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EQUIVARIANT LINEARISATION CONTACT-SYMPLECTIC OF SINGULAR LAGRANGIAN-LEGENDRIAN FOLIATION PAIRS

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ABSTRACT. On a differential manifold $(M^{2h+2k+1}, \omega, \eta)$ equipped to contact-symplectic pair, we consider a restricted completely integrable Hamiltonian system pair with $(k+h)$ degrees of freedom whose first integrals are invariant under the contact-symplectic action of a compact Lie group G. We prove that the singular Lagrangian-Legendrian foliation pair associated to this restricted completely integrable Hamiltonian system is contact-symplectically equivalent, in a G-equivariant way, to the linearised foliation pair in a neighbourhood of a nondegenerate singular compact orbit pair.

1. INTRODUCTION

In this paper we study the geometry of integrable Hamiltonian systems. An integrable Hamiltonian system on a $2n$-dimensional symplectic manifold is given by (n) first integrals f_i with the property that integral is preserved by the Hamiltonian flow of the other integrals. This condition is classically known as involutivity of the first integrals and can be written in terms of the Poisson bracket as

$$\{f_i, f_j\} = 0.$$

The study of the integrability of such systems is relevant in many areas of mathematics and has its own story. In June $29th$ of 1853 Joseph Liouville presented a communication entitled "Sur l'intégration des équations différentielles de la Dynamique". In the resulting note [10] he relates the notion of integrability of the system to the existence of $n-$integrals in involution with respect to the Poisson bracket attached to the symplectic form. A lot of work has been done in the subject after Liouville. Let us outline some of the remarkable achievements from a geometrical and topological point of view. Consider a completely integrable Hamiltonian system. The Hamiltonian vector fields of the Hamiltonian function f_i define an involutive distribution. Let $\mathcal{O}$ be a regular compact orbit of this distribution then this orbit is a Lagrangian submanifold. Moreover, it is a torus and the neighbouring orbits are also tori. Those tori are called Liouville tori. This is the topological contribution of a theorem which has been known in the literature as

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Arnold-Liouville theorem. The geometrical contribution of the above-mentioned theorem ensures the existence of symplectic normal forms in the neighbourhood of a compact regular orbit. The works of Henri Mineur [11],[12],[13] already gave a complete description of the Hamiltonian system in a neighbourhood of a compact regular orbit. That is why we will refer to the classical Arnold-Liouville theorem as Liouville-Mineur-Arnold theorem. Let recall that the theorem below:

Theorem 1.1. *Given an completely integrable Hamiltonian system on symplectic manifold* (M^{2n}, ω)*, and* $\mathcal{O}$ *a regular compact orbit; There is a symplectomorphism* ϕ *from a neighbourhood* $U(\mathcal{O})$ *of* $\mathcal{O}$ *in* (M^{2n}, ω) *to* $(\mathbb{D}^n \times \mathbb{T}^n, \sum_{i=1}^{n} d\mu_i \wedge d\beta_i)$ *where* $(\mu_i), 1 \leq i \leq n$ *is a coordinate on the ball* $\mathbb{D}^n$*, and* $(\beta_i), 1 \leq i \leq n$ *is a periodic coordinate system on the torus* $\mathbb{T}^n$ *such that* $\phi^* F$ *is the map which depend only on the coordinate* $\phi^*(\mu_i)$*. The functions* $\phi^* \beta_i$ *are called angle variables and the functions* $\phi^*(\mu_i)$ *are called action variables.*

The existence of action-angle coordinates in a neighbourhood of a regular compact orbit provides a symplectic model for the Lagrangian foliation $\mathfrak{F}$ determined by the Hamiltonian vector fields of the $n-$component functions f_i of the moment map μ. In fact, Liouville-Mineur-Arnold theorem entails a uniqueness result for the symplectic structures making $\mathfrak{F}$ into a Lagrangian foliation. In other words, if ω_1 and ω_2 are two symplectic structures defined in a neighbourhood of $\mathcal{O}$ for which the foliation $\mathfrak{F}$ is Lagrangian then there exists a symplectomorphism preserving the foliation, fixing $\mathcal{O}$ and carrying ω_1 to ω_2. So if the orbit is regular the existence of action-angle coordinates enables to classify the symplectic germs, up to foliation-preserving symplectomorphism, for which $\mathfrak{F}$ is Lagrangian in a neighbourhood of a compact orbit. After this review for symplectic linearisation in a neighbourhood of regular orbit, the following question arises: What can be said about the corresponding classification problem for symplectic germs if the completely integrable systems has singularities? This question is quite natural because singularities are present in many well known examples of integrable systems. In fact, if the completely integrable system is defined on a compact manifold then the singularities cannot be avoided. The symplectic linearisation in a neighbourhood of an singular orbit $\mathcal{O}$ with $dim\mathcal{O} > 0$ is due to Ito in the analytic case [6]. Partial results in the smooth case (with $dim\mathcal{O} = 1$ in a manifold of dimension 4) were obtained by Currás-Bosch and Eva Miranda in [4] and independently by Colin de Verdière and San Vu Ngoc in [3]. The final result in any dimension was obtained by Nguyen Tien Zung and Eva Miranda in [14].
In [14] it is also included a G-equivariant version of the symplectic linearisation. Symmetries are present in many physical problems and therefore they show up in integrable systems theory as well. Those symmetries are encoded in actions of Lie groups. A special emphasis has been given to Hamiltonian actions of tori in symplectic geometry. Along the way many results of symplectic uniqueness are obtained. A good example of this is Delzant's theorem [5] which enables to recover information of a compact $2n$-dimensional manifold by looking at the image of the moment map of a Hamiltonian torus action which is, surprisingly, a convex polytope in $\mathbb{R}^n$. A lot of contributions in the area of Hamiltonian actions of Lie groups have been done ever since. Let us mention some of the references of the large list of results in that direction: the works of Lerman and Tolman to extend those result to symplectic orbifolds [9] and the works of Karshon and Tolman for complexity one Hamiltonian group actions [7],[8]. In this article we consider a particular class of manifolds which have been called in the literature contact-symplectic pairs. Contact-symplectic pairs were introduced by G.BANDE in [1], where they study further differential objects associated to them. On a such manifolds we define a restricted completely integrable Hamiltonian system pair, then prove an analogue to the symplectic

linearisation result which was mentioned above but in the case of restricted completely integrable Hamiltonian systems pairs in manifold equipped to contact pair. Precisely, we show that for a given two contact-symplectic pairs (ω, η) and (ω', η') for which the system is a restricted completely integrable Hamiltonian system pair in a neighbourhood of compact nondegenerate orbit pair $(\mathcal{O}_1, \mathcal{O}_2)$ there is a diffeomorphism preserving the system, fixing $(\mathcal{O}_1, \mathcal{O}_2)$, and sending (ω, η) on (ω', η'). We also take into account the possible symmetries of the system. Namely, we will show that in the case there exists a contact-symplectic action of a compact Lie group G in a neighborhood of $(\mathcal{O}_1, \mathcal{O}_2)$ preserving the moment map, this linearisation can be carried out in an equivariant way.

2. Preliminaries

In this section, we recall some basics definitions and properties for contact-symplectic pairs.

2.1. **Hamiltonian vector fields and Poisson bracket.**

Definition 2.1. *A contact-symplectic pair of type (k, h) on a manifold $M^{2k+2h+1}$ of dimension $(2k + 2h + 1)$ is a pair (ω, η) where η is pfaffian form and ω is a closed 2-form such that:*

$$\eta \wedge (d\eta)^h \wedge \omega^k \text{ is a volume form on } M^{2k+2h+1}, \text{ and } d\eta^{h+1} = 0,\ \omega^{k+1} = 0.$$

The forms η and ω are so necessarily constant class $2h + 1$ and $2k$. For $k = 0$ we find a contact structures, and for $h = 0$ the cosympletic structures. To a such structure are naturally associated two distributions: the distribution of vector fields which annul η and $d\eta$, and the distribution of vector fields which annul ω. This distributions are completely integrable because the forms η and ω are constant class $2h + 1$ and $2k$. They determine the characteristic foliations of η and ω, which we will note $\mathfrak{H}$ and $\mathfrak{G}$.
The characteristic foliation of η is of dimension $2k$ and her leaves are symplectic manifolds of symplectic structure associated ω. The characteristic foliation of ω is of dimension $(2h + 1)$ and her leaves are contact manifold of contact form associated η.
The foliations $\mathfrak{H}$ and $\mathfrak{G}$ are hollowing out transversal and additional. On a Such manifolds it exists a vector field Z_η called the Reeb vector field satisfying $\eta(Z_\eta) = 1$ and $i_{Z_\eta}(d\eta^h \wedge \omega^k) = 0$. Z_η is tangent to characteristic foliation of ω and it is the Reeb vector field (in the classical sense) of contact form induced by η on each leaf of this foliation(see[2]).

