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ON THE NUMERICAL QUENCHING TIME AT BLOW-UP

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ABSTRACT. This paper deals with the study of the numerical approximation for the following boundary value

$$\begin{cases} v_t = v_{xx} + \varepsilon(1-v)^{-\beta}, & (x,t) \in \Omega \times (0,T), \\ v(\pm 1, t) = 0, & t > 0, \\ v(x,0) = v_0(x) > 0, & x \in \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\beta > 0$, and $\varepsilon > 0$. By a transformation, we obtain some conditions under which the solution v_t of the above problem blows up in finite time and estimate its semidiscrete blow-up time. We also establish the convergence of the semidiscrete blow-up time to the real one when the mesh size goes to zero. Finally, we give some numerical experiments to illustrate our analysis.

1. INTRODUCTION

Consider the problem

$$\begin{aligned} v_t - v_{xx} &= f(v) \quad in \quad (-1,1)\times(0,T), \\ v(\pm 1,t) &= 0 \qquad if \ \ t \geq 0, \\ v(x,t) &= v_0(x) \quad for \ \ |x| \leq 1, \end{aligned} \tag{1.1}$$

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where $f(v)=\frac{\varepsilon}{(1-v)^\beta}$, $\beta>0$, $\varepsilon>0$, $0\le v_0<1$, and $v_0(\pm1)=0$.
This type of reaction diffusion equation with a singular reaction term arises in connection with the diffusion equation generated by a polarization phenomena in ionic conductors, see [16, 25]. The problem can also be considered as a limiting case of models in chemical catalyst kinetics (Langmuir-Hinshelwood model) or of models in enzyme kinetics, see [22, 5]. The problem (1.1) has been extensively studied under assumptions implying that the solution $v(x,t)$ approaches one in finite time. The reaction term then tends to infinity and the smooth solution ceases to exist. This phenomenon is called quenching. For more general problems of parabolic type, some results were obtained by several authors, see [1, 16, 11, 10, 9, 12, 13, 3, 15, 14, 7]. There is also a large number of partial differential equations of parabolic type whose solution for a given initial data tends to infinity in finite time T. Such a phenomenon is called blow-up and T is called the blow-up time. Blow-up is known to occur in various equations including those in combustion theory, chemotaxis models and equations describing crystalline formation involving curvature-driven motion, see [21, 2, 4, 23, 27, 26, 24]. The study of blow-up phenomena is not only interesting from the mathematical point of view but also important for deep understanding of the nature of the phenomena which those equations describe. Throughout this paper we assume that v quenches at finite time T, and that v_0 is smooth and satisfies

$$v_0''+\frac{\varepsilon}{(1-v_0)^\beta}\ge 0\,,$$

i.e., $v_t\ge0$ at $t=0$, where v_0'' is the second derivative of v_0 with respect to x. By means of transformation $u=\frac{1}{(1-v)}$, the differential equation in (1.1) becomes:

$$u_t-u_{xx} = -\frac{2{u_x}^2}{u}+\varepsilon u^{2+\beta} \qquad in\ \ (-1,1)\times(0,T), \tag{1.2}$$

$$u(\pm1,t) = 1 \qquad if\ \ t\ge0, \tag{1.3}$$

$$u(x,t) = u_0(x) \qquad for\ \ |x|\le1, \tag{1.4}$$

where $u_0(x)=\dfrac{1}{1-v_0(x)}\ge1.$
Blow-up of solutions of this problem is equivalent to the quenching of solutions of 1.1 see([11, 1, 16, 17]). In [11], J.S. Guo has shown that

the solution u of problem (1.2-1.4) blows up in finite time T, and that $u \leq B(T-t)^{-\gamma}$, $0 \leq t < T$, for some positive constant B and $\gamma = \frac{1}{\beta+1}$, but Compared with the theoretical study, numerical analysis of the blow-up problem (1.2-1.4) does not seem to be explored enough. In the present work, we consider semidiscrete problem based on uniform discretization as in [6, 20, 12], but we are mainly concerned with its estimating the blow-up time.

