INTUITIONISTIC FUZZY STRONG IMPLICATIVE FILTERS OF LATTICE WAJSBERG ALGEBRAS

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Abstract: In this paper, we introduce the notion of an intuitionistic fuzzy strong implicative filter of lattice Wajsberg algebra. Also, we investigate some properties with illustrations. Further, we obtain the relation between an intuitionistic fuzzy implicative filter and anti intuitionistic fuzzy strong implicative filter in lattice Wajsberg algebra. Finally, we establish the equivalent condition of an intuitionistic fuzzy strong implicative filter.

Keywords: Wajsberg algebra; Lattice Wajsberg algebra; Implicative filter; Strong Implicative filter; Fuzzy implicative filter; Fuzzy Strong implicative filter; Intuitionistic fuzzy implicative filter; Intuitionistic fuzzy strong implicative filter.

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1. Introduction

The concept of the fuzzy set was introduced by Zadeh [10] in 1965. After the introduction of the concept of fuzzy sets by Zadeh, several researches were conducted on the generalizations of the notion of fuzzy sets. The concept of intuitionistic fuzzy sets was first introduced by Atanassov [1, 2] in 1986 which is a generalization of the fuzzy sets. Mordchaj Wajsbreg [9] introduced the concept of Wajsberg algebras in 1935 and studied by Font, Rodriguez and Torrens in [5]. They [5] defined lattice structure of Wajsberg algebras and also, they introduced the notion of an implicative filter of lattice Wajsberg algebras and discussed some of their properties. Basheer Ahamed and Ibrahim [3, 4] introduced the definitions of fuzzy implicative filter and an anti fuzzy implicative filter of lattice Wajsberg algebras and obtained
some properties with illustrations. The authors \[6, 7, 8\] introduced the notions of strong implicative, fuzzy strong implicative, an anti fuzzy strong implicative, an intuitionistic fuzzy implicative and an intuitionistic anti fuzzy implicative filters of lattice Wajsberg algebra, and investigated some properties of them.

In this paper, we introduce the notion of an intuitionistic fuzzy strong implicative filter of lattice Wajsberg algebra. We establish the intuitionistic fuzzification of the concept of strong implicative filters in Wajsberg algebras, and investigate some of their properties. We obtain the relation between an intuitionistic fuzzy strong implicative filter and an intuitionistic fuzzy implicative filter.

2. Preliminaries

In this section, we recall some basic definitions and properties which are useful to develop the main results.

**Definition 2.1.** [5] Let \((A, \rightarrow, \ast, 1)\) be an algebra with a binary operation \(\rightarrow\) and a quasi complement \(*\) is called a Wajsberg algebra if and only if it satisfies the following axioms for all \(x, y, z \in A\),

\[
\begin{align*}
(i) \quad & 1 \rightarrow x = x \\
(ii) \quad & (x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1 \\
(iii) \quad & (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x \\
(iv) \quad & (x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = 1.
\end{align*}
\]

**Proposition 2.2.** [5] The Wajsberg algebra \((A, \rightarrow, \ast, 1)\) satisfies the following axioms for all \(x, y, z \in A\),

\[
\begin{align*}
(i) \quad & x \rightarrow x = 1 \\
(ii) \quad & \text{If } x \rightarrow y = y \rightarrow x = 1 \text{ then } x = y \\
(iii) \quad & x \rightarrow 1 = 1 \\
(iv) \quad & x \rightarrow (y \rightarrow x) = 1 \\
(v) \quad & \text{If } x \rightarrow y = y \rightarrow z = 1 \text{ then } x \rightarrow z = 1 \\
(vi) \quad & (x \rightarrow y) \rightarrow ((z \rightarrow x) \rightarrow (z \rightarrow y)) = 1 \\
(vii) \quad & x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) \\
(viii) \quad & x \rightarrow 0 = x \rightarrow 1^* = x^* \\
(ix) \quad & (x^*)^* = x \\
(x) \quad & x^* \rightarrow y^* = y \rightarrow x.
\end{align*}
\]

**Definition 2.3.** [5] The Wajsberg algebra \((A, \rightarrow, \ast, 1)\) is called a lattice Wajsberg algebra if it satisfies the following axioms for all \(x, y, z \in A\),

\[
\begin{align*}
(i) \quad & \text{A partial ordering } \leq \text{ on a lattice Wajsberg algebra } A \text{ such that } x \leq y \text{ if and only if } x \rightarrow y = 1. \\
(ii) \quad & (x \lor y) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1 \\
(iii) \quad & (x \land y) = ((x^* \rightarrow y^*) \rightarrow y^*). \text{ Thus, we have } (A, \lor, \land, \ast, 0, 1) \text{ is a lattice Wajsberg algebra with lower bound } 0 \text{ and upper bound } 1.
\end{align*}
\]