Example 1.
(1) *On $\mathbb{R}^{2k+2h+1}$ equipped to coordinates system $(x_1, \cdots, x_k, y_1, \cdots, y_k, p_1, \cdots, p_h, q_1, \cdots, q_h, z)$ the pair (ω, η) defined by:*
$$\omega = \sum_{i=1}^k dx_i \wedge dy_i,\ \eta = \sum_{i=1}^h p_i dq_i + dz$$
is a contact-symplectic pair. The Reeb vector fields is $Z_\eta = \partial_z$.
(2) *On $\mathbb{T}^2 \times \mathbb{S}^2 \times \mathbb{S}^1$ equipped to coordinates system $(\alpha, \beta, \phi, \theta_0, \theta_1)$ the pair (ω, η) defined by*
$$\eta = \cos(\theta_1)d\phi + \sin(\theta_1)d\theta_0,\ \omega = d\alpha \wedge d\beta$$
is a contact-symplectic pair of type $(1, 1)$. The Reeb vector field is defined by $Z = \partial_{\theta_1}$.

Contrary to Riemannian manifolds, the contact-symplectic pairs have not local invariants. That is to do to theorem called the Darboux theorem (see[2]). It establishes an unique local model of contact-symplectic pairs. All contact-symplectic pair on a manifold $M^{2k+2h+1}$ induced an isomorphism of $C^\infty(M^{2k+2h+1}, \mathbb{R})$-module $\flat_{(\eta,\omega)}$: $\chi(M^{2k+2h+1}) \longrightarrow \Omega^1(M^{2k+2h+1})$ defined by the following proposition.

Proposition 2.1. *Let (ω, η) be a contact-symplectic pair on $M^{2k+2h+1}$.*
The map $\flat_{(\eta,\alpha)} : \chi(M^{2k+2h+1}) \longrightarrow \Omega^1(M^{2k+2h+1})$ defined by:

$$\flat_{(\eta,\omega)}(X) = \eta(X).\eta + i_X(d\eta + \omega), \forall X \in \chi(M^{2k+2h+1})$$

is an isomorphism of $C^\infty(M^{2k+2h+1}, \mathbb{R})$-module.

Proof. Observe that it suffices to show that the map $\flat_{(\eta,\omega)}$ is injective.
Let $(U, x_1, \cdots, x_k, y_1, \cdots, y_k, p_1, \cdots, p_h, q_1, \cdots, q_h, z)$ be a Darboux coordinates system (see[2]) and $X = \Sigma_{i=1}^k a_i \partial x_i + \Sigma_{i=1}^k b_i \partial y_i + \Sigma_{i=1}^h c_i \partial p_i + \Sigma_{i=1}^h d_i \partial q_i + e\partial z$ be a vectors field on U. We assume that

$$\flat_{(\eta,\omega)}(X) = 0.$$

So we have

$$\begin{array}{rcl} \eta(X).\eta(X) + i_X(d\eta + \omega)(X) & = & 0 \\ \eta^2(X) & = & 0 \\ \eta(X) & = & 0. \end{array}$$

Thus, we obtain $i_X(d\eta + \omega) = 0$. Consequently,

$$\begin{array}{rcl} i_X(d\eta + \omega)(\partial x_i) & = & 0 \\ b_i & = & 0, \end{array} \tag{2.1}$$

$$\begin{array}{rcl} i_X(d\eta + \omega)(\partial y_i) & = & 0 \\ a_i & = & 0, \end{array} \tag{2.2}$$

$$\begin{array}{rcl} i_X(d\eta + \omega)(\partial p_i) & = & 0 \\ d_i & = & 0, \end{array} \tag{2.3}$$

$$\begin{array}{rcl} i_X(d\eta + \omega)(\partial q_i) & = & 0 \\ c_i & = & 0, \end{array} \tag{2.4}$$

$$\begin{array}{rcl} \eta(X) & = & 0 \\ e & = & 0. \end{array} \tag{2.5}$$

According to the relations (2.1), (2.2), (2.3), (2.4) and (2.5) we deduce that $X = 0$ and later on the map $\flat_{(\eta,\omega)}$ is injective. □

Thanks to this isomorphism, we can associate at every function $f \in C^\infty(M^{2k+2h+1}, \mathbb{R})$ an unique vectors field $gradf \in \chi(M)$, called gradient of f, which is defined by

Definition 2.2. *Let $(M^{2k+2h+1}, \omega, \eta)$ be a contact-symplectic pair and $f \in \mathcal{C}^\infty(M^{2k+2h+1}, \mathbb{R})$ a function. The single vectors field $gradf$ defined by*

$$gradf = \flat^{-1}_{(\eta,\omega)}(df)\,,$$

is called gradient vectors field of f, . Likewise

$$\left\{ \begin{array}{l} i_{gradf}(\omega + d\eta) = df - Z_\eta(f)\eta \\ \eta(gradf) = Z_\eta(f). \end{array} \right.$$

We define also the Hamiltonian vectors field associated to $f \in C^\infty(M^{2k+2h+1}, \mathbb{R})$.

Definition 2.3. *Let $(M^{2k+2h+1}, \omega, \eta)$ be a contact-symplectic pair and $f \in \mathcal{C}^\infty(M^{2k+2h+1}, \mathbb{R})$ a function. The single vectors field X_f defined by:*

$$X_f = \flat^{-1}_{(\eta,\omega)}(df - Z_\eta(f)\eta)\,,$$

is called Hamiltonian vectors field of f. Likewise,

$$\begin{cases} i_{X_f}(d\eta + \omega) = df - Z_\eta(f)\eta \\ \eta(X_f) = 0 \end{cases} .$$

The gradient vectors field $gradf$ and the Hamiltonian vector field X_f have the same horizontal component and

$$X_f = gradf \quad \text{if only if} \quad Z_\eta(f) = 0\,. \tag{2.6}$$

The Hamiltonian vectors fields which verify the relation (2.6) are called the restricted Hamiltonian vectors fields. Moreover, the Hamiltonian vectors fields verifies the following properties.

Property 2.1. *For all f and $g \in \mathcal{C}^\infty(M^{2k+2h+1}, \mathbb{R})$ we have:*

(1) $X_{f+g} = X_f + X_g$.
(2) $X_{fg} = fX_g + gX_f$.
(3) *If the Hamiltonians f and g are associated to the same Hamiltonian vector field X, then the difference $(f - g)$ is constant locally.*

Proof. Let $f, g \in \mathcal{C}^\infty(M^{2k+2h+1}, \mathbb{R})$. For the first property, we have

$$\begin{aligned} \eta(X_{f+g})\eta + i_{X_{f+g}}(d\eta + \omega) &= d(f+g) \\ &= df + dg \\ &= \eta(X_f)\eta + i_{X_f}(d\eta + \omega) + \eta(X_g)\eta + i_{X_g}(d\alpha + d\eta) \\ &= \eta(X_f + X_g)\eta + i_{X_f + X_g}(d\eta + \omega) \end{aligned}$$
$$\flat_{(\eta,\omega)}(X_{f+g}) = \flat_{(\eta,\omega)}(X_f + X_g). \tag{2.7}$$

Since the map $\flat_{(\eta,\omega)}$ is an isomorphism, so according to the relation (2.7), we obtain $X_{f+g} = X_f + X_g$. For the second property, we have

$$\begin{aligned} \eta(X_{fg})\eta + i_{X_{fg}}(d\eta + \omega) &= d(fg) \\ &= gdf + fdg \\ &= g(\eta(X_f)\eta + i_{X_f}(d\eta + \omega)) + f(\eta(X_g)\eta + i_{X_g}(d\eta + \omega)) \\ &= \eta(gX_f + fX_g)\eta + i_{gX_f + fX_g}(d\eta + \omega) \end{aligned}$$
$$\flat_{(\eta,\omega)}(X_{fg}) = \flat_{(\eta,\omega)}(gX_f + fX_g). \tag{2.8}$$

Since the map $\flat_{(\eta,\omega)}$ is an isomorphism, so according to the relation (2.7), we obtain $X_{fg} = gX_f + fX_g$. For the third property, we have:

$$\begin{aligned} d(f-g) &= df - dg \\ &= \eta(X)\eta + i_X(d\eta + \omega) - \eta(X)\eta + i_X(d\eta + \omega) \\ &= 0\,. \end{aligned}$$

□

The proposition that follows, show that it exists a Poisson bracket $\{,\}$ on $C^\infty(M^{2k+2h+1}, \mathbb{R})$ such that the map $(C^\infty(M^{2k+2h+1}, \mathbb{R}), \{,\}) \longrightarrow (\chi(M^{2k+2h+1}), [,])$ defined by $f \longrightarrow X_f$

is a Lie algebra anti-homomorphism with respect to the Poisson bracket and the commutator of vector fields when it is restricted to the set of Hamiltonian functions f satisfying $df(Z_\eta) = 0$.