Let I be a positive integer, we set $h = \frac{2}{I}$ and define the grid $x_i = ih - 1$, for $i = 0, ..., I$. Let δ^2 denote the standard second order difference operator. We approximate the solution u of the problem (1.2-1.4) by the solution $U_h(t) = (U_0(t), U_2(t), ..., U_I(t))^T$ of the semidiscrete equations :

$$\frac{d}{dt}U_i(t) = \delta^2 U_i(t) - 2\frac{\left(\delta^0 U_i(t)\right)^2}{U_i(t)} + \varepsilon U_i^{\beta+2}(t), \quad 1 \leq i \leq I-1,\ t \geq 0, \tag{1.5}$$

$$U_0(t) = U_I(t) = 1, \qquad t \geq 0, \tag{1.6}$$

$$U_i^0 = \varphi_i \geq 1, \qquad 0 \leq i \leq I, \tag{1.7}$$

where:

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \qquad 1 \leq i \leq I-1,\ t \geq 0,$$

$$\delta^0 U_i(t) = \frac{U_{i+1}(t) - U_i(t)}{h}, \qquad 1 \leq i \leq I-1,\ t \geq 0,$$

$$\varphi_0 = 1, \quad \varphi_I = 1, \quad \varphi_i = \varphi_{I-1},\ 0 \leq i \leq I, \quad \delta^+\varphi_i = \frac{\varphi_{i+1} - \varphi_i}{h},$$

$$\delta^+\varphi_i > 0,\ 0 \leq i \leq k-1,$$

and k is the integer part of number $I/2$.

Our paper is written in the following manner. In the next section, we give some properties concerning our semidiscrete scheme. Section 3 is consecrated to the study of the convergence of the semidiscrete blow-up time. In Section 4, we use an efficient algorithm to estimate the blow-up time and give some numerical results to illustrate our analysis.

2. Properties of the Semidiscrete Scheme

In this section, we give somme lemmas which will be used later. The following lemma is a semidiscrete form of the maximum principle.

Lemma 2.1. *Let* $a_h(t)$, $b_h(t) \in C([0,T],\mathbb{R}^{I+1})$ *and let* $V_h(t) \in C^1([0,T],\mathbb{R}^{I+1})$ *where* $b_h(t)\delta^0 V_h(t) \le 0$*, such that for all* $0 \le i \le I$*,*

$$\begin{aligned} \frac{d}{dt}V_i(t) - \delta^2 V_i(t) + b_i(t)\delta^0 V_i(t) + a_i(t)V_i(t) &\ge 0,\ t \in]0,T[, \\ V_0(t) \ge 0, \quad V_I(t) &\ge 0, \\ V_i(0) &\ge 0. \end{aligned} \tag{2.1}$$

Then,

$$V_i(t) \ \ge\ 0, \quad 0 \le i \le I, \quad t \in]0,T[.$$

Proof. Let $T_0 < T$ and Define the vector $Z_h(t) = e^{\gamma t}V_h(t)$ where γ is sufficiently small such that $(a_i(t) - \gamma) > 0$ for $0 \le i \le I$, $t \in [0,T_0]$. Let $m = \min_{0 \le i \le I, 0 \le t \le T_0} Z_i(t)$. Since, for $i \in \{0,\dots,I\}$, $Z_i(t)$ is a continuous function on the compact $[0,T_0]$, there exist $t_0 \in [0,T_0]$ and $i_0 \in \{0,\dots,I\}$ such that $m = Z_{i_0}(t_0)$. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{\epsilon \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - \epsilon)}{\epsilon} \le 0, \quad 0 \le i_0 \le I, \tag{2.2}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \ge 0, 1 \le i_0 \le I-1. \tag{2.3}$$

From (2.1),we obtain the following inequality

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + b_{i_0}(t_0)\delta^0 Z_{i_0}(t_0) + (a_{i_0}(t_0) - \gamma)Z_{i_0}(t_0) \ge 0.$$

It follows from (2.2)-(2.3) that $(a_{i_0}(t_0) - \gamma)Z_{i_0}(t_0) \ge 0$, which implies that $Z_{i_0}(t_0) \ge 0$ because $(a_{i_0}(t_0 - \gamma) > 0$. We deduce that $V_h(t) \ge 0$ for $t \in [0,T_0]$ and the proof is complete. □

Lemma 2.2. *Let* $V_h(t)$, $W_h(t) \in C^1([0,T],\mathbb{R}^{I+1})$ *and* $f \in C^0(\mathbb{R}\times\mathbb{R},\mathbb{R})$ *such that:*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + V_i^q(t)\delta^0 V_i(t) + f(V_i(t),t) <$$
$$< \frac{dW_i(t)}{dt} - \delta^2 W_i(t) + W_i^q(t)\delta^0 W_i(t) + f(W_i(t),t),$$
$$V_0(t) < W_0(t), \quad V_I(t) < W_I(t) \quad t \in]0,T[$$
$$V_i(0) < W_i(0), \quad 0 \le i \le I.$$

Then $V_i(t) < W_i(t), \quad 0 \le i \le I, \quad t \in]0,T[$.