**Proposition 2.4.** \([/5]\) The Wajsberg algebra \((A, \rightarrow, \ast, 1)\) satisfies the following axioms for all \(x, y, z \in A\),

\[
\begin{align*}
(i) \quad & \text{If } x \leq y \text{ then } x \rightarrow z = y \rightarrow z \\
(ii) \quad & \text{If } x \leq y \text{ then } z \rightarrow x = z \rightarrow y \\
(iii) \quad & x \leq y \rightarrow z \text{ if and only if } y \leq x \rightarrow z \\
(iv) \quad & (x \lor y)^* = (x^* \land y^*) \\
(v) \quad & (x \land y)^* = (x^* \lor y^*) \\
(vi) \quad & (x \lor y) \rightarrow z = ((x \rightarrow z) \land (y \rightarrow z)) \\
(vii) \quad & x \rightarrow (y \land z) = (x \rightarrow y) \land (x \rightarrow z) \\
(viii) \quad & (x \rightarrow y) \lor (y \rightarrow x) = 1 \\
(ix) \quad & x \rightarrow (y \lor z) = (x \rightarrow y) \lor (x \rightarrow z) \\
(x) \quad & (x \land y) \rightarrow z = (x \rightarrow y) \lor (x \rightarrow z) \\
(xi) \quad & (x \land y) \lor z = (x \lor z) \land (y \lor z) \\
(xii) \quad & (x \land y) \rightarrow z = (x \rightarrow y) \rightarrow (x \rightarrow z).
\end{align*}
\]
Definition 2.5. [6] A lattice Wajsberg algebra $(A, \to, *, 1)$ is called a lattice $H$-Wajsberg algebra, if it satisfies $x \lor y \lor (x \land y) = 1$ for all $x, y, z \in A$. In a lattice $H$-Wajsberg algebra $A$, the following hold

(i) $x \to (x \to y) = (x \to y)$
(ii) $x \to (y \to z) = (x \to y) \to (x \to z)$ for all $x, y, z \in A$.

Definition 2.6. [6] Let $(A_1, \to, *, 1)$ and $(A_2, \to, *, 1)$ be lattice Wajsberg algebras, $f : A_1 \to A_2$ be a mapping from $A_1$ to $A_2$, if for any $x, y \in A_1, f(x \to y) = f(x) \to f(y)$ holds, then $f$ is called an implication homomorphism from $A_1$ to $A_2$.

Definition 2.7. [5] Let $(A, \to, *, 1)$ be a lattice Wajsberg algebra. A subset $F$ of $A$ is called an implicative filter of $A$ if it satisfies the following axioms for all $x, y \in A$.

(i) $1 \in F$
(ii) $x \in F$ and $x \to y \in F$ imply $y \in F$.

Definition 2.8. [10] Let $X$ be a set. A function $\mu : X \to [0, 1]$ is called a fuzzy subset on $X$, for all $x \in X$ the value of $\mu(x)$ describes a degree of membership of $x$ in $\mu$.

Definition 2.9. [10] Let $\mu$ be a fuzzy subset of $X$ then the complement of $\mu$ is denoted by $\mu^c$ and defined as $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$.

Definition 2.10. [10] Let $\mu$ be a fuzzy set in a set $A$. Then for $t \in [0, 1]$, the set $\mu_t = \{x \in A / \mu(x) \geq t\}$ is called a level subset of $\mu$.

Definition 2.11. [10] Let $\mu$ be a fuzzy set in a set $A$. Then for $t \in [0, 1]$, the set $\mu^c = \{x \in A / \mu(x) \leq t\}$ is called a lower $t$-level cut of $\mu$.

Definition 2.12. [3] Let $(A, \to, *, 1)$ be a lattice Wajsberg algebra. A fuzzy subset $\mu$ of $A$ is called a fuzzy implicative filter of $A$ if it satisfies the following axioms for all $x, y, z \in A$.

(i) $\mu(1) \geq \mu(x)$
(ii) $\mu(z) \geq \min\{\mu(y), \mu(y \to z)\}$.

Proposition 2.14. [3] Let $\mu$ be a fuzzy implicative filter of a lattice Wajsberg algebra $A$, then $x \leq y$ implies $\mu(x) \leq \mu(y)$ for all $x, y \in A$.

Definition 2.15. [6] Let $(A, \to, *, 1)$ be a lattice Wajsberg algebra. A subset $F$ of $A$ is called a strong implicative filter of $A$ if it satisfies the following axioms for all $x, y, z \in A$

(i) $1 \in F$
(ii) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$.

Definition 2.16. [7] Let $(A, \to, *, 1)$ be a lattice Wajsberg algebra. A fuzzy subset $\mu$ of $A$ is called a fuzzy strong implicative filter of $A$ if it satisfies the following for all $x, y, z \in A$,

(i) $\mu(1) \geq \mu(x)$
(ii) $\mu(x \to z) \geq \min\{\mu(x\to y), \mu(x \to (y \to z))\}$.

Definition 2.17. [1] An intuitionistic fuzzy set $S$ of a non-empty set $A$ is an object having the form $S = \{(x, \mu_S(x), \gamma_S(x)) / x \in A\} = (\mu_S, \gamma_S)$ where the functions $\mu_S : A \to [0, 1]$ and $\gamma_S : A \to [0, 1]$ denotes the
In the same example 3.2, consider the intuitionistic fuzzy set of

$$S((i))$$  

Then, we have

$$\mu_S((i)) \geq \mu_S((ii)) = \min \{\mu_S((i)), \mu_S((iii))\}$$

(iii) $$\gamma_S((ii)) = \max \{\gamma_S((i)), \gamma_S((iii))\}$$.