Proposition 2.2. *It exists a Poisson algebra structure* $\{,\}$ *on* $C^\infty(M^{2k+2h+1}, \mathbb{R})$ *such that for all* f, g *verifying* $df(Z_\eta) = dg(Z_\eta) = 0$*, we have*

$$X_{\{f,g\}} = -[X_f, X_g]\,.$$

Proof. We put

$$\{f, g\} = (d\eta + \omega)(gradf, gradg).$$

If we take into account to the relation $i_{X_f}(d\eta + \omega) = i_{gradf}(d\eta + \omega)$, we obtain

$$\{f, g\} = (d\eta + \omega)(X_f, X_g).$$

Since ω and $d\eta$ are bilinear and antisymmetric, then $\{,\}$ is bilinear and antisymmetric.
For Leibniz identity, we consider f, g and $h \in \mathcal{C}^\infty$. We have

$$\begin{aligned}
\{f, gh\} &= (\omega + d\eta)(X_f, X_{gh}) \\
&= (\omega + d\eta)(X_f, gX_h + hX_g) \\
&= (\omega + d\eta)(X_f, gX_h) + (\omega + d\eta)(X_f, hX_g) \\
&= g(\omega + d\eta)(X_f, X_h) + h(\omega + d\eta)(X_f, X_g) \\
&= g\{f, h\} + h\{f, g\}
\end{aligned}$$

For Jacobi identity, we consider f, g and $h \in \mathcal{C}^\infty(M^{2k+2h+1})$. For all $f \in \mathcal{C}^\infty(M^{2k+2h+1}, \mathbb{R})$,the equation

$$i_Y(\omega + d\eta) = df$$

has a unique well-defined solution when restricted to $(\ker\eta \cap \ker d\eta, \omega)$ and $(\ker\omega, d\eta)$. We denote by Y_f (resp.Z_f) the Hamiltonian vector fields of function f with respect to the symplectic structure ω on $\ker\eta \cap \ker d\eta$ (resp. $d\eta$ on $\ker\omega$) With all these information at hand we can write

$$X_f = Y_f + Z_f$$

Thus, we obtain

$$\begin{aligned}
\{f, \{g, h\}\} &= (\omega + d\eta)(Y_f + Z_f, Y_{\{g,h\}} + Z_{\{g,h\}}) \\
&= (\omega + d\eta)(Y_f, Y_{\{g,h\}}) + (\omega + d\eta)(Y_f, Z_{\{g,h\}}) + (\omega + d\eta)(Z_f, Y_{\{g,h\}}).
\end{aligned}$$

But since Y_f,$Y_{\{g,h\}}$ are tangent to $\ker\eta \cap \ker d\eta$ and Z_f,$Z_{\{g,h\}}$ tangent to $\ker\omega$. Therefore we have

$$(d\eta + \omega)(Y_f, Y_{\{g,h\}}) = \omega(Y_f, Y_{\{g,h\}}),\ (d\eta + \omega)(Z_f, Z_{\{g,h\}}) = d\eta(Z_f, Z_{\{g,h\}}).$$

and

$$(d\eta + \omega)(Y_f, Z_{\{g,h\}}) = 0,\ (d\eta + \omega)(Z_f, Y_{\{g,h\}}) = 0\,.$$

We have:

$$\begin{aligned}
\{f, \{g, h\}\} &= \omega(Y_f, Y_{\{g,h\}}) + d\eta(Z_f, Z_{\{g,h\}}). \\
&= -\omega(Y_g, Y_{\{h,f\}}) - \omega(Y_h, Y_{\{f,g\}}) - d\eta(Z_g, Z_{\{h,f\}}) - d\eta(Z_h, Z_{\{f,g\}}) \\
&= -(d\eta + \omega)(Y_g + Z_g, Y_{\{h,f\}} + Z_{\{h,f\}}) - (d\eta + \omega)(Y_h + Z_h, Y_{\{f,g\}} + Z_{\{f,g\}}) \\
&= -(d\eta + \omega)(X_g, X_{\{h,f\}}) - (d\eta + d\omega)(X_h, X_{\{f,g\}}) \\
&= -\{g, \{h, f\}\} - \{h, \{f, g\}\}\,.
\end{aligned}$$

Let f, g be a differential functions verifying $df(Z_\eta) = dg(Z_\eta) = 0$. We have

$$\begin{array}{rcl} i_{[X_f,X_g]}(d\eta+\omega) & = & L_{X_f}i_{X_g}(d\eta+\omega) - i_{X_g}L_{X_f}(d\eta+d\omega) \\ & = & L_{X_f}dg - i_{X_g}d(i_{X_f}(d\eta+\omega)) \\ & = & di_{X_f}dg - i_{X_g}d(df) \\ & = & d(dg(X_f)) \\ & = & -d\{f,g\}. \end{array}$$

Thus we obtain,

$$X_{\{f,g\}} = -[X_f, X_g].$$

□

Let us recall the notion of contact-symplectomorphism and contact-symplectic action.

Definition 2.4. *Let (M, ω, η), (M', ω', η'), be two contact-symplectic pairs. A diffeomorphism $\phi : M \longrightarrow M'$ is called a contact-symplectomorphism, if*

$$\begin{cases} \phi^\star(\omega') = \omega \\ \phi^\star(\eta') = \eta \end{cases}.$$

Example 2. *We consider $M_0 = \mathbb{S}^2 \times \mathbb{T}^3$ equipped to contact-symplectic pair $\omega = d\theta \wedge d\phi$, $\eta = \cos(\theta_0)d\theta_1 + \sin(\theta_0)d\theta_2$. The map $\psi_t : M_0 \longrightarrow M_0$ defined by: for all $(\theta, \phi, \theta_0, \theta_1, \theta_2) \in M_0$,*

$$\psi_t(\theta, \phi, \theta_0, \theta_1, \theta_2) = (e^t\theta, \frac{\phi}{e^t}, \frac{\pi}{2} - \theta_0, \theta_2 - t\pi, \theta_1 - t\frac{\pi}{2})$$

is a contact-symplectic map.

Definition 2.5. *Let (M, ω, η) be a contact-symplectic pairs. An action $\rho : G \times M \longrightarrow M$ of Lie group G is called contact-symplectic, if for all $g \in G$*

$$\begin{cases} \rho_g^\star(\omega') = \omega \\ \rho_g^\star(\eta') = \eta \end{cases}.$$

Example 3. *We consider $M_0 = \mathbb{S}^2 \times \mathbb{T}^3$ equipped to contact-symplectic pair $\omega = d\theta \wedge d\phi$, $\eta = \cos(\theta_0)d\theta_1 + \sin(\theta_0)d\theta_2$. The map $\rho : \mathbb{S}^1 \times M_0 \longrightarrow M_0$ defined by: for all $(\theta, \phi, \theta_0, \theta_1, \theta_2) \in M_0$,*

$$\rho_t(\theta, \phi, \theta_0, \theta_1, \theta_2) = (e^t\theta, \frac{\phi}{e^t}, \theta_0, \theta_1 - t\pi, \theta_2 - t\frac{\pi}{2})$$

is a contact-symplectic action.

2.2. **Completely integrable Hamiltonian systems pairs and Lagrangian-Legendrian foliations pairs.**

Definition 2.6. *Let $(M^{2k+2h+1}, \omega, \eta)$ be a contact-symplectic pair and $f_1, \cdots, f_k, g_1 \cdots, g_h$ $(k+h)$-Hamiltonian functions on $(M^{2k+2h+1}, \eta, \omega)$. We say that $(f_1, \cdots, f_k, g_1 \cdots, g_h)$ is a restricted completely integrable Hamiltonian system pair if the following conditions are verified:*

1. *The Hamiltonian vector fields X_{f_i} are tangent to $\mathfrak{G}$ and the Hamiltonian vector fields X_{g_i} are tangent to $\mathfrak{H}$.*
2. $df_i(Z_\eta) = dg_i(Z_\eta) = 0$.
3. $\{f_i, f_j\} = \{g_i, g_j\} = \{f_i, g_j\} = 0$ *for all i, j.*

(4) *The system* $(df_1, \cdots, df_k, dg_1, \cdots, dg_h)$ *is linearly independent almost everywhere.*

The functions f_i, g_i are called first integrals of the integrable system. Given a restricted completely integrable Hamiltonian system pair,there is a local Hamiltonian $\mathbb{R}^{k+h}$-action of momentum map $\mu = (f_1, \cdots, f_k, g_1, \cdots, g_h)$, and two foliations naturally attached to it.

Proposition 2.3. *Let* $(f_1, \cdots, f_k, g_1, \cdots, g_h)$ *be a restricted completely integrable Hamiltonian system pair on contact-symplectic pair* $(M^{2k+2h+1}, \omega, \eta)$. *Assume that* $p \in M^{2k+2h+1}$ *is a point for which* $d_pf_1 \wedge \cdots \wedge d_pf_k \wedge dg_1 \wedge \cdots \wedge dg_h \neq 0$. *Then the distributions* $\mathcal{D}_1 =< X_{f_1}, \cdots, X_{f_k} >$ *and* $\mathcal{D}_2 =< X_{g_1}, \cdots, X_{g_h} >$ *are involutive and the tangent spaces at* p *to leaves through* p *are respectively a Lagrangian subspace of* $(\ker \eta \cap \ker d\eta(p), \omega(p))$ *and a Legendrian subspace of* $(\ker \omega(p), \eta(p))$.