Proof. Introduce the vector $Z_h(t) = W_h(t) - V_h(t)$. Let t_0 the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0,t_0[$, $0 \le i \le I$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0,...,I\}$. We observe that

$$\begin{aligned} \frac{dZ_{i_0}(t_0)}{dt} &= \lim_{\epsilon\to 0}\frac{Z_{i_0}(t_0) - Z_{i_0}(t_0-\epsilon)}{\epsilon} \le 0, \quad 0 \le i_0 \le I \\ \delta^2 Z_{i_0}(t_0) &= \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \ge 0, \quad 1 \le i_0 \le I-1\,. \end{aligned}$$

Therefore we have

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + W_{i_0}^q(t_0)\delta^0 Z_{i_0}(t_0) + q\mu_{i_0}^{q-1}(t_0) Z_{i_0}(t_0)\delta^0 V_{i_0}(t_0) +$$
$$f(W_{i_0}(t_0),t) - f(V_{i_0}(t_0),t) \le 0,$$

where $\mu_{i_0}(t_0)$ is an intermediate value between $W_{i_0}(t_0)$ and $V_{i_0}(t_0)$. But this inequality contradicts the first strict differential inequality of the lemma 2.1 and the proof is complete. □

Lemma 2.3. *Let* U_h *be the solution of problem (1.6–1.7). Then we have,*

$$U_i(t) > 0 \quad for \quad 0 \le i \le I, t \in]0,T[.$$

Proof. Assume that there exists a time $t_0 \in]0,T[$ such that $U_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0,...,I\}$. We remark that:

$$\begin{aligned} \frac{dU_{i_0}(t_0)}{dt} &= \lim_{\epsilon\to 0}\frac{U_{i_0}(t_0) - U_{i_0}(t_0-\epsilon)}{\epsilon} \le 0, \ 0 \le i_0 \le I, \\ \delta^2 U_{i_0}(t_0) &= \frac{U_{i_0+1}(t_0) - 2U_{i_0}(t_0) + U_{i_0-1}(t_0)}{h^2} > 0, \ 1 \le i_0 \le I-1, \end{aligned}$$

which implies:

$$\frac{dU_{i_0}(t_0)}{dt} - \delta^2 U_{i_0}(t_0) + U_{i_0}^q(t_0)\delta^0 U_{i_0}(t_0) - \varepsilon U_{i_0}^{\beta+2}(t_0) < 0,\ 1 \le i_0 \le I-1,$$

But this inequality contradicts (1.6) and we obtain the desired result. □

The following lemma reveals that the solution U_h of the semidiscrete problem is symetric and $\delta^0 U_i(t)$ is positive when i is between 0 and $k-1$.

Lemma 2.4. *Let U_h be the solution of (1.6)-(1.7). Then for $t \in (0,T)$ we have:*

$$U_{I-i}(t) = U_i(t), \quad 0 \le i \le I \ \textit{and} \quad \delta^+ U_i(t) > 0, \quad 0 \le i \le k-1.$$

Proof. Introduce the vector $V_h(t)$ defined by $V_i(t) = U_{I-i}(t)$ for $0 \le i \le I$. It is not hard to see that $V_h(t)$ is a solution of (1.6)-(1.7). It follows from lemma 2.2 that $V_h(t)= U_h(t)$. Now, define the vector $Z_h(t)$ such that

$$Z_i(t) = U_{i+1}(t) - U_i(t), \quad 0 \le i \le k-1,$$

and let t_0 be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$. Without loss of the generality, we assume that i_0 is the smallest integer which guarantees the equality. If $i_0 = 0$ then we have $U_1(t_0) = U_0(t_0) = 0$, which is a contradiction because from lemma 2.3, $U_1(t_0) > 0$. It is easy to see that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) = 0, \quad if\ \ 1 \le i_0 \le k-1. \tag{2.4}$$

