ON COMMUTATIVITY OF PRIME NEAR-RINGS WITH GENERALIZED DERIVATIONS

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Abstract: Let $R$ be a prime near-ring. The commutativity of $R$ satisfying the conditions:

$(i) D([x, y]) = \pm x^k[x^m, y]x^l$

$(ii) D(x \circ y) = \pm x^k(x^m \circ y)x^l$

where $k \geq 0, l \geq 0, m \geq 1$, are fixed integers is studied. Further, some interesting relations between the prime graph and Zero-divisor graph of $R$ are studied.

1. Introduction

Let $R$ be a right near-ring. $R$ is called zero-symmetric if $x \circ 0 = 0 \forall x \in R$ (Recall that in a right near-ring $R$, $0.x = 0$ is true for all $x \in R$). A near ring $R$ is said to be prime if

$xRy = 0 \text{for} x, y \in R \text{implies} x = 0 \text{or} y = 0$ [Here : $xRy = \{xry \mid r \in R\}$

An endomorphism $d$ of $R$ is called a derivation if

$(i) d(x + y) = d(x) + d(y)$ and

$(ii) d(xy) = xd(y) + d(x)y \text{for all} x, y \in R$

Implies $x = 0 \text{or} y = 0$ [Here: $xRy = \{xry \mid r \in R\}$]

An endomorphism $D$ of $R$ is called a generalized endomorphism associated with a non-zero derivation $d$ of $R$, if

$(i) D(x + y) = D(x) + D(y) \text{and} (ii) D(xy) = D(x)y + yD(x) \forall x, y \in R$

Let $Z(R)$ denote the centre of $R$. For all $x, y \in R$, let $[x, y] = xy - yx$, called the commutator of $x$ and $y$ and $x \circ y = xy + yx$, called the anti-commutator of $x$ and $y$. In [11] the authors showed that a prime ring $R$ must be commutative if $R$ admits a derivation $d$ such that $d([x, y]) = [x, y]$ or $d([x, y]) = [x, y]$ or $d([x, y]) = -[x, y]$ for all $x, y \in l$ where is non-zero ideal of $R$. In [15] Yilun shang proved that a prime near-ring $R$ which admits a generalized derivation $D$ associated with a non-zero derivation $d$ satisfying either

$(i) D([x, y]) = x^k[x, y]x^l \text{for all} x, y \in R$ (or)

$(ii) D([x, y]) = -x^k[x, y]x^l \text{for all} x, y \in R$
Then $R$ is a commutative ring
He also proved that if $R$ is a prime near-ring which admits a generalized derivation $D$ associated with a non-zero derivation $d$ satisfying either

\[(i)D(x \cdot y) = x^k(x \cdot y)x^l \text{ for all } x, y \in R\]

or \[(ii)D(x \cdot y) = -x^k(x \cdot y)x^l \text{ for all } x, y \in R\]

then $R$ is a commutative ring.

In this paper we investigate the commutativity of a prime near ring $R$ satisfying the following conditions.

\[(i)D([x, y]) = \pm x^k[x^m, y]x^l \text{ for all } x, y \in R\]

\[(ii)D(x \cdot y) = \pm x^k(x \cdot y)x^l \text{ for all } x, y \in R\]

Where $k \geq 0, l \geq 0, m \geq 1$ are fixed integers.

**Lemma 1.1** [6]
Let $R$ be a prime near-ring. If $R$ admits a non-zero derivation $d$ for which $d(R) \subseteq z(R)$, then $R$ is a commutative ring.

2. **Main results**

Through out this paper $R$ denote a prime near-ring(right). $Z(R)$ denote the centre of $R$. Let $R^m = \{x^m \mid x \in R\}$

**Theorem 2.1**
Let $R$ be a prime near-ring. If there exists integers $k \geq 0, l \geq 0, m \geq 1$ such that $R$ admits a generalized derivation $D$ associated with a non-zero derivation $d$ satisfying either

\[(i)D([x, y]) = x^k[x^m, y]x^l \text{ for all } x, y \in R\]

\[(ii)D(x \cdot y) = -x^k(x \cdot y)x^l \text{ for all } x, y \in R\]

then $R$ is a commutative ring.