Proof. On the one hand, since $[X_{f_i}; X_{f_j}] = -X_{\{f_i,f_j\}}$ (see2.2), the condition $\{f_i, f_j\} = 0$ implies $[X_{f_i}, X_{f_j}] = 0$ for all i, j and the distribution $\mathcal{D}_1$ is involutive. On the other hand, since $[X_{g_i}; X_{g_j}] = -X_{\{g_i,g_j\}}$, the condition $\{g_i, g_j\} = 0$ implies $[X_{g_i}, X_{g_j}] = 0$ for all i, j and the distribution $\mathcal{D}_2$ is also involutive. Let $\mathcal{F}$ and $\mathcal{G}$ be a leaves through at p of distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. From the definitions of Hamiltonian vectors field and restricted completely integrable Hamiltonian system pair, for all $X, X' \in T_p\mathcal{F}$ and $Y \in T_p\mathcal{G}$ we have $\eta(Y) = \omega(X, X') = 0.$. The condition $d_pf_1\wedge\cdots\wedge d_pf_k\wedge d_pg_1\wedge\cdots\wedge d_pg_h \neq 0$ implies that the Hamiltonian vector fields X_{f_i} span an k-dimensional vector space at the point p and the Hamiltonian vector fields X_{g_i} span an h-dimensional vector space at the point p. Therefore the tangent space at p of the leaf $\mathcal{F}$ is Lagrangian subspace to $(\ker \eta \cap d\eta(p), \omega(p))$ and the tangent space at p of the leaf $\mathcal{G}$ is Legendrian subspace to $(\ker \omega(p), \eta(p))$. □

In all we note $\mathfrak{F}_1$ the foliation defined by $\mathcal{D}_1$ and $\mathfrak{F}_2$ the foliation defined by $\mathcal{D}_2$. The pair $(\mathfrak{F}_1, \mathfrak{F}_2)$ is called the Lagrangian-Legendrian foliation pair attached to restricted integrable Hamiltonian system pair

Example 4. (1) *On* $\mathbb{R}^{2k+2h+1}$ *equipped to standard contact-symplectic pair*

$$\omega = \sum_{i=1}^k dx_i \wedge dy_i,\ \eta = \sum_{i=1}^h p_i dq_i + dz$$

The system $f_1 = x_1^2 + y_1^2, \cdots, f_k = x_k^2 + y_k^2, g_1 = p_1^2 + q_1^2, \cdots, g_h = p_h^2 + q_h^2$ *is a restricted completely integrable Hamiltonian system pair. The foliation* $\mathfrak{F}_1$ *is generated by vector fields* $X_i = 2(y_i\partial x_i - x_i\partial y_i)$ *for* $1 \leq i \leq k$ *and the foliation* $\mathfrak{F}_2$ *is generated by vector fields* $X_i = 2(q_i\partial p_i - p_i\partial q_i)$ *for* $1 \leq i \leq h$.

(2) *On* $\mathbb{T}^3 \times \mathbb{S}^2$ *equipped to contact-symplectic pair*

$$\eta = \cos(\theta_0)d\theta_1 + \sin(\theta_0)d\theta_2,\ \omega = d\alpha \wedge d\phi.$$

The system $f_1 = \alpha\phi, f_2 = \beta\theta$ *is a restricted completely integrable Hamiltonian pair. The foliation* $\mathfrak{F}$ *is generated by* $X_1 = \alpha\partial\alpha - \phi\partial\phi$, $X_2 = \beta\partial\beta - \theta\partial\theta$.

Definition 2.7. *Let* $(f_1, \cdots, f_k, g_1, \cdots, g_h)$ *be a restricted completely integrable Hamiltonian system pair on contact-symplectic pair* $(M^{2k+2h+1}, \omega, \eta)$ *and* ρ *a contact-symplectic action of compact Lie group* G *on* $(M^{2k+2h+1}, \omega, \eta)$. *We say that the system* $(f_1, \cdots, f_k, g_1, \cdots, g_h)$ *is invariant under action* ρ *if for all* $g \in G$, $f_i \circ \rho_g = f_i$.

Example 5. *We consider* $M_0 = \mathbb{S}^2 \times \mathbb{T}^3$ *equipped to contact-symplectic pair* $\omega = d\theta \wedge d\phi$, $\eta = \cos(\theta_0)d\theta_1 + \sin(\theta_0)d\theta_2$. *The map* $\rho : \mathbb{S}^1 \times M_0 \longrightarrow M_0$ *defined by: for all* $(\theta, \phi, \theta_0, \theta_1, \theta_2) \in M_0$,

$$\rho_t(\theta, \phi, \theta_0, \theta_1, \theta_2) = (e^t\theta, \frac{\phi}{e^t}, \theta_0, \theta_1 - t\pi, \theta_2 - t\frac{\pi}{2})$$

is a contact-symplectic action. The system (f, g) where $f = \theta \times \phi$ and $g = \theta_1 - 2\theta_2$ is a restricted completely integrable Hamiltonian system pair invariant under action ρ_t

2.3. **Nondegenerate singulars orbits pairs.** Let $(f_1, \cdots, f_k, g_1, \cdots, g_h)$ be a restricted completely integrable Hamiltonian system pair of momentum map μ on contact pair $(M^{2k+2h+1}, \omega, \eta)$, p a point in $M^{2k+2h+1}$, $\mathcal{O}_1$ the orbit of $\mathcal{D}_1$ through p, and $\mathcal{O}_2$ the orbit of $\mathcal{D}_2$ through p. We denote by $\pi_1 : \mathbb{R}^{k+h} \longrightarrow \mathbb{R}^k$ and $\pi_2 : \mathbb{R}^{k+h} \longrightarrow \mathbb{R}^h$ the canonical projections.

Definition 2.8. *We say that $(\mathcal{O}_1, \mathcal{O}_2)$ is singular orbit pair of type (r, s) if $rank(d_p\pi_1 \circ \mu) = r$ and $rank(d_p\pi_2 \circ \mu) = s$ with $r < k, s < h$.*

Remark 2.1. *If a point in orbit pair $(\mathcal{O}_1, \mathcal{O}_2)$ is singular orbit pair of type (r, s) then all point in $(\mathcal{O}_1, \mathcal{O}_2)$ is singular of type (r, s). Because singularity is a property which is invariant under the local Hamiltonian $\mathbb{R}^{k+h}$-action.*

Proposition 2.4. *Let $(\mathcal{O}_1, \mathcal{O}_2)$ be a singular compact orbit pair of type (r, s). We assume that the first integrals $f_{r+1}, \cdots, f_k$ have a non-degenerate singularity in a Bott-Morse sense a long $\mathcal{O}_1$ and the first integrals $g_{s+1}, \cdots, g_h$ have a nondegenerate singularity in a Boot-Morse sense a long $\mathcal{O}_2$. Then it exist two triplets of natural numbers (ke, kh, kf), (ke', kh', kf'), where $(ke+ke')$ (resp.,$(kh+kh')$, $(kf+kf')$) is the number of component elliptic (respectively hyperbolic, foyer-foyer) such that $ke + kh + 2kf = k - r$, $ke' + kh' + 2kf' = h - s$ and a group Γ attached to $((ke, kh, kf), (ke', kh', kf'))$.*

The pair of triplets enters $((ke, kh, kf), (ke', kh', kf'))$ is called the pair of Williamson type of $(\mathcal{O}_1, \mathcal{O}_2)$ and Γ the twisting group attached to $((ke, kh, kf), (ke', kh', kf'))$.