3. Intuitionistic fuzzy strong implicative filters

In this section, we introduce an intuitionistic fuzzy strong implicative filter of lattice Wajsberg algebra with illustrations and investigate some properties.

**Definition 3.1.** Let $$(A, \rightarrow, *, 1)$$ be a lattice Wajsberg algebra. An intuitionistic fuzzy set $$S = (\mu_S, \gamma_S)$$ of $$A$$ is called an intuitionistic fuzzy implicative filter of $$A$$ if it satisfies the following inequalities for all $$x, y, z \in A$$,

(i) $$\mu_S(1) \geq \mu_S(x)$$ and $$\gamma_S(1) \leq \gamma_S((i))$$

(ii) $$\mu_S((i)) \geq \mu_S((ii)) = \min \{\mu_S((i)), \mu_S((iii))\}$$

(iii) $$\gamma_S((ii)) = \max \{\gamma_S((i)), \gamma_S((iii))\}$$.

**Example 3.2.** Consider a set $$A = \{0, a, b, c, d, 1\}$$ with Figure 3.1 as a partial ordering. Define a quasi complement “*” and a binary operation “$$\rightarrow$$” on $$A$$ as in Tables 3.1 and 3.2.

<table>
<thead>
<tr>
<th>x</th>
<th>x*</th>
<th>→</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>a</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>a</td>
<td>C</td>
<td>b</td>
<td>c</td>
<td>1</td>
<td>1</td>
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<td>1</td>
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</tr>
<tr>
<td>b</td>
<td>D</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>b</td>
<td>1</td>
<td>a</td>
<td>1</td>
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<tr>
<td>c</td>
<td>A</td>
<td>d</td>
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<td>1</td>
<td>a</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>d</td>
<td>B</td>
<td>1</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>1</td>
</tr>
</tbody>
</table>

Table:3.1

Table:3.2

Figure:3.1

Define 'v' and '∧' operations on $$A$$ as follows:

$$(x \lor y) = (x \rightarrow y) \rightarrow y; (x \land y) = ((x^* \rightarrow y^*)^*)$$ for all $$x, y \in A$$.

Then, $$A$$ is a lattice Wajsberg algebra.

Consider an intuitionistic fuzzy set $$S = (\mu_S, \gamma_S)$$ on $$A$$ as

$$\mu_S(x) = \begin{cases} 1 & \text{if } x \in \{a, 1\}, \\ 0.4 & \text{otherwise} \end{cases} \quad \gamma_S(x) = \begin{cases} 0 & \text{if } x \in \{a, 1\}, \\ 0.6 & \text{otherwise} \end{cases}$$

is an intuitionistic fuzzy strong implicative filter of $$A$$.

In the same example 3.2, consider the intuitionistic fuzzy set $$S = (\mu_S, \gamma_S)$$ on $$A$$ as,

$$\mu_S(x) = \begin{cases} 0.7 & \text{if } x \in \{1, a, c\}, \\ 0.3 & \text{if } x \in \{0, b, d\} \end{cases} \quad \gamma_S(x) = \begin{cases} 0.4 & \text{if } x \in \{1, a, c\}, \\ 0.6 & \text{if } x \in \{0, b, d\} \end{cases}$$

Then, we have $$S = (\mu_S, \gamma_S)$$ is not an intuitionistic fuzzy strong implicative filter of lattice Wajsberg algebra $$A$$. Since, we have $$\mu_S(c \rightarrow b) = 0.3 \geq \min \{\mu_S(c \rightarrow d), \mu_S(c \rightarrow (d \rightarrow b))\} = \min \{\mu_S(a), \mu_S(1)\} = 0.7$$.

Finally, we have $$\gamma_S(c \rightarrow b) = 0.6 \leq \max \{\gamma_S(c \rightarrow d), \gamma_S(c \rightarrow (d \rightarrow b))\} = \max \{\gamma_S(a), \gamma_S(1)\} = 0.4$$. 

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Proposition 3.3. Let $A$ be a lattice Wajsberg algebra, and let $S = (\mu_S, \gamma_S)$ be an intuitionistic fuzzy strong implicative filter of $A$, then $S = (\mu_S, \gamma_S)$ is an intuitionistic fuzzy implicative filter of $A$.

Proof. Let $S = (\mu_S, \gamma_S)$ be an intuitionistic fuzzy strong implicative filter of a lattice Wajsberg algebra $A$.

From the Definition 3.1, we have $\mu_S(x \rightarrow z) \geq \min \{\mu_S(x \rightarrow y), \mu_S(x \rightarrow (y \rightarrow z))\}$ for all $x, y, z \in A$.

Take $x = 1$, then from (i) of Definition 2.1, we have

$$\mu_S(1 \rightarrow z) \geq \min \{\mu_S(1 \rightarrow y), \mu_S(1 \rightarrow (y \rightarrow z))\}.$$  \hspace{1cm} (3.1)

Hence, $\mu_S(z) \geq \min \{\mu_S(y), \mu_S(y \rightarrow z)\}$. \hspace{1cm} (3.2)

From the Definition 3.1, we have $\gamma_S(x \rightarrow z) \leq \max \{\gamma_S(x \rightarrow y), \gamma_S(x \rightarrow (y \rightarrow z))\}$ for all $x, y, z \in A$.