**Proof:** We first assume that (i) holds

\[(i)D([x, y]) = x^k[x^m, y]x^l \text{ for all } x, y \in R\] \hspace{1cm} (1)

Replace $y$ by $yx$ in (1)

\[D([x, yx]) = x^k[x^m, yx]x^l \text{ for all } x, y \in R\] \hspace{1cm} (2)

Since $[x, yx] = [x, y]x \forall x, y \in R$, (2) becomes

\[D([x, y]) = x^k[x^m, y]x^{l+1} \forall x, y \in R\] \hspace{1cm} (3)

By definition we have:

\[(i)\ x^k[x^m, y]x^{l+1} = D([x, y])x + [x, y]d(x) \quad \text{ (using (3))}

\[x^k[x^m, y]x^{l+1} = (x^k[x^m, y]x^l)x + [x, y]d(x) \quad \text{ (using (1))}

\[x^k[x^m, y]x^{l+1} = x^k[x^m, y]x^{l+1} + [x, y]d(x) \]

\[\Rightarrow x^k[x^m, y]x^{l+1} = x^k[x^m, y]x^{l+1} = [x, y]d(x) \]

\[\Rightarrow [x, y]d(x) = 0 \quad \text{ (4)}

Replacing $y$ by $zy$ we have:

\[0 = [x, zy]d(x) = [z[x, y] + [x, z]y]d(x) = [z]d(x) + [x, z]yd(x) = [x, z]yd(x) \quad \text{ (using (4))} \quad \forall x, y, z \in R\] \hspace{1cm} (5)

This implies: $[x, z]Rd(x) = 0 \quad \forall x, z \in R$ \hspace{1cm} (6)

Since $R$ is prime (6) yields that for each $x \in R$

\[d(x) = 0 \quad \text{ (or) } [x, z] = 0 \quad \forall z \in R\]

\[(ie) \text{ for each } x \in R, d(x) = 0 \quad \text{ (or) } x \in z(R) \quad \forall x \in R \] \hspace{1cm} (7)

If $x \in z(R)$, then $xy = yx$ for all $y \in R$

Then \[d(x) = d(y) \]

\[d(x)y + xd(y) = d(y)x + yd(x) \]

\[d(x)y + xd(y) = xd(y) + yd(x) \quad \therefore x \in z(R)\]
\[ d(x)y = yd(x) \text{ for all } y \in R, (x) \in z(R) \]

Thus: \( x \in z(R) \Rightarrow d(x) \in z(R) \)

So, by (7) and (8) we get that

\[ d(x) \in z(R), \forall x \in R \]

(ie) \( d(R) \subseteq z(R) \)

Then by Lemma 1.1, R is a commutative ring.

For (ii) we assumethat it holds:

\[ D([x,y]) = -x^k[x^m,y]x^l \forall x, y \in R \]

Replace \( y \) by \( yx \) in (10)

\[ D([x,yx]) = -x^k[x^m,xy]x^l \forall x, y \in R \]

Since \([x,yx] = [x,y]x \forall x, y \in R \) (11) becomes

\[ D([x,y]x) = -x^k[x^m,y]x^{l+1} \forall x, y \in R \]

By definition, we have:

\[ D([x,y]x) = D([x,y])x + [x,y]d(x) \]

\[ -x^k[x^m,y]x^{l+1} = -x^k([x^m,y]x^l)x + [x,y]d(x) \] (using (10))

\[ -x^k[x^m,y]x^{l+1} = -x^k[x^m,y]x^{l+1} + [x,y]d(x) \]

\[ -x^k[x^m,y]x^{l+1} + x^k[x^m,y]x^{l+1} = [x,y]d(x) \]

\[ [x,y]d(x) = 0 \]

\[ \forall x, y \in R \]

Replacing \( y \) by \( yx \), we have:

\[ 0 = [x,xy]d(x) = [x,y] + [x,y]d(x) \]

\[ = z[x,y]d(x) + [x,z]ydx(x) \]

\[ = [x,z]ydx(x) \text{ using (13) } x, y, z \in R \]

This implies \([x,z]Rd(x) = 0 \)

\[ \forall x, z \in R \]

Since \( R \) is prime (15) yields that for each \( x \in Rd(x) = 0 \) (or) \( x, z = 0 \)

(ie) for each \( x \in Rd(x) = 0 \) (or) \( x \in z(R) \)

\[ \forall x \in z(R) \]

Now \( x \in z(R) \) then \( d(x) \in z(R) \)

So, by (16) and (17) we get that

\[ d(x) \in z(R) \forall x \in R \]

(ie) \( d(R) \subseteq z(R) \)

Then by Lemma 1.1, R is a commutative ring.

**Note:** If \( m = 1 \), we get Theorem 1[15]

**Definition 2.2. [9]**

The prime graph of a near-ring \( R \) denoted by \( G(R) \) is a graph with vertices as the set of elements of \( R \) and edges as the set of vertex pair \( \{x, y\} \) such that \( xy = 0 \) or \( yRx = 0 \), It is easy to check that \( R \) is prime if and only if \( G(R) \) is a star graph.

**Definition 2.3 [9]**

The Zero-divisor graph of a commutative ring \( R \) is a graph with the set of non-zero zero divisors of \( R \) as the vertices and any two vertices \( x, y \) are adjacent if and only if \( d(x) \) and \( yx \neq 0 \)

**Corollary 2.4**

Let \( R \) be a prime near-ring. If the prime graph \( G(R) \) is a star and there exist \( k, l, m \in N \) such that \( R \) admits a generalized derivation \( d \) satisfying either (i) (or) (ii) of Theorem 2.1 then the zero divisor graph of \( R \) is a sub graph of \( G(R) \)

**Remark 2.5:** The condition \( R \) is prime in Theorem 2.1 is necessary even in the case of arbitrary rings as seen in the following example.