Proof. Since the orbits $\mathcal{O}_1$, $\mathcal{O}_2$ are compact then the produce $\mathcal{O}_1 \times \mathcal{O}_2$ is a closed of $M^{2k+2h+1}$. The Manifold $M^{2k+2h+1}$ being normal, then it exist a neighbourhood U around $(\mathcal{O}_1 \times \mathcal{O}_2)$ in $M^{2k+2h+1}$ such that

$$U \cap (\mathcal{O}_1 \times \mathcal{O}_2) = \mathcal{O}_1 \times \mathcal{O}_2.$$

Moreover, since the Reeb vector fields have not singularity then according to redressely theorem its exist local coordinate system $(U, x_1, \cdots, x_k, y_1, \cdots, y_k, p_1, \cdots, p_h, q_1, \cdots, q_h, z)$ such that $Z_\eta = \partial z$. Now, the pfaffian form η can be written as $\eta = dz + \overline{\eta}$. Observe that since Z_η is the Reeb vector fields in particular we obtain

$$i_{Z_\eta} d\overline{\eta} = 0, \, i_{Z_\eta}\omega.$$

Using Cartan's formula $L_{Z_\eta}\overline{\eta} = i_{Z_\eta} d\overline{\eta} + di_{Z_\eta}\overline{\eta}$, $L_{Z_\eta}\omega = i_{Z_\eta}\omega + di_{Z_\eta}\omega$ we deduce that $\overline{\eta}$ and ω does not depend to z.
Let N_1, N_2 be the submanifolds defined by

$$N_1 : \{p_1 = \cdots = p_h = q_1 = \cdots = q_h = z = 0\} \text{ and}$$
$$N_2 : \{x_1 = \cdots = x_k = y_1 = \cdots = y_h = 0\}.$$

Observe that ω and η are respectively a symplectic and contact forms on N_1 and N_2 respectively. Thus, the equations systems

$$\begin{cases} i_X(\omega) = df_i - Z_\eta(f_i)\eta \\ \eta(X) = 0 \end{cases}$$

$$\begin{cases} i_X(d\eta) = dg_i - Z_\eta(g_i)\eta \\ \eta(X) = 0 \end{cases}$$

have a unique well-defined solution when restricted to the sub manifolds (N_1, ω) and (N_2, η) respectively. We denote respectively by Y_{f_i} and Z_{g_i} the solution of this equations systems. With all these information at hand we can write $X_{f_i} = Y_{f_i}$ and $X_{g_i} = Z_{g_i}$ where X_{f_i} and X_{g_i} are the Hamiltonian vector fields with respect to contact-symplectic pair (η, ω). Observe that, the Hamiltonian vector fields $Y_{f_1}, \cdots, Y_{f_k}$ define a restricted completely integrable Hamiltonian system on (N_1, ω) and the Hamiltonian vector fields $Z_{g_1}, \cdots, Z_{g_h}$ define a restricted completely integrable Hamiltonian system on (N_2, η). Indeed, since $d\eta(Y_{f_i}, Y_{f_j}) = \omega(Z_{g_i}, Z_{g_j}) = 0$ we have

$$\begin{aligned} \{f_i, f_j\}_{N_1} &= \omega(Y_{f_i}, Y_{f_j}) \\ &= (d\eta + \omega)(X_{f_i}, X_{f_j}) \\ &= \{f_i, f_j\} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \{g_i, g_j\}_{N_2} &= d\eta(Z_{g_i}, Z_{g_j}) \\ &= (d\eta + \omega)(X_{g_i}, X_{g_j}) \\ &= \{g_i, g_j\} \\ &= 0 \end{aligned}$$

Moreover, $\mathcal{O}_2$ is a singular non degenerate compact orbit of rank s of restricted integrable Hamiltonian system $Z_{g_1}, \cdots, Z_{g_h}$ in (N_2, η). Consequently, if we denote by M_2 the submanifold of N_2 given by $z = 0$, we have $Z_{g_1}, \cdots, Z_{g_h}$ is a completely integrable Hamiltonian system on symplectic manifold $(M_2, d\eta)$, $\mathcal{O}_{2|M_2}$ a singular nondegenerate compact orbit of rang s of this system. Thus, according to Eva.Miranda and Nguyen Tien Zung theorem (see[14]), there exists a finite group Γ_2 and a triplet natural numbers (ke', kh', kf') such that $ke' + kh' + 2kf' = h - s$. $\mathcal{O}_1$ is a singular nondegenerate compact orbit of rank r of integrable Hamiltonian system $Z_{g_1}, \cdots, Z_{g_h}$ in symplectic manifold (N_1, ω). Thus, according to Eva.Miranda and Nguyen Tien Zung theorem (see[14]), there exists a finite group Γ_1 and a triplet natural numbers (ke, kh, kf) such that $ke + kh + 2kf = k - r$. To achieve this proposition we put $\Gamma = \Gamma_1 \times \Gamma_2$. □

In the following section we study the contact-symplectic pair linearisation problem in a neighbourhood of singular compact orbit pair $(\mathcal{O}_1, \mathcal{O}_2)$.

3. Linearisation contact-symplectic in a neighbourhood of nondegenerate singular compact orbit pair

In all that follows, $(M^{2k+2h+1}, \omega, \eta)$ designate a contact-symplectic pair denoted by $(f_1, \cdots, f_k, g_1, \cdots, g_h)$, a restricted completely integrable Hamiltonian system pair on $(M^{2k+2h+1}, \omega, \eta)$, μ the momentum map, $(\mathfrak{F}_1, \mathfrak{F}_2)$ the Lagrangian-Legendrian foliation pair, $(\mathcal{O}_1, \mathcal{O}_2)$ a singular compact orbit pair of type (r, s) with $r < k, s < h$. We assume that, the first integrals $f_{r+1}, \cdots, f_k$ and $g_{s+1}, \cdots, g_h$ have a nondegenerate singularity in the Morse-Bott sense along $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. Thus according to proposition 2.4, there exists two triplets of natural numbers (ke, kh, kf),(ke', kh', kf') such that $ke + kh + 2kf = k - r$ and $ke' + kh' + 2kf' = h - s$. We recall the notion of linear model. Denote by $(x_1, \cdots, x_r)$ a linear coordinate system of a small ball $\mathbb{D}^r$ of dimension r, $(\alpha_1, \cdots, \alpha_r)$ a standard periodic coordinate system of the torus $\mathbb{T}^r$, $(y_1, \gamma_1, \cdots, y_{k-r}, \gamma_{k-r})$ a linear coordinate system of a small ball $\mathbb{D}^{2(k-r)}$ of dimension $2(k-r)$, $(p_1, \cdots, p_s)$ a linear coordinate system of a small ball $\mathbb{D}^s$ of dimension s, $(\beta_1, \cdots, \beta_s)$ a standard periodic coordinate system of the torus $\mathbb{T}^s$, $(q_1, \mu_1, \cdots, q_{h-s}, \mu_{h-s}, z_2)$ a linear coordinate system of a small ball $\mathbb{D}^{2(h-s)+1}$ of dimension $2(h-s)+1$. Consider the manifold

$$M_0^{2k+2h+1} = \mathbb{D}^r \times \mathbb{T}^r \times \mathbb{D}^{2k-2r+1} \times \mathbb{D}^s \times \mathbb{T}^s \times \mathbb{D}^{2h-2s+1}$$

with the standard contact-symplectic pair

$$\omega_0 = \Sigma_{i=1}^r dx_i \wedge d\alpha_i + \Sigma_{i=1}^{k-r} dy_i \wedge d\gamma_i, \eta_0 = \Sigma_{i=1}^s p_i d\beta_i + \Sigma_{i=1}^{h-s} p_i d\mu_i + dz$$

and the following momentum map:

$$\mu_0 = (x_1, \cdots, x_r, f_{0_{r+1}}, \cdots, f_{0_k}, p_1, \cdots, p_s, g_{0_{s+1}}, \cdots, g_{0_h})$$

where

$$\begin{aligned} f_{0_{i+k}} &= y_i^2 + \gamma_i^2, for, 1 \leq i \leq ke_1, \\ f_{0_{i+k}} &= y_i \gamma_i, for, ke_1 + 1 \leq i \leq ke_1 + kh_1, \\ f_{0_{i+k}} &= y_i \gamma_{i+1} - y_{i+1} \gamma_i, and \\ f_{0_{i+k+1}} &= y_i \gamma_i + y_{i+1} \gamma_{i+1}, for, i = ke_1 + kh_1 + 2j - 1, 1 \leq j \leq kf_1 \end{aligned}$$

and

$$\begin{aligned} g_{0_{i+h}} &= q_i^2 + \mu_i^2, for, 1 \leq i \leq ke_2, \\ g_{0_{i+k}} &= q_i \mu_i, for, ke_2 + 1 \leq i \leq ke_2 + kh_2, \\ g_{0_{i+k}} &= q_i \mu_{i+1} - q_{i+1} \mu_i, and \\ g_{0_{i+k+1}} &= q_i \mu_i + q_{i+1} \mu_{i+1}, for, i = ke_2 + kh_2 + 2j - 1, 1 \leq j \leq kf_2. \end{aligned}$$