On the other hand, we observe:

$$\begin{aligned}
\frac{dZ_{i_0}(t_0)}{dt} &= \lim_{\epsilon \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0-\epsilon)}{\epsilon} \le 0,\\
\delta^2 Z_{i_0}(t_0) &= \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} > 0,\ 1 \le i_0 \le k-2,
\end{aligned}$$

and we know if $i_0 = k-1$,

$$\begin{aligned}
\delta^2 Z_{k-1}(t_0) &= \delta^2 U_k(t_0) - \delta^2 U_{k-1}(t_0)\\
&= \frac{U_{k+1}(t_0) - 2U_k(t_0) + U_{k-1}(t_0) - U_k(t_0) + 2U_{k-1}(t_0) - U_{k-2}(t_0)}{h^2}.
\end{aligned}$$

Since k is the integer part of the number $I/2$, using the fact that the discrete solution is symmetric, we have either $U_{k+1}(t) = U_{k-1}(t)$ or $U_{k+1}(t) = U_k(t)$.

In the both cases, we find that

$$\delta^2 Z_k(t_0) = \frac{Z_{k-2}(t_0)}{h^2} > 0.$$

The above inequalities imply that $\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) < 0$, which a contradiction because of (2.4) and the proof is complete. □

Lemma 2.5. *Let U_h be the solution. Then, we have:*

$$\frac{dU_i(t)}{dt} > 0 \quad for \quad 0 \leq i \leq I, \quad t \in]0, T[.$$

Proof. Consider the vector $Z_h(t)$ with $Z_i(t) = \frac{d}{dt} U_i(t)$, $0 \leq i \leq I$. Let t_0 be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0, t_0[$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{1, ..., I\}$. Whithout loss of the generality, we assume that i_0 is the smallest integer which satisfies the above equality. We get:

$$\begin{aligned} \frac{dZ_{i_0}(t_0)}{dt} &= \lim_{\epsilon \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - \epsilon)}{\epsilon} \leq 0, \ 0 \leq i_0 \leq I, \\ \delta^2 Z_{i_0}(t_0) &= \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} > 0, \ 1 \leq i_0 \leq I - 1, \end{aligned}$$

which implies that:

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + U_{i_0}^q(t_0)\delta^0 Z_{i_0}(t_0) + (qU_{i_0}^{q-1}(t_0)\delta^0 U_{i_0}(t_0) - \varepsilon(\beta+2)U_{i_0}^{\beta+1}(t_0))Z_{i_0}(t_0) < 0, \ if \ \ 1 \leq i_0 \leq I - 1\,.$$

Therefore, we have a contradiction because of (1.6–1.7) and leads to the desired result. □

The next theorem establishes that, for each fixed time interval $[0, T]$ where u is defined, the solution of semidiscrete problem approximates u, as $h \longrightarrow 0$.

Theorem 2.1. *Assume that (1.2–1.4) has a solution $u \in C^{4,1}([-1, 1] \times [0, T])$ and the initial condition φ_h at (1.7) satisfies:*

$$\|\varphi_h - u_h(0)\|_\infty = o(1), \ as \quad h \to 0,$$

where $u_h(t) = (u(x_0, t), ..., u(x_I))^T$, $t \in [0, T]$. Then, for h sufficiently small, problem (1.6)-(1.7) has a unique solution $U_h \in C^1([0, T], \mathbb{R}^{I+1})$ such that

$$\max_{t \in [0,T]} \|U_h(t) - u_h(t)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + h^2), \ h \to 0.$$

The proof of the theorem of convergence of the solution U_h is similar to those given in [19, 18], so we omit it here.

3. Convergence of semidiscrete blow-up time

In this section, under some assumptions we show that the semidiscrete solution U_h of problem (1.6–1.7) blows up in a finite time then we estimate its semidiscrete blow-up time and we prove that this time converges to the real one when the mesh size goes to zero.