Take $x = 1$, then from (i) of Definition 2.1, we have

$$\gamma_S(1 \rightarrow z) \leq \max \{\gamma_S(1 \rightarrow y), \gamma_S(1 \rightarrow (y \rightarrow z))\}.$$  \hspace{1cm} (3.3)

Hence, $\gamma_S(z) \leq \max \{\gamma_S(y), \gamma_S(y \rightarrow z)\}$. \hspace{1cm} (3.4)

From (3.1) and (3.2), we have $S = (\mu_S, \gamma_S)$ is an intuitionistic fuzzy implicative filter of $A$. \hspace{1cm} \blacksquare

Proposition 3.4. Let $A$ be a lattice Wajsberg algebra. Then $A$ is a lattice $H$-Wajsberg algebra if and only if each intuitionistic fuzzy implicative filter of $A$ is an intuitionistic fuzzy strong implicative filter.

Proof. Let $A$ be a lattice $H$-Wajsberg algebra and $S = (\mu_S, \gamma_S)$ be an intuitionistic fuzzy implicative filter of $A$.

From (i) of Definition 2.18, we have $\mu_S(1) \geq \mu_S(x)$ and $\gamma_S(1) \leq \gamma_S(x)$ for all $x \in A$. \hspace{1cm} (3.5)

From (ii) of Definition 2.5, we have $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$. Therefore, $\mu_S(x \rightarrow y), \mu_S(x \rightarrow (y \rightarrow z)) = \min \{\mu_S(x \rightarrow y), \mu_S(x \rightarrow (y \rightarrow z))\} \leq \mu_S(x \rightarrow z)$ and $\max \{\gamma_S(x \rightarrow y), \gamma_S(x \rightarrow (y \rightarrow z))\} = \gamma_S((x \rightarrow y) \rightarrow (x \rightarrow z)) \geq \gamma_S(x \rightarrow z)$, for all $x, y, z \in A$. \hspace{1cm} (3.6)

Thus, $\mu_S(x \rightarrow z) \geq \min \{\mu_S(x \rightarrow y), \mu_S(x \rightarrow (y \rightarrow z))\}$ and $\gamma_S(x \rightarrow z) \leq \max \{\gamma_S(x \rightarrow y), \gamma_S(x \rightarrow (y \rightarrow z))\}$ for all $x, y, z \in A$. \hspace{1cm} (3.7)

From (3.5) and (3.6), we have $S = (\mu_S, \gamma_S)$ is an intuitionistic fuzzy strong implicative filter.

Conversely, we consider an intuitionistic fuzzy implicative filter $S = (\mu_S, \gamma_S)$ of $A$ is an intuitionistic fuzzy strong implicative filter.

Define a mappings $\mu_S : A \rightarrow [0, 1]$ and $\gamma_S : A \rightarrow [0, 1]$ as

$$\mu_S(x) = \begin{cases} 0.8 & \text{if } x = 1, \\ 0 & \text{if } x \neq 1 \end{cases} \quad \gamma_S(x) = \begin{cases} 0 & \text{if } x = 1, \\ 0.7 & \text{if } x \neq 1 \end{cases}$$

Then, $\mu$ is a fuzzy implicative filter of $A$ and hence it is a fuzzy strong implicative filter. It follows that $\{1\} = \mu_{0.8}$ and $\{0\} = \gamma_{0.7}$ are implicative filters of $A$. This implies that $A$ is a lattice $H$-Wajsberg algebra. \hspace{1cm} \blacksquare

Proposition 3.5. Let $A$ be a lattice Wajsberg algebra and $S = (\mu_S, \gamma_S)$ be an intuitionistic fuzzy set of $A$, if $S = (\mu_S, \gamma_S)$ is an intuitionistic fuzzy strong implicative filter, then the following are satisfied and equivalent:

(i) $S = (\mu_S, \gamma_S)$ is an intuitionistic fuzzy strong implicative filter and for all $x, y \in A$,

$$\mu_S(x \rightarrow y) \geq \mu_S(x \rightarrow (x \rightarrow y)) \quad \gamma_S(x \rightarrow y) \leq \gamma_S(x \rightarrow (x \rightarrow y)).$$

(ii) $S = (\mu_S, \gamma_S)$ is an intuitionistic fuzzy implicative filter and for all $x, y, z \in A$,

$$\mu_S((x \rightarrow y) \rightarrow (x \rightarrow z)) \geq \mu_S((x \rightarrow y) \rightarrow (x \rightarrow z)) \quad \gamma_S((x \rightarrow y) \rightarrow (x \rightarrow z)) \leq \gamma_S((x \rightarrow y) \rightarrow (x \rightarrow z)).$$

(iii) $\mu_S(x \rightarrow y) \geq \min \{\mu_S(x), \mu_S(x \rightarrow z)\}$ and $\gamma_S((x \rightarrow y) \rightarrow (x \rightarrow z)) \leq \max \{\gamma_S(x \rightarrow y), \gamma_S(x \rightarrow z)\}$ for all $x, y, z \in A$.

Proof. \hspace{1cm} (i) \hspace{1cm} (ii).

Let $S = (\mu_S, \gamma_S)$ be an intuitionistic fuzzy strong implicative filter of $A$.