We denote by $(\mathfrak{F}_{01}, \mathfrak{F}_{02})$ the linear Lagrangian-Legendrian foliation pair given respectively by the orbits of the linear distributions $\mathcal{D}_{01} =< X_{x_1}, \cdots, X_{x_r}, X_{f_{0r+1}}, \cdots, X_{f_{0k}} >$ and $\mathcal{D}_{02} =< X_{p_1}, \cdots, X_{p_s}, X_{g_{0s+1}}, \cdots, X_{g_{0h}} >$ where X_{x_i}, X_{p_i}, $X_{f_{0i}}$ and $X_{g_{0i}}$ being respectively the Hamiltonian vector fields of x_i, p_i, f_{0i} and g_{0i} in the contact-symplectic pair model $(M_0^{2k+2h+1}, \omega_0, \eta_0)$. Let Γ be a group with a contact-symplectic action $\rho(\Gamma)$ on $M_0^{2k+2h+1}$, which preserves the momentum map μ_0. We will say that the action of Γ on $M_0^{2k+2h+1}$ is linear if it satisfies the following property: Γ acts on the product $M_0^{2k+2h+1}$ componentwise; the action of Γ on $\mathbb{D}^r$ and $\mathbb{D}^s$ is trivial, its action on $\mathbb{T}^r$ and $\mathbb{T}^s$ is by translations (with respect to the coordinate system $(\alpha_1, \cdots, \alpha_r)$,$(\beta_1, \cdots, \beta_s)$) and its action on $\mathbb{D}^{2k-2r}$ and $\mathbb{D}^{2h-2s+1}$ is linear (with respect to the coordinate system $(y_1, \gamma_1, \cdots, y_{k-r}, \gamma_{k-r})$,$(q_1, \mu_1, \cdots, q_{k-r}, \mu_{k-r})$). Suppose now that Γ is a finite group with a free contact-symplectic action $\rho(\Gamma)$ on $M_0^{2k+2h+1}$, which preserves the momentum map and which is linear. Then we can form the quotient contact-symplectic pair manifold $\widetilde{M_0} = M_0^{2k+2h+1}/\Gamma$, with an integrable system on it given by the induced momentum map as above:

$$\mu_0 = (x_1, \cdots, x_r, f_{0r+1}, \cdots, f_{0k}, p_1, \cdots, p_s, g_{0s+1}, \cdots, g_{0h}).$$

The set pair $(\{x_i = y_i = \gamma_i = p_i = q_i = \mu_i = \beta_i = 0\}, \{x_i = y_i = \gamma_i = p_i = q_i = \mu_i = \alpha_i = 0\}) \subset \widetilde{M_0}$ is a compact orbit pair of Williamson type $((ke_1, kh_1, kf_1), (ke_2, kh_2, kf_2))$ of the above system. We will call the above system on $\widetilde{M_0}$, together with its associated singular Lagrangian-Legendrian foliation pair, the linear system (or linear model) of Williamson pair type $((ke_1, kh_1, kf_1), (ke_2, kh_2, kf_2))$ and twisting group Γ (or more precisely, twisting action $\rho(\Gamma)$). We will also say that it is a direct model if Γ is trivial, and a twisted model if Γ is non trivial.

Now, under the above hypotheses, we can formulate and show, the symplectic linearisation theorem for compact nondegenerate singular orbits pair of restricted integrable Hamiltonian systems pair.

Theorem 3.1. *Then there exists a finite group Γ and a contact-symplectomorphism ϕ defined in a neighbourhood of $(\mathcal{O}_1, \mathcal{O}_2)$ to $(M_0^{2h+2k+1}/\Gamma, \omega_0, \eta_0)$ such that:*

(1) *It sends $(\mathcal{O}_1, \mathcal{O}_2)$ to $(\mathbb{T}^r, \mathbb{T}^s)$.*
(2) *It sends $(\mathfrak{F}_1, \mathfrak{F}_2)$ to $(\mathfrak{F}_{01}, \mathfrak{F}_{02})$.*

Proof. Since the orbits $\mathcal{O}_1$, $\mathcal{O}_2$ are compact then the produce $\mathcal{O}_1 \times \mathcal{O}_2$ is a closed of $M^{2k+2h+1}$. The Manifold $M^{2k+2h+1}$ being normal, then it exist a neighbourhood U around $(\mathcal{O}_1 \times \mathcal{O}_2)$ in $M^{2k+2h+1}$ such that

$$U \cap (\mathcal{O}_1 \times \mathcal{O}_2) = \mathcal{O}_1 \times \mathcal{O}_2.$$

Moreover, since the Reeb vector field have not singularity then according to redressely theorem its exist local coordinate system $(U, x_1, \cdots, x_k, y_1, \cdots, y_k, p_1, \cdots, p_h, q_1, \cdots, q_h, z)$ such that $Z_\eta = \partial z$. Now, the pfaffian form η can be written as $\eta = dz + \overline{\eta}$. Observe that since Z_η is the Reeb vector fields in particular we obtain

$$i_{Z_\eta} d\overline{\eta} = 0, i_{Z_\eta}\omega = 0.$$

Using Cartan's formula $L_{Z_\eta}\overline{\eta} = i_Z d\overline{\eta} + d i_{Z_\eta}\overline{\eta}$, $L_{Z_\eta}\omega = i_Z d\omega + d i_{Z_\eta}\overline{\omega}$ we deduce that $\overline{\eta}$ and ω does not depend to z_1 and z_2 respectively. Let N_1, N_2 be the submanifold defined by

$$N_1 : \{p_1 = \cdots = p_h = q_1 = \cdots = q_h = z = 0\} \text{ and}$$
$$N_2 : \{x_1 = \cdots = x_k = y_1 = \cdots = y_h = 0\}.$$

Observe that ω and η are respectively a symplectic and contact forms on N_1 and N_2 respectively. Thus, the equations systems

$$\begin{cases} i_X(\omega) = df_i - Z_\eta(f_i)\eta \\ \eta(X) = 0 \end{cases}$$

$$\begin{cases} i_X(d\eta) = dg_i - Z_\eta(g_i)\eta \\ \eta(X) = 0 \end{cases}$$

have a unique well-defined solution when restricted to the symplectic and contact submanifolds (N_1, ω) and (N_2, η) respectively. We denote respectively by Y_{f_i} and Z_{f_i} the solution of this equations systems. With all these information at hand we can write $X_{f_i} = Y_{f_i}$ and $X_{g_i} = Z_{g_i}$ where X_{f_i} and X_{g_i} are the Hamiltonian vector fields with respect to contact-symplectic pair (ω, η). Observe that, the Hamiltonian vector fields $Y_{f_1}, \cdots, Y_{f_k}$ define a completely integrable Hamiltonian system on symplectic manifold (N_1, ω) and the Hamiltonian vector fields $Z_{g_1}, \cdots, Z_{g_h}$ define a restricted completely integrable Hamiltonian system on (N_2, η). Indeed, since $d\eta(Y_{f_i}, Y_{f_j}) = \omega(Z_{g_i}, Z_{g_j})$ we have

$$\begin{aligned} \{f_i, f_j\}_{N_1} &= \omega(Y_{f_i}, Y_{f_j}) \\ &= (\omega + d\eta)(X_{f_i}, X_{f_j}) \\ &= \{f_i, f_j\} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned}\{g_i, g_j\}_{N_2} &= d\eta(Z_{g_i}, Z_{g_j})\\ &= (\omega + d\eta)(X_{g_i}, X_{g_j})\\ &= \{g_i, g_j\}\\ &= 0\end{aligned}$$

Moreover, $\mathcal{O}_{1|N_1}$ is a singular non degenerate compact orbit of rank r of integrable Hamiltonian system $Y_{f_1}, \cdots, Y_{f_k}$ in symplectic manifold (N_1, ω) and $\mathcal{O}_2$ is a singular non degenerate compact orbit of rank s of restricted integrable Hamiltonian system $Z_{g_1}, \cdots, Z_{g_h}$ in (N_2, η). Consequently, if we denote by M_2 the submanifold of N_2 given by $z = 0$, we have $Z_{g_1}, \cdots, Z_{g_h}$ is a completely integrable Hamiltonian system on symplectic manifold $(M_2, d\eta)$, and $\mathcal{O}_{2|M_2}$ a singular non degenerate compact orbit of rang s of this system. Thus, according to Eva.Miranda and Nguyen Tien Zung theorem (see[14]), there exists a finite group Γ_1 and a diffeomorphism ϕ taking the foliation $\mathfrak{F}_1$ to the linear foliation $\mathfrak{F}_{01}$ on $(\mathbb{D}^r \times \mathbb{T}^r \times \mathbb{D}^{2k-2r})/\Gamma_1$ and taking ω to ω_0 which send $\mathcal{O}_{1|N_1}$ to the torus $\mathbb{T}^r$. Also,there exists a finite group Γ_2 and a diffeomorphism ϕ' taking the foliation $\mathfrak{F}_2$ to the linear foliation $\mathfrak{F}_{02}$ on $(\mathbb{D}^s \times \mathbb{T}^s \times \mathbb{D}^{2h-2s})/\Gamma_2$, taking $d\eta$ to $d\eta_0$ and sends $\mathcal{O}_{2|M_2}$ to the torus $\mathbb{T}^s$. Now, we define the map $\phi'' : U(\mathcal{O}_{2|M_2}) \times \mathbb{D}^1 \longrightarrow (\mathbb{D}^s \times \mathbb{T}^s \times \mathbb{D}^{2h-2s+1})/\Gamma_2$ by

$$\phi''(x, z) = (\phi'(x), z), \text{ for all } (x, z) \in U(\mathcal{O}_{2|M_2}) \times \mathbb{D}^1.$$