Lemma 3.1. *Let $U_h \in \mathbb{R}^{I+1}$ such that $U_h > 0$. Then, we have*

$$\delta^2 U_i^\beta \geq \beta U_i^{\beta-1}\delta^2 U_i \quad for \quad 0 \leq i \leq I, \quad \beta > 0.$$

Proof. Using Taylor's expansion, we obtain:

$$\begin{aligned}
\delta^2 U_0^\beta &= \beta U_0^{\beta-1}\delta^2 U_0 + (U_1 - U_0)^2\frac{\beta(\beta-1)}{h^2}\theta_0^{\beta-2},\\
\delta^2 U_i^\beta &= \beta U_i^{\beta-1}\delta^2 U_i + (U_{i+1} - U_i)^2\frac{\beta(\beta-1)}{2h^2}\theta_i^{\beta-2} + (U_{i-1} - U_i)^2\frac{\beta(\beta-1)}{2h^2}\xi_i^{\beta-2},\\
&\quad 1 \leq i \leq I-1,\\
\delta^2 U_I^\beta &= \beta U_I^{\beta-1}\delta^2 U_I + (U_{I-1} - U_I)^2\frac{\beta(\beta-1)}{2h^2}\theta_i^{\beta-2},
\end{aligned}$$

where θ_i is an intermediate value between U_i and U_{i+1} and ξ_i is an intermediate value between U_{i-1} and U_i. Using the fact that $U_h > 0$, we have the desired result. □

Theorem 3.1. *Let U_h be the solution U_h of problem (1.6–1.7). Suppose that there exists a positive integer λ such that:*

$$\delta^2\varphi_i - \gamma_i\delta^0\varphi_i + \varepsilon\varphi_i^{\beta+2} \geq \lambda\varphi_i^{\beta+2}, 0 \leq i \leq I. \tag{3.1}$$

Then, the solution U_h of problem (1.6–1.7) blows up in a finite time T_b^h and we have the following estimate :

$$U_i(t) \leq B(T_b^h - t)^{-\gamma},$$

for $0 \leq t < T_b^h$, $0 \leq i \leq I$, and a positive constant B.

Proof. Let $[0, T_b^h[$ be the maximal time interval on which $\|U_h(t)\|_\infty < \infty$. Our aim is show that T_b^h is finite and satisfies the above inequality. We introduce the vector $J_h(t)$ such that:

$$J_i(t) = \frac{d}{dt}U_i(t) - \lambda U_i^{\beta+2}(t), \quad 0 \le i \le I,\ t \ge 0.$$

Then we have:

$$\frac{d}{dt}J_i - \delta^2 J_i = \frac{d}{dt}\Big(\frac{d}{dt}U_i - \lambda U_i^{\beta+2}\Big) - \delta^2\Big(\frac{d}{dt}U_i - \lambda U_i^{\beta+2}\Big).$$

Using lemma 3.1, a straightforward calculation gives:

$$\frac{d}{dt}J_i - \delta^2 J_i + 4\frac{\delta^0 U_i}{U_i}\delta^0 J_i + \Big(\varepsilon(\beta+2)U_i^{\beta+1} + 2\Big(\frac{\delta^0 U_i}{U_i}\Big)^2\Big)J_i \ge \lambda\beta(\beta+1)U_i^{\beta}(\delta^0 U_i)^2.$$

Setting $\gamma_i = 4\frac{\delta^0 U_i}{U_i}$ and $b_i = -\left(\varepsilon(\beta+2)U_i^{\beta+1} + 2\left(\frac{\delta^0 U_i}{U_i}\right)^2\right)$ we obtain:

$$\frac{d}{dt}J_i - \delta^2 J_i + \gamma_i \delta^0 J_i + b_i J_i \ge \lambda\beta(\beta+1)U_i^{\beta}(\delta^0 U_i)^2 \ge 0.$$

From (3.1), we observe that:

$$J_i(0) = \delta^2 U_i(0) - \gamma_i(0)\delta^0 U_i(0) + \varepsilon U_i^{\beta+2}(0) - \lambda U_i^{\beta+2}(0) \ge 0, \quad 0 \le i \le I.$$

We deduce from lemma 2.1 that $J_h(t) \ge 0$ for $t \in [0, T_b^h)$, which implies that

$$\frac{dU_i(t)}{dt} \ge \lambda U_i^{\beta+2}(t), \quad 0 \le i \le I,\ t \ge 0.$$