Then, from the Definition 3.1, we have $\mu_S(1) \geq \mu_S(x)$ and $\gamma_S(1) \leq \gamma_S(x)$, $\mu_S(x \rightarrow z) \geq \min \{\mu_S(x \rightarrow y), \mu_S(x \rightarrow (y \rightarrow z))\}$ and $\gamma_S(x \rightarrow z) \leq \max \{\gamma_S(x \rightarrow y), \gamma_S(x \rightarrow (y \rightarrow z))\}$. Put $x = 1$ and (i) of Proposition 2.1, we get $\mu_S(1) \geq \min \{\mu_S(x), \mu_S(x \rightarrow z)\}$ and $\gamma_S(x \rightarrow z) \leq \max \{\gamma_S(x \rightarrow y), \gamma_S(x \rightarrow (y \rightarrow z))\}$ for all $x, y, z \in A$.

Therefore, $S = (\mu_S, \gamma_S)$ is an intuitionistic fuzzy implicative filter of $A$.

From $\mu_S(x \rightarrow z) \geq \mu_S(x \rightarrow (x \rightarrow z)), \gamma_S(x \rightarrow z) \leq \gamma_S(x \rightarrow (x \rightarrow z))$, put $z = y$, we have $\mu_S(x \rightarrow y) \geq \mu_S((x \rightarrow y) \rightarrow (x \rightarrow z)) \quad \gamma_S((x \rightarrow y) \rightarrow (x \rightarrow z)) \leq \gamma_S((x \rightarrow y) \rightarrow (x \rightarrow z))$. If for any $x, y, z \in A$, $\mu_S(x \rightarrow y) \geq \mu_S((x \rightarrow y) \rightarrow (x \rightarrow z)) \quad \gamma_S((x \rightarrow y) \rightarrow (x \rightarrow z)) \leq \gamma_S((x \rightarrow y) \rightarrow (x \rightarrow z))$.

Then, we have $\mu_S((x \rightarrow y) \rightarrow (x \rightarrow z)) \geq \mu_S((x \rightarrow y) \rightarrow (x \rightarrow z)) \geq \mu_S((x \rightarrow y) \rightarrow (x \rightarrow z))$, \hspace{1cm} \blacksquare
and \( x \rightarrow ((x \rightarrow y) \rightarrow z) = x \rightarrow ((x \rightarrow y) \rightarrow z) = x \rightarrow (y \rightarrow z \rightarrow y) \). From the Proposition 2.14, we have \( \mu_S(x \rightarrow ((x \rightarrow y) \rightarrow z)) \geq \mu_S(x \rightarrow (y \rightarrow z)) \).

Therefore, \( \mu_S(x \rightarrow (x \rightarrow y)) \geq \mu_S(x \rightarrow (y \rightarrow z)) \).

Similarly, \( \gamma_S((x \rightarrow y) \rightarrow (x \rightarrow z)) = \gamma_S((x \rightarrow (y \rightarrow z)) \leq \gamma_S((x \rightarrow (y \rightarrow z)) \) and \( \gamma_S((x \rightarrow (y \rightarrow z)) \leq \gamma_S((x \rightarrow y) \rightarrow z)) \). Thus, \( \gamma_S((x \rightarrow y) \rightarrow (x \rightarrow z)) \leq \gamma_S((x \rightarrow y) \rightarrow z)) \).

(ii) \rightarrow (iii). Let (ii) be hold. Then, it is clear that \( \mu_S(1) \geq \mu_S(x) \) and \( \gamma_S(1) \leq \gamma_S(x) \).

If for any \( x, y, z \in A \), we have \( \mu_S((x \rightarrow y) \rightarrow (x \rightarrow z)) \geq \mu_S(x \rightarrow (y \rightarrow z)) \)

and \( \gamma_S((x \rightarrow y) \rightarrow (x \rightarrow z)) \leq \gamma_S(x \rightarrow (y \rightarrow z)) \).

Put \( y = x \), then, we get \( \mu_S((x \rightarrow x) \rightarrow (x \rightarrow z)) \geq \mu_S(x \rightarrow (x \rightarrow z)) \).

That is, \( \mu_S(x \rightarrow z) \geq \mu_S(x \rightarrow (x \rightarrow z)) \). [From (i) of Proposition 2.2 and (i) of Definition 2.1]

Thus for any \( x, y \in A \), \( \mu_S((x \rightarrow y) \rightarrow (x \rightarrow z)) \geq \mu_S(x \rightarrow (x \rightarrow y)) \).

Since \( \mu_S \) is a fuzzy implicative filter and then, we have

\[
\mu_S(x \rightarrow (x \rightarrow y)) \geq \min \{ \mu_S(z \rightarrow (x \rightarrow y)), \mu_S(z) \}.
\]

Hence \( \mu_S((x \rightarrow y) \rightarrow y) \geq \min \{ \mu_S(z \rightarrow (x \rightarrow y)), \mu_S(z) \} \).

Similarly, \( \gamma_S(x \rightarrow y) \leq \gamma_S((x \rightarrow y) \rightarrow y) \) and \( \gamma_S((x \rightarrow y) \rightarrow y) \leq \min \{ \gamma_S(z \rightarrow (x \rightarrow y)), \gamma_S(z) \} \).