Observe that since

$$\phi'^*(d\eta_0) = d\overline{\eta}$$

then

$$\phi'^*(\eta_0 + dH) = \overline{\eta}$$

this yields,

$$\phi''^*(\eta_0 + dH) = dz + \overline{\eta}.$$

Now, consider the path of pfaffian forms

$$\eta_t = dz + \textstyle\sum_{i=1}^{s} p_i d\beta_i + \sum_{i=1}^{h-s} q_i d\mu_i + t dH.$$

Let ξ_t be the flow of vector field $X = -HZ_\eta$ respectively. Note that as matter of fact

$$\xi_1(x_i, \alpha_i, y_i, \gamma_i, p_i, \beta_i, q_i, \mu_i, z) = (x_i, \alpha_i, y_i, \gamma_i, p_i, \beta_i, q_i, \mu_i, z - H)$$

Thus, $\xi_1^*(\omega) = \omega_0$ and $\xi_1^*(\eta_1) = \eta_0$. The functions $\xi_1^*(f_i)$ and $\xi_1^*(g_i)$ does not depend on t. In fact, we have

$$\begin{aligned}L_X(f_i) &= df_i(-HZ_\eta)\\ &= -Hdf_i(Z_\eta)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}L_X(g_i) &= dg_i(-HZ_\eta)\\ &= -Hdg_i(Z_\eta)\\ &= 0\,.\end{aligned}$$

Thus, $\xi_1^*(f_i) = f_i$ and $\xi_1^*(g_i) = g_i$. Now, we put $\Gamma = \Gamma_1 \times \Gamma_2$ and consider the map $\Phi : U(\mathcal{O}_{1N_1}) \times U(\mathcal{O}_{2M_2}) \times \mathbb{D}^1 \longrightarrow (\mathbb{D}^r \times \mathbb{T}^r \times \mathbb{D}^{2k-2r} \times \mathbb{D}^s \times \mathbb{T}^s \times \mathbb{D}^{2h-2s+1})/\Gamma$ by

$$\Phi(x, y, z) = (\xi_1^{-1} \circ \phi(x), \xi_1^{-1}\phi''(y, z)), \text{ for all } (x, y, z) \in U(\mathcal{O}_{1|N_1}) \times U(\mathcal{O}_{2|M_2}) \times \mathbb{D}^1.$$

This map is a contact-symplectomorphism. In fact, we have

$$\begin{aligned} \Phi^*(\omega_0) &= (\xi_1^{-1} \circ \phi)^*(\omega_0) \\ &= \phi^* \circ \xi_1^{-1*}\omega_0 \\ &= \phi^*\omega \\ &= \omega \end{aligned}$$

and

$$\begin{aligned} \Phi^*(\eta_0) &= (\xi_1^{-1} \circ \phi'')^*(\eta_0) \\ &= \phi''^* \circ \xi_1^{-1*}\eta_0 \\ &= \phi''^*\eta_1 \\ &= \eta. \end{aligned}$$

Moreover $\Phi^*(\mu)$ is a map which depend only on the variable $(x_1 \cdots, x_r, p_1 \cdots, p_s)$, Γ is a finite group and Φ a contact-symplectomorphism taking the Lagrangian- Legendrian foliation pair $(\mathfrak{F}_1, \mathfrak{F}_2)$ to the linear Lagrangian-Legendrian foliation pair $(\mathfrak{F}_{01}, \mathfrak{F}_{02})$ on $M_0^{2k+2h+1}/\Gamma$, and sends $(\mathcal{O}_1, \mathcal{O}_2)$ to $(\mathbb{T}^r, \mathbb{T}^s)$. This ends the prove of the theorem. □

This theorem permit to classify the contact-symplectic pair germs, up to foliation-preserving contact-symplectomorphism, for which $(\mathfrak{F}_1, \mathfrak{F}_2)$ is a Lagrangian-Legendrian foliation pair in a neighbourhood of a singular nondegenerate compact orbit pair $(\mathcal{O}_1, \mathcal{O}_2)$. There is just one class of contact-symplectic pair germs for which $(\mathfrak{F}_1, \mathfrak{F}_2)$ is a Lagrangian-Legendrian foliation pair.

Theorem 3.2. *If (ω', η') is another contact-symplectic pair for which $(\mathfrak{F}_1, \mathfrak{F}_2)$ is a Lagrangian-Legendrian foliation pair, then there exists a diffeomorphism ϕ defined in a neighbourhood of $(\mathcal{O}_1, \mathcal{O}_2)$ such that:*

(1) *It fix the orbit pair $(\mathcal{O}_1, \mathcal{O}_2)$.*
(2) *It preserves the foliation pair $(\mathfrak{F}_1, \mathfrak{F}_2)$.*
(3) $\phi^*(\eta') = \eta$, $\phi^*(\omega') = \omega$.

Proof. According to theorem 3.1, it exists two diffeomorphism ϕ_1, ϕ_2 sending respectively (ω, η) and (ω', η') to (ω_0, η_0), sending $(\mathfrak{F}_1, \mathfrak{F}_2)$ to $(\mathfrak{F}_{01}, \mathfrak{F}_{02})$ and sending $(\mathcal{O}_1, \mathcal{O}_2)$ to $(\mathbb{T}^r, \mathbb{T}^s)$. Now, we put $\phi_2 \circ \phi_1^{-1}$. It clear that $\phi_2 \circ \phi_1^{-1}$ verified the conditions of theorem. □

In this case we say that the contact-symplectic pairs are equivalent, and we note $(\omega, \eta) \sim_{(\mathfrak{F}_1, \mathfrak{F}_2)} (\omega', \eta')$.

4. Equivariant linearisation contact-symplectic in a neighbourhood of nondegenerate singular compact orbit pair

In this section we consider a compact Lie group G acting on a contact model manifold in such a way that preserves the n first integrals of the Reeb vector field and preserves the contact form as well. We want to prove that there exists a diffeomorphism in a neighbourhood of $(\mathcal{O}_1, \mathcal{O}_2)$ preserving the $(k+h)$ first integrals , preserving the contact form and linearising the action of the group. This result is a consequence of the equivariant symplectic linearisation theorem. The notion of linear action of a Lie group on the contact-symplectic model manifold is analogous to the equivalent notion for the symplectic model manifold . Let G be a group with a contact-symplectic action $\rho(G)$ on $M_0^{2k+2h+1}$, which preserves the momentum map μ_0. We will say that the action of G on $M_0^{2k+2h+1}$ is linear if it satisfies the following property: G acts on the product $M_0^{2k+2h+1}$ componentwise; the action of G on $\mathbb{D}^r$ and $\mathbb{D}^s$ is trivial, its action on $\mathbb{T}^r$ and $\mathbb{T}^s$ is by translations (with respect to the coordinate system $(\alpha_1, \cdots, \alpha_r)$,$(\beta_1, \cdots, \beta_s)$) and its action on $\mathbb{D}^{2k-2r}$ and $\mathbb{D}^{2h-2s+1}$ is linear (with respect to the coordinate system $(y_1, \gamma_1, \cdots, y_{k-r}, \gamma_{k-r})$,$(q_1, \mu_1, \cdots, q_{k-r}, \mu_{k-r})$). A symplectic action of a compact group G on M_0/Γ which preserves the momentum map μ_0 will be called linear if it comes from a linear symplectic action of G on M_0 which commutes with the action of Γ. Now, under the above notations and assumptions, we can formulate and show our main result, which is the equivariant contact-symplectic linearisation theorem for compact nondegenerate singular orbits pair of restricted integrable Hamiltonian systems pair.

Theorem 4.1. *Let ρ be a contact-symplectic action of compact Lie group G on $(M_0^{2h+2k+1}/\Gamma, \omega_0, \eta_0)$ which preserve the momentum map μ_0. Then it exists a contact-symplectomorphism ϕ defined in a neighbourhood of $(\mathbb{T}^r, \mathbb{T}^s)$ in $(M_0^{2h+2k+1}/\Gamma, \omega_0, \eta_0)$ such that:*

1. *ϕ fix $(\mathbb{T}^r, \mathbb{T}^s)$*
2. *ϕ preserves $(\mathfrak{F}_{01}, \mathfrak{F}_{02})$*
3. *ϕ linearises the action ρ of G. That is to say $\phi \circ \rho_g = \rho_g^{(1)} \circ \phi$.*