Integrating the above inequality over (t, T_b^h), we arrive at

$$T_b^h - t \le \frac{1}{\lambda}\frac{\big(U_i(t)\big)^{-(\beta+1)}}{\beta+1}, \tag{3.2}$$

which implies that: $U_i(t) \le B(T_b^h - t)^{-\gamma}$ where $B = \Big(\lambda(\beta+1)\Big)^{-\gamma}$ and $\gamma = \frac{1}{\beta+1}$, completing the proof. □

Remark 3.1. *The inequality (3.2) implies that:*

$$T_b^h - t_0 \le \frac{1}{\lambda}\frac{\|U_h(t_0)\|_\infty^{-(\beta+1)}}{\beta+1} \quad \textit{if} \quad 0 \le t_0 < T_b^h.$$

Theorem 3.2. *Suppose that the solution of (1.2)–(1.4) blows up in a finite time T_b such that $u \in C^{4,1}([0,1] \times [0,T[,\mathbb{R})$ and the initial condition at (1.7) satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \text{ as } h \to 0.$$

Assume that there exists a positive constant λ such that:

$$\delta^2\varphi_i - \gamma_i\delta^0\varphi_i + \varepsilon\varphi_i^{\beta+2} \geq \lambda\varphi_i^{\beta+2}, 0 \leq i \leq I.$$

Then the solution U_h of (1.6)–(1.7) blows up in a finite time T_b^h and

$$\lim_{h\to 0} T_b^h = T_b.$$

Proof. Let $\varepsilon > 0$. There exists a positive constante N such that:

$$\frac{1}{\lambda}\frac{y^{-(\beta+1)}}{(\beta+1)} \leq \frac{\varepsilon}{2} < \infty \quad for \quad y \in [N, +\infty[. \tag{3.3}$$

Since $\lim_{t\to T_b} \max_{x\in[0,1]} |u(x,t)| = +\infty$, then there exists T_1 such that:

$$|T_1 - T_b| \leq \frac{\varepsilon}{2} \quad and \quad \|u(x,t)\|_\infty \geq 2N \quad for\ t \in [T_1, T_b].$$

Let $T_2 = \dfrac{T_1 + T_b}{2}$, then $\sup_{t\in[0,T_2]} |u(x,t)| < \infty$. It follows from Theorem 2.1 that $\sup_{t\in[0,T_2]} |U_h(t) - u_h(t)|_\infty \leq N$. Applying the triangular inequality, we get

$$\|U_h(t)\|_\infty \geq \|u_h(t)\|_\infty - \|U_h(t) - u_h(t)\|_\infty,$$

which implies $\|U_h(t)\|_\infty \geq N$ for $t \in [0, T_2]$. From theorem 3.1, $U_h(t)$ blows up in a finite time T^h. We deduce from remark 3.1 and (3.3) that

$$|T_b - T_b^h| \leq |T_b - T_2| + |T_2 - T_b^h| \leq \frac{\varepsilon}{2} + \frac{1}{\lambda}\frac{\|U_h(T_2)\|_\infty^{-(\beta+1)}}{\beta+1} \leq \varepsilon\,,$$

which completes the proof. □

4. Numerical experiments

In this section, we estimate the numerical blow-up time and present some numerical results to the blow-up time of (1.2)-(1.4) with initial condition $\varphi(x) = \dfrac{1}{1-u(x)}$ where $u(x) = 0.001 * \left(1 - e^{x^2-1} + 0.5 * \cos(\frac{\pi}{2}x)\right)$ by using the algorithm proposed by C. Hirota and K. Ozawa [4]. The main idea of this method is to transform the ODE into a tractable form by the arc length transformation technique and to generate a linearly convergent sequence to the blow-up time. The sequence is then accelerated by the Aitken Δ^2 method. The present method is applied to the blow-up problems of PDEs by discretising the equations in space and integrating the resulting ODEs by an ODE solver, see [4, 12, 14, 15]. For our experiments we use the DOP54, see [8], and we set the three tolerances parameters $AbsTol = RelTol = 1.d15$, $InitialStep = 0$. Then we define our geometric sequence s_ℓ by $s_\ell = 2^{15}.2^\ell$, $(\ell = 0, 1, ..., 12)$. And finally to show that T_b^h converges actually to T, we varied I, ε and β. In the following, we present some tables containing the numerical blow-up times, values of I, the steps and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256, 512, and 1024 and some figures to illustrate our analysis. The order(s) of the method is computed from $s = \dfrac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{log(2)}$.