Thus \( \gamma_S(x \rightarrow y) \leq \min \{ \gamma_S(z \rightarrow (x \rightarrow y)), \gamma_S(z) \} \).

(iii) \rightarrow (i). Let (iii) be hold.

Put \( x = 1 \), then, we get \( \mu_S(1) \geq 1 \) and \( \gamma_S(1) \leq 1 \).

From (i) of Proposition 2.2 and (i) of Definition 2.1

Also, \( \mu_S(1) \geq \mu_S(x) \) and \( \gamma_S(1) \leq \gamma_S(x) \). Hence \( S = (\mu_S, \gamma_S) \) is an intuitionistic fuzzy implicative filter.

Put \( z = 1 \), then, we get \( \mu_S(x \rightarrow y) \geq \mu_S(x \rightarrow (x \rightarrow y)) \) and \( \gamma_S(x \rightarrow y) \leq \gamma_S(x \rightarrow (x \rightarrow y)) \). [From (i) of Definition 2.1 and (1) of Definition 2.1 and \( \gamma_S(1) \leq 1 \) of \( \gamma_S(x) \)].

Proposition 3.6. Let \( S = (\mu_S, \gamma_S) \) be an intuitionistic fuzzy strong implicative filter of a lattice Wajsberg algebra \( A \) if and only if the fuzzy subsets \( \mu_S, \gamma_S \) are fuzzy strong implicative filters of \( A \), where \( \gamma_S(x) = 1 - \gamma_S(1) \) for all \( x \in A \).

Proof. Let \( S = (\mu_S, \gamma_S) \) be an intuitionistic fuzzy strong implicative filter of \( A \).

From the Definition 3.1, we have the fuzzy subset \( \mu_S \) is a fuzzy strong implicative filter of \( A \).

Now \( \gamma_S(x \rightarrow z) = 1 - \gamma_S(y \rightarrow z) \geq 1 - \gamma_S(x \rightarrow y) \gamma_S(y \rightarrow z) \)

\[ = \min \{ 1 - \gamma_S(x \rightarrow y), 1 - \gamma_S(x \rightarrow (y \rightarrow z)) \} \]

\[ = \min \{ \gamma_S(x \rightarrow y), \gamma_S(x \rightarrow (y \rightarrow z)) \}. \]

Thus, \( \gamma_S(x \rightarrow z) \leq \min \{ \gamma_S(x \rightarrow y), \gamma_S(x \rightarrow (y \rightarrow z)) \}. \)

Hence, \( \gamma_S \) is a fuzzy strong implicative filter of \( A \).

Conversely, if \( \mu_S \) and \( \gamma_S \) are fuzzy strong implicative filters of \( A \).

Then we have \( \gamma_S(1) \geq \mu_S(x) \) and

\[ 1 - \gamma_S(1) = \gamma_S(1) \geq \gamma_S(x) = \gamma_S(1). \]

From (i) and (ii) of Definition 2.16, we have \( \mu_S(x \rightarrow z) \geq \min \{ \mu_S(x \rightarrow y), \mu_S(x \rightarrow (y \rightarrow z)) \} \) and \( \gamma_S(x \rightarrow z) \leq \mu_S(x \rightarrow z), \gamma_S(x \rightarrow (y \rightarrow z)) \}

\[ 1 - \gamma_S(x \rightarrow z) \leq \min \{ 1 - \mu_S(x \rightarrow y), 1 - \mu_S(x \rightarrow (y \rightarrow z)) \} \]

\[ = 1 - \gamma_S(x \rightarrow z) \leq \min \{ \mu_S(x \rightarrow y), \mu_S(x \rightarrow (y \rightarrow z)) \}. \]

Thus, \( S = (\mu_S, \gamma_S) \) is an intuitionistic fuzzy strong implicative filter of \( A \).■

Proposition 3.7. Let \( S = (\mu_S, \gamma_S) \) be an intuitionistic fuzzy strong implicative filter of a lattice Wajsberg algebra \( A \) if and only if \( (\mu_S, \gamma_S) \) and \( (\gamma_S, \gamma_S) \) are intuitionistic fuzzy strong implicative filters of \( A \).

Proof. Let \( S = (\mu_S, \gamma_S) \) be an intuitionistic fuzzy strong implicative filter of \( A \).

Then, \( 0 \leq \mu_S(x) + \gamma_S(x) = 1 \) and \( 0 \leq \gamma_S(x) + \gamma_S(x) = 1 \).

Thus, \( \mu_S(x) + \gamma_S(x) \) and \( \gamma_S(x), \gamma_S(x) \) are intuitionistic fuzzy strong filters of \( A \).

We have \( \mu_S(1) \geq \mu_S(x), \mu_S(1) \leq \mu_S(x) \geq \mu_S(x) \).

\[
\begin{align*}
\mu_S(x \rightarrow y) \geq & \min \{ \mu_S(x \rightarrow y), \mu_S(x \rightarrow (y \rightarrow z)) \} \\
\gamma_S(x \rightarrow y) \leq & 1 - \min \{ \mu_S(x \rightarrow y), \mu_S(x \rightarrow (y \rightarrow z)) \}
\end{align*}
\]
Therefore,

\[ n_0 = \max \{ \mu_\omega(x \to y), \mu_\omega(x \to (y \to z)) \} \]

Thus \( \mu_\omega(x \to z) \leq \max \{ \mu_\omega(x \to y), \mu_\omega(x \to (y \to z)) \} \).