Proof. Let $\pi_1 : M_0^{2k+2h+1}/\Gamma \longrightarrow (\mathbb{D}^r \times \mathbb{T}^r \times \mathbb{D}^{2k-2r})/\Gamma$,$\pi_2 : M_0^{2k+2h+1}/\Gamma \longrightarrow (\mathbb{D}^s \times \mathbb{T}^s \times \mathbb{D}^{2h-2s+1})/\Gamma$ be the canonical projections and $J_1 : (\mathbb{D}^r \times \mathbb{T}^r \times \mathbb{D}^{2k-2r})/\Gamma \longrightarrow M_0^{2k+2h+1}/\Gamma$, $J_2 : (\mathbb{D}^s \times \mathbb{T}^s \times \mathbb{D}^{2h-2s+1})/\Gamma \longrightarrow M_0^{2k+2h+1}/\Gamma$ the canonical injection. Observe that the map $\pi_1 \circ \rho_g \circ J_1$ and $\pi_2 \circ \rho_g \circ J_2$ define respectively a contact-symplectomorphism action of G on $((\mathbb{D}^r \times \mathbb{T}^r \times \mathbb{D}^{2k-2r})/\Gamma, \omega_0)$ and $((\mathbb{D}^s \times \mathbb{T}^s \times \mathbb{D}^{2h-2s+1})/\Gamma, \eta_0)$. In fact we have,

$$\begin{aligned}
\pi_1 \circ \rho_e \circ J_1(x) &= \pi_1 \circ \rho_e(x, 0) \\
&= \pi_1(x, 0) \\
&= x,
\end{aligned}$$

$$\begin{aligned}
\pi_2 \circ \rho_e \circ J_2(x) &= \pi_2 \circ \rho_e(0, x) \\
&= \pi_2(0, x) \\
&= x,
\end{aligned}$$

$$\begin{aligned}
\pi_1 \circ \rho_{gg'} \circ J_1(x) &= \pi_1 \circ \rho_{gg'}(x, 0) \\
&= \pi_1 \circ \rho_g \circ \rho_{g'}(x, 0) \\
&= \pi_1 \circ \rho_g(\rho_{g'}(x), \rho_{g'}(0)) \\
&= \rho_g \circ \rho_{g'}(x) \\
&= \pi_1 \circ \rho_g \circ J_1 \circ \pi_1 \circ \rho_{g'} \circ J_1(x)
\end{aligned}$$

$$\begin{aligned}
\pi_2 \circ \rho_{gg'} \circ J_2(x) &= \pi_2 \circ \rho_{gg'}(0,x) \\
&= \pi_2 \circ \rho_g \circ \rho_{g'}(0,x) \\
&= \pi_2 \circ \rho_g(\rho_{g'}(0), \rho_{g'}(x)) \\
&= \rho_g \circ \rho_{g'}(x) \\
&= \pi_2 \circ \rho_g \circ J_2 \circ \pi_2 \circ \rho_{g'} \circ J_2(x)
\end{aligned}$$

and

$$\begin{aligned}
(\pi_1 \circ \rho_g \circ J_1)^*\omega_0 &= J_1^* \circ \rho_g^* \circ \pi_1^*\omega_0 \\
&= J_1^* \circ \rho_g^*\omega_0 \\
&= J_1^*\omega_0 \\
&= \omega_0 .
\end{aligned}$$

$$\begin{aligned}
(\pi_2 \circ \rho_g \circ J_2)^*\eta_0 &= J_2^* \circ \rho_g^* \circ \pi_2^*\eta_0 \\
&= J_2^* \circ \rho_g^*\eta_0 \\
&= J_2^*\eta_0 \\
&= \eta_0 .
\end{aligned}$$

Moreover, the map $\pi_1 \circ \rho_g \circ J_1$ and $\pi_2 \circ \rho_g \circ J_2$ preserve respectively the induced momentum map $\mu_0 \circ J_1$ et $\mu_0 \circ J_2$. The induced action ρ' of G on $(S = (\mathbb{D}^r \times \mathbb{T}^r \times \mathbb{D}^{2k-2r})/\Gamma, \omega_0)$ is a symplectic action. The induced action ρ'' of G on $(\mathbb{D}^s \times \mathbb{T}^s \times \mathbb{D}^{2h-2s+1})/\Gamma$ can be stand on a natural ways to a symplectic action of G on $S' = (\mathbb{D}^s \times \mathbb{T}^s \times \mathbb{D}^{2h-2s+1} \times]\epsilon, \epsilon[)/\Gamma, \omega_0' = dt \wedge dz + d\eta_0)$ that follow,

$$\widehat{\rho''} : (g, x, t) \in G \times S' \longmapsto (\pi_2 \circ \rho_g \circ J_2(x), t) \in S' .$$

On the manifold S and S' we consider respectively the induced momentum map $\mu_0 \circ J_1$ and $(\mu_0 \circ J_2, t)$. Thus we can applied the equivariant symplectic linearisation theorem to obtain two symplectomorphism ϕ' and $\widehat{\phi''}$ preserving respectively the momentum map $\mu_0 \circ J_1$ and $(\mu_0 \circ J_2, t)$, the orbits $\mathbb{T}^r$ and $\mathbb{T}^s$ and linearising the action ρ' and $\widehat{\rho''}$. According the definition of action $\widehat{\rho''}$ and the momentum map $(\mu_0 \circ J_2, t)$, the symplectomorphism $\widehat{\phi''}$ descend to a diffeomorphism ϕ'' on $(\mathbb{D}^s \times \mathbb{T}^s \times \mathbb{D}^{2h-2s+1})/\Gamma$ which linearise action $\pi_2 \circ \rho_g \circ J_2(x)$ and satisfied $\phi''^* d\eta_0 = d\eta_0$. Then

$$\phi''^*\eta_0 = \eta_0 + dH .$$

Finally the diffeomorphism,

$$\begin{aligned}
&\varphi(p_1, \cdots, p_s, \beta_1, \cdots, \beta_s, q_1, \mu_1, \cdots, q_{h-s}, \mu_{h-s}, z) = \\
&= (p_1, \cdots, p_s, \beta_1, \cdots, \beta_s, q_1, \mu_1, \cdots, q_{h-s}, \mu_{h-s}, z - H)
\end{aligned}$$

linearise action $\pi_2 \circ \rho_g \circ J_2(x)$ and satisfied

$$\varphi^*(\eta_0 + dH) = \eta_0 .$$

Now, we consider the map $\phi = (\phi', \varphi)$. It clear that ϕ is a diffeomorphism verifying conditions of theorem. □

This previous theorem permit us to obtain the following theorem.

Theorem 4.2. *Let ρ be a contact-symplectic action of compact Lie group G on $(M^{2k+2h+1}, \omega, \eta)$ preserving the momentum μ. Then it exists a contact-symplectomorphism ϕ in a neighbourhood $(U(\mathcal{O}_1, \mathcal{O}_2))$ of $(\mathcal{O}_1, \mathcal{O}_2)$ in $(M^{2k+2h+1}, \omega, \eta)$ to $(M_0^{2k+2h+1}/\Gamma, \omega_0, \eta_0)$, such that:*

(1) ϕ *sends* $(\mathfrak{F}_1, \mathfrak{F}_2)$ *to* $(\mathfrak{F}_{01}, \mathfrak{F}_{02})$.
(2) ϕ *sends* $(\mathcal{O}_1, \mathcal{O}_2)$ *to* $(\mathbb{T}^r, \mathbb{T}^s)$.
(3) ϕ *linearises the action* ρ *of* G*.That is to say* $\phi \circ \rho_g = \rho_g^{(1)} \circ \phi$.

The following theorem permit us to classify the germs of contact-symplectic pairs via a contact-symplectomorphism G-equivariant preserving the pair of foliation $(\mathfrak{F}_1, \mathfrak{F}_2)$, for which $(\mathfrak{F}_1, \mathfrak{F}_2)$, is Lagrangian-Legendrian.

Theorem 4.3. *If* (ω', η') *is an other contact-symplectic for which the foliation pair* $(\mathfrak{F}_1, \mathfrak{F}_2)$ *is Lagrangian-Legendrian, then it exist a diffeomorphism* ϕ *defined in a neighbourhood of* $(\mathcal{O}_1, \mathcal{O}_2)$ *such that:*

(1) ϕ *fix* $(\mathcal{O}_1, \mathcal{O}_2)$.
(2) ϕ *preserves* $(\mathfrak{F}_1, \mathfrak{F}_2)$.
(3) ϕ *linearises action* ρ_g *de* G*. That is to say* $\phi \circ \rho_g = \rho_g^{(1)} \circ \phi$.

Proof. According to theorem 4.2, it exist two diffeomorphism ϕ_1, ϕ_2 sending respectively (ω, η) and (ω', η') to (ω_0, η_0), sending $(\mathfrak{F}_1, \mathfrak{F}_2)$ to $(\mathfrak{F}_{01}, \mathfrak{F}_{02})$, sending $(\mathcal{O}_1, \mathcal{O}_2)$ to $(\mathbb{T}^r, \mathbb{T}^s)$ and linearising the action ρ of G. Now, we put $\phi_2 \circ \phi_1^{-1}$. It clear that $\phi_2 \circ \phi_1^{-1}$ verified the conditions of theorem. □

In this case, we say that the contact-symplectic pairs are G-equivalent, and we note $(\omega, \eta)_G \sim_{(\mathfrak{F}_1, \mathfrak{F}_2)} (\omega', \eta')$.

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