Table 1 : Numerical blow-up times, numbers of iterations, and orders of the approximations for $\varepsilon = 6, \quad \beta = 3$

I	T_b^h	$Steps$	s
16	0.0416756045	1652	-
32	0.0416703945	1981	-
64	0.0416692123	2229	2.13
128	0.0416689256	2470	2.04
256	0.0416688545	2977	2.01
512	0.0416688367	5117	2.01
1024	0.0416688323	14890	2.01

Table 2 : Numerical blow-up times, numbers of iterations, and orders of the approximations for $\varepsilon = 9, \quad \beta = 0.86$

I	T_b^h	$Steps$	s
16	0.059813403	2310	-
32	0.059793630	2648	-
64	0.059788614	2955	1.97
128	0.059787359	3229	1.99
256	0.059787046	3813	2.00
51	0.059786969	7060	2.02
1024	0.059786950	21084	2.02

Table 3 : Numerical blow-up times, numbers of iterations, and orders of the approximations for $\varepsilon = 10, \quad \beta = 0.86$

I	T_b^h	$Steps$	s
16	0.0538100852	2332	-
32	0.053795621	2684	-
64	0.053791808	2998	1.93
128	0.053790856	3308	2.00
256	0.053790619	3444	2.00
512	0.053790560	6660	2.00
1024	0.053790546	192522	2.06

Remark 4.1. *From these tables, we can assure the convergence of T_b^h to the blow-up time of the solution of (1.2-1.4), since the rate of convergence is near 2, which is just the accuracy of the difference approximation in space. For other illustrations, we also give some plots. From the Figures below, we can observe the rapidly growing behaviour of the solution and the blow-up point of the semidiscrete solution, which is in agreement with the theoretical results, see* [11].

Figure 1 :Evolution of the semidiscrete solution for $I = 64, \ \ \varepsilon = 6, \ \ \beta = 3$

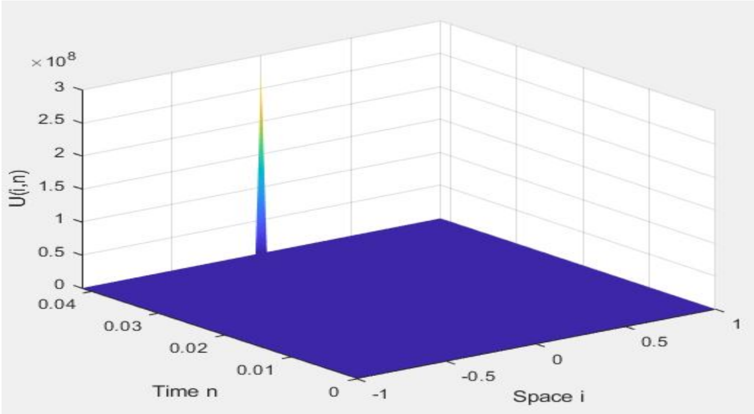
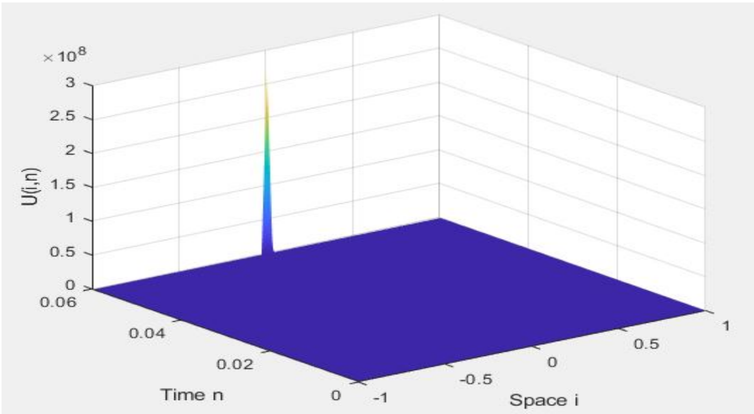


Figure 2 :Evolution of the semidiscrete solution for
$I = 256, \ \ \varepsilon = 9, \ \ \beta = 0.86$



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