}$ converges compactly to the function F on U, then the $r-$elongation of $\{S_{n_k}\}$, $\{g_k\}$ is so, where $r = \{r_k\} = \{n_k - n_{k-1} + m_k\}$. By the regularity of the matrix $A = (a_{n,k})$, the sequence $\{h_n\}$ defined by

$$h_n(z) := \sum_{i=0}^{\infty} a_{n,i}g_i(z)$$

converges compactly to the function F on U. If we prove that

$$\sum_{k=0}^{\infty} \sum_{i=n_{k-1}+1}^{n_k} a_{n,i+M_{k-1}} \left(S_i(z) - S_{n_k}(z)\right)$$

tends to zero on each compact subset K_k, when $n \to \infty$, the assertion follows.

Let $\{\gamma_n\}$ be a sequence of nonnegative numbers which tends to zero. By the Silverman-Toeplitz conditions for the regularity of matrix transformations, we can choose the natural numbers m_k, $k \in \mathbb{N}_0$, such that the inequality

$$\sum_{i=n_{k-1}+1}^{n_k} |a_{n,i+M_{k-1}}| \leq \frac{\gamma_n}{N_k 2^k}, \tag{2.1}$$

holds, where

$$N_k = \max_{n_{k-1}+1 \leq i \leq n_k} \sup_{z \in K_k} |S_i(z)|.$$

Consequently, by the inequality (2.1) it is obtained that

$$\begin{aligned} |\sum_{k=0}^{\infty} \sum_{i=n_{k-1}+1}^{n_k} a_{n,i+M_{k-1}} (S_i(z) - S_{n_k}(z))| &\leq \sum_{k=0}^{\infty} 2N_k \sum_{i=n_{k-1}+1}^{n_k} |a_{n,i+M_{k-1}}| \\ &\leq \sum_{i=0}^{n_0} 2N_0 |a_{n,i}| + \sum_{k=1}^{\infty} \frac{\gamma_n}{2^{k-1}} \\ &= \sum_{i=0}^{n_0} 2N_0 |a_{n,i}| + \gamma_n. \end{aligned}$$

Using the fact that γ_n tends to zero, when $n \to \infty$ and (1.6), we have proven the theorem. □

The other result gives a partially converse of the above-mentioned theorem.

Theorem 2.2. *Let $U \subset \Omega_R$ be an open set and $k \in \mathbb{N}$. If there exists an elongation of the sequence (1.3) such that its sequence of (C,k)-means converges compactly to a function F in U, then, the p-Faber series (1.2) is compactly overconvergent to the function F in U.*

Proof. Suppose that there exists an $m = \{m_n\}_{n=0}^{\infty}$ elongation of the sequence $\{S_n(z)\}$ whose the (C,k)-means converges compactly in U, that is, the sequence $\{\widetilde{\sigma}_n^{(k)}(z)\}$ defined by

$$\widetilde{\sigma}_n^{(k)}(z) := \frac{1}{\binom{n+k}{n}} \sum_{i=0}^{n} \binom{n-i+k-1}{n-i} \widetilde{S}_i(z),$$

where

$$\{\widetilde{S}_k(z)\} = (\underbrace{S_0, S_0, ..., S_0}_{m_0-\text{times}}, \underbrace{S_1, S_1, ..., S_1}_{m_1-\text{times}}, ..., \underbrace{S_n, S_n, ..., S_n}_{m_n-\text{times}}, ...),$$

is convergent uniformly on every compact subset of U.

Let us consider a special subsequence of the sequence $\{\widetilde{\sigma}_n^{(k)}(z)\}$ which has the form

$$\sigma_n(z) := \widetilde{\sigma}_{M_n}^{(k)}(z) = \frac{1}{\binom{M_n+k}{M_n}} \sum_{i=0}^{n} \left(\sum_{j=M_{i-1}+1}^{M_i} \binom{M_n-j+k-1}{M_n-j} \right) S_i(z)$$

where

$$M_k = \sum_{i=0}^{k} m_i, \qquad M_{-1} := m_{-1} = 0.$$

If we set

$$\beta_j := \sum_{j=M_{i-1}+1}^{M_i} \binom{M_n-j+k-1}{M_n-j} \quad j = 0, 1, 2, ... \quad \text{and} \quad B_n := \binom{M_n+k}{M_n},$$

then, we have $B_n = \sum_{j=0}^{n} \beta_j$, and therefore the relation

$$S_n(z) = \frac{B_n}{\beta_n}\{\sigma_n(z) - \sigma_{n-1}(z)\} + \sigma_{n-1}(z).$$

holds. If we show that the sequence $\left\{\frac{B_n}{\beta_n}\right\}$ has a bounded subsequence, then the desired result will be clear. Let us now show that.

For any $z \in U$, the (C,k)-means of $\{\widetilde{S}_n(z)\}$ is convergent; hence we obtain, as already mentioned in section 1, that $\left\{\frac{\widetilde{S}_n(z)}{n^k}\right\}$ is bounded. Since $\widetilde{S}_{B_n}(z) = S_n(z)$, there is a constant $c > 0$ with

$$|S_n(z)| \leq cB_n^k \tag{2.2}$$

for every $n \in \mathbb{N}_0$. Assume that $\left\{\frac{B_n}{\beta_n}\right\}$ does not have a bounded subsequence, i.e. $\frac{B_n}{\beta_n} \to \infty$, as $n \to \infty$. This implies

$$\frac{B_{n-1}}{B_n} = \frac{B_n - \beta_n}{B_n} = 1 - \frac{\beta_n}{B_n} \to 1, \qquad \text{as} \quad n \to \infty,$$

thus

$$(B_n)^{\frac{1}{n}} \to 1, \qquad \text{as} \quad n \to \infty.$$

From (2.2), we get

$$\limsup_{n\to\infty} |S_n(z)|^{\frac{1}{n}} \leq 1,$$

which contradicts (1.4). This completes the proof of the theorem.

□

Combining these two theorems we get the following:

Corollary 2.1. *Let $U \subset \Omega_R$ be an open set and $k \in \mathbb{N}$. There exists an elongation of the sequence (1.3) that (C,k)-convergent compactly to a function F in U iff the p-Faber series (1.2) is compactly overconvergent to the function F in U.*

Remark 2.1. *For the case $k = 1$ and G is an open disc, we obtain the results of Drobot* [1] *and Gharibyan and Luh* [2] *from Corollary 2.1 when p tends to infinity.*

Remark 2.2. *For the case $k = 1$, we obtain the result of Luh and Nieß in* [6] *from Corollary 2.1 when p tends to infinity.*

Remark 2.3. *In* [15]*, it was obtained the similar result to Corollary 2.1 with the power series instead of p-Faber series and regular Riesz means instead of (C,k)-means.*

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DEPARTMENT OF MATHEMATICS
FACULTY OF ARTS AND SCIENCES, BITLIS EREN UNIVERSITY
BITLIS, 13000, TURKEY
E-mail address: ndaral@beu.edu.tr

DEPARTMENT OF MATHEMATICS
FACULTY OF ARTS AND SCIENCES, MERSIN UNIVERSITY
MERSIN, 33343, TURKEY
E-mail address: ttunc@mersin.edu.tr