From (3.5), (3.6) and (3.7), we have \( (\mu_\omega, \mu_{\omega}^c) \) is an intuitionistic fuzzy strong implicative filter of \( A \).

Similarly, we have \( \gamma_\omega(x) \) is an intuitionistic fuzzy strong implicative filter of \( A \).

Thus, \( \mu_\omega, \mu_{\omega}^c, \gamma_\omega, \gamma_{\omega}^c \) are intuitionistic fuzzy strong implicative filters of \( A \) then \( \mu_\omega(1) \geq 1 \) and \( \gamma_\omega(1) \leq 1 \), \( \mu_\omega(x) \geq \min \{ \mu_\omega(x \to y), \mu_\omega(x \to (y \to z)) \} \),

and \( \gamma_\omega(x) \leq \max \{ \gamma_\omega(x \to y), \gamma_\omega(x \to (y \to z)) \} \).

Hence, \( S = (\mu_\omega, \gamma_\omega) \) is an intuitionistic fuzzy strong implicative filter of \( A \).

Proposition 3.8. An intuitionistic fuzzy set \( S = (\mu_\omega, \gamma_\omega) \) of a lattice Wajsberg algebra \( A \) is an intuitionistic fuzzy strong implicative filter of \( A \) if and only if for each \( m, n \in [0, 1], \) the sets \( \mu_{\omega}^m \) and \( \gamma_{\omega}^m \) is either empty or fuzzy strong implicative filters of \( A \).

Proof. Let \( S = (\mu_\omega, \gamma_\omega) \) be an intuitionistic fuzzy strong implicative filter of \( A \) and \( \mu_{\omega}^m \neq \emptyset \), \( \gamma_{\omega}^m \neq \emptyset \) for all \( m, n \in [0, 1] \). Let \( x \to y \in \mu_{\omega}^m \) and \( x \to (y \to z) \in \mu_{\omega}^m \) for all \( x, y, z \in A \).

Then, we have \( \mu_\omega(x \to y) \geq n \) and \( \mu_\omega(x \to (y \to z)) \geq n \).

Thus, we get \( \mu_\omega(x \to y) \geq \min \{ \mu_\omega(x \to y), \mu_\omega(x \to (y \to z)) \} = n \), that is, \( x \to z \in \mu_{\omega}^m \).

Thus, \( \mu_{\omega}^m \) is a fuzzy strong implicative filter of \( A \).

Similarly, let \( x, y, z \in A \) be such that \( x \to y \in \gamma_{\omega}^m \) and \( x \to (y \to z) \in \gamma_{\omega}^m \).

Then, we have \( \gamma_\omega(x \to y) \leq m \) and \( \gamma_\omega(x \to (y \to z)) \leq m \).

Thus, we get \( \gamma_\omega(x \to y) \leq \max \{ \gamma_\omega(x \to y), \gamma_\omega(x \to (y \to z)) \} \leq m \), that is, \( x \to z \in \gamma_{\omega}^m \).

Thus, \( \gamma_{\omega}^m \) is a fuzzy strong implicative filter of \( A \).

Conversely, for each \( m, n \in [0, 1], \) the sets \( \mu_{\omega}^m \) and \( \gamma_{\omega}^m \) is either empty or implicative filters of \( A \). For any \( x \in A, \) let \( \mu_\omega(x) = m \) and \( \gamma_\omega(x) = n \). Then, \( x \in \mu_{\omega}^m \cap \gamma_{\omega}^m \) and so that \( \mu_{\omega}^m \neq \emptyset \neq \gamma_{\omega}^m \).

Since \( \mu_{\omega}^m \) and \( \gamma_{\omega}^m \) are fuzzy strong implicative filters of \( A \), therefore \( 1 \in \mu_{\omega}^m \) and \( 1 \in \gamma_{\omega}^m \).

Hence, \( \mu_\omega(1) \geq n = \mu_{\omega}^m(1) \).

To prove: (i) \( \mu_\omega(x \to y) \geq \min \{ \mu_\omega(x \to y), \mu_\omega(x \to (y \to z)) \} \) and

(ii) \( \gamma_\omega(x \to z) \leq \max \{ \gamma_\omega(x \to y), \gamma_\omega(x \to (y \to z)) \} \).

(i) If not, then there exists \( p, q, r \in A \) such that \( \mu_\omega(p \to q) < \min \{ \mu_\omega(p \to q), \mu_\omega(p \to (q \to r)) \} \).

Taking \( m_0 = \frac{1}{2} [\mu_\omega(p \to r) + \min \{ \mu_\omega(p \to q), \mu_\omega(p \to (q \to r)) \} \),

we have \( \mu_\omega(p \to r < m_0 < \min \{ \mu_\omega(p \to q), \mu_\omega(p \to (q \to r)) \} \).

Therefore, \( p \to q \in \mu_{\omega}^m \) and \( p \to (q \to r) \in \mu_{\omega}^m \) but \( p \to r \notin \mu_{\omega}^m \).

Thus, \( \mu_{\omega}^m \) is not a fuzzy strong implicative filter of \( A \), which is a contradiction.

Therefore, \( \mu_\omega(x \to z) \geq \min \{ \mu_\omega(x \to y), \mu_\omega(x \to (y \to z)) \} \).

(ii) If not, then there exists \( u, v, w \in A \) such that \( \gamma_\omega(u \to v) > \max \{ \gamma_\omega(u \to v), \gamma_\omega(u \to (v \to w)) \} \).

Taking \( m_0 = \frac{1}{2} [\mu_\omega(u \to w) + \max \{ \gamma_\omega(u \to v), \gamma_\omega(u \to (v \to w)) \} \),

we have \( \gamma_\omega(u \to v) < m_0 < \min \{ \gamma_\omega(u \to v), \gamma_\omega(u \to (v \to w)) \} \).

Therefore, \( u \in \gamma_{\omega}^m \) and \( v \to u \in \gamma_{\omega}^m \) but \( u \notin \gamma_{\omega}^m \).

Thus, \( \gamma_{\omega}^m \) is not a fuzzy strong implicative filter of \( A \), which is a contradiction.

Therefore, \( \gamma_\omega(x \to z) \leq \max \{ \gamma_\omega(x \to y), \gamma_\omega(x \to (y \to z)) \} \).

Hence, \( S = (\mu_\omega, \gamma_\omega) \) is an intuitionistic fuzzy strong implicative filter of \( A \).

Proposition 3.9. Let \( A \) be a lattice Wajsberg algebra, \( V \) a non-empty subset of \( [0, 1] \) and \( I_t = \{ x \in J(x) \geq t, x \in A \} \) such that \( t \in V \) and \( J \) is a fuzzy strong implicative filter of \( A \) satisfies the following:

(i) \( A = \bigcup_{t \in V} I_t \)

(ii) \( r > t \) if and only if \( I_r \subseteq I_t \) for all \( r, t \in V \)

(iii) Let \( \mu_\omega(x) = \sup \{ t \in V \mid x \in I_t \} \) and \( \gamma_\omega(x) = \inf \{ t \in V \mid x \in I_t \} \) for all \( x \in A \) then \( P = (\mu_\omega, \gamma_\omega) \) is an intuitionistic fuzzy strong implicative filter of \( A \).

Proof. Let \( P = (\mu_\omega, \gamma_\omega) \) be an intuitionistic fuzzy set of \( A \).
We have to prove: \( P = (\mu_p, \gamma_p) \) is an intuitionistic fuzzy strong implicative filter of \( A \).

(i) \( \mu_p(x) = \sup \{ t \in V / 1 \leq t \} \) for all \( x \in A \).

(ii) If \( \mu_p(x \rightarrow z) < \min \{ \mu_p(x), \mu_p(x \rightarrow (y \rightarrow z)) \} \) for all \( x, y, z \in A \).

Let \( t_1 = \min \{ \mu_p(x), \mu_p(x \rightarrow y) \} \). Therefore, \( \mu_p(x \rightarrow y) \geq t_1, \mu_p(x \rightarrow (y \rightarrow z)) \geq t_1 \).

Then there exists a fuzzy strong implicative filter \( f_{t_1} \), such that \( x \rightarrow y \in f_{t_1} \) and \( x \rightarrow (y \rightarrow z) \in f_{t_1} \). So, \( x \rightarrow z \in f_{t_1} \).

We have to prove:

\[
\gamma_p(x \rightarrow y) = \inf \{ t \in V / y \rightarrow z \} \leq t_2 \quad \text{and} \quad \gamma_p(x \rightarrow (y \rightarrow z)) \leq t_3.
\]

From (3.8), (3.9) and (3.10), we have

Thus, we have \( \gamma_p(x \rightarrow y) = \inf \{ t \in V / y \rightarrow z \} \) is an intuitionistic fuzzy strong implicative filter of \( A \).

\[ f^{-1}(\mu_T)(x \rightarrow z) = \mu_T(f(x \rightarrow z)) = \mu_T(f(x) \rightarrow f(z)) \geq \min \{ \mu_T(f(x \rightarrow y)), \mu_T(f(x) \rightarrow f(y \rightarrow z)) \} \]

\[ = \min \{ \mu_T(f(x \rightarrow y)), \mu_T(f(x) \rightarrow f(y \rightarrow z)) \}. \quad (3.12) \]

Thus, \( f^{-1}(\mu_T)(x \rightarrow z) \geq \min \{ f^{-1}(\mu_T)(x \rightarrow y), f^{-1}(\mu_T)(x \rightarrow (y \rightarrow z)) \} \). (3.11)

From (3.11), (3.12) and (3.13), we have \( P = (\mu_p, \gamma_p) \) is an intuitionistic fuzzy strong implicative filter of \( A \).

4. Conclusion

We have introduced the notion of an intuitionistic fuzzy strong implicative filter of lattice Wajsberg algebra. Also, we investigated some properties with interesting illustrations. We obtained the relation between an intuitionistic fuzzy implicative filter and strong implicative filter in lattice Wajsberg algebra. Finally, we establish the equivalent condition of an intuitionistic fuzzy strong implicative filter.

References