**Example 2.6.**

Let \( R \) be a Commutative ring.
Let $R^* = \left\{ \begin{pmatrix} x & y \\ z & 0 \end{pmatrix} \right\} | x, y, z \in R$. Then $R^*$ is a ring with respect to usual matrix addition and matrix multiplication.

Define $d: R^* \to R^*$ by

$$d \left( \begin{pmatrix} x & y \\ z & 0 \end{pmatrix} \right) = \left( \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \right),$$

where $y \neq 0$, then $AR^*A = 0$, which proves $R^*$ is not Prime. Define $D: R^* \to R^*$ as $D \left( \begin{pmatrix} x & y \\ z & 0 \end{pmatrix} \right) = \left( \begin{pmatrix} 0 & y + z \\ 0 & 0 \end{pmatrix} \right)$

We shall show that $D$ is a generalized derivation on $R^*$ with an associated derivation $d$ on $R^*$.

Let $A = \left( \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \right) \in R^*$, $B = \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) \in R^*$. Then $AB = \left( \begin{pmatrix} xa & xb + yc \\ 0 & zc \end{pmatrix} \right)$ and

$$D(AB) = \left( \begin{pmatrix} 0 & xb + yc + zc \\ 0 & 0 \end{pmatrix} \right)$$

Also, $D(A)B + Ad(B) = \left( \begin{pmatrix} 0 & y + z \\ 0 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) + \left( \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \right) = \left( \begin{pmatrix} 0 & yc + zc + xb \\ 0 & 0 \end{pmatrix} \right)$. Hence $D(AB) = D(A)B + Ad(B)$. Then $D([[A, B]]) = [A, B]$ for all $A, B \in R^*$. But $R^*$ is a non-commutative ring.

**Theorem 2.7**

Let $R$ be a prime near ring. If there exist integers $k \geq 0, l \geq 0, m \geq 1$, such that $R$ admits a generalized derivation $D$ associated with a non-zero derivation $d$ satisfying either

(i) $D(x \circ y) = x^k(x^{m \circ y}x^l)$ for all $x, y \in R, (or)$

(ii) $D(x \circ y) = -x^k(x^{m \circ y}x^l)$ for all $x, y \in R$,

Then $R$ is a commutative ring.

**Proof:** We first assume that (i) holds.

$$D(x \circ y) = x^k(x^{m \circ y}x^l) \forall x, y \in R \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (18)$$

Replace $y$ by $yx$ in (18)

$$D(x \circ yx) = x^k(x^{m \circ yx}x^l) \forall x, y \in R \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (19)$$

Since $(x \circ yx) = (x \circ y)x, \forall x, y \in R$, (19) becomes

$$D(x \circ y)x = x^k(x^{m \circ y}x^l) \forall x, y \in R \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (20)$$

By definition we have:

$$D(x \circ y)x = D(x \circ y)x + (x \circ y)d(x)$$

(i) $x^k(x^{m \circ y}x^l) = D(x \circ y)x + (x \circ y)d(x)$ \quad \text{(using (20))}

$x^k(x^{m \circ y}x^l) = x^k(x^{m \circ y}x^l) + (x \circ y)d(x)$ \quad \text{(using (18))}

$x^k(x^{m \circ y}x^l) = x^k(x^{m \circ y}x^l) + (x \circ y)d(x)$

$\Rightarrow x^k(x^{m \circ y}x^l) - x^k(x^{m \circ y}x^l) = (x \circ y)d(x)$

Replacing $y$ by $zy$, we have:

$$0 = (x \circ zy) d(x) = [z(x \circ y) + (x \circ z)y] d(x)$$

$$= z(x \circ y) d(x) + (x \circ z)yd(x)$$

This implies: $(x \circ z) Rd(x) = 0 \quad \forall x, z \in R \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (22)$

Since $R$ is prime(23) yields that for each $x \in R$ $d(x) = 0 \quad (or) (x \circ z) = 0 \quad \forall z \in R$

(i.e) for each $x \in d(x) = 0 \quad (or) x \in z(R)$ \quad \text{.................................(24)}

Here $z(R)$ is the centre of $R$. If $x \in z(R)$ then $xy = yx$, then $d(xy) = d(yx)$,

$$d(x)y + xd(y) = d(y)x + yd(x)$$

$\Rightarrow d(x)y + xd(y) = xd(y) + yd(x)$ \quad \forall x \in z(R)$

$$\Rightarrow d(x) = yd(x) \forall y \in R$$

$$d(x) \in z(R)$$

Thus: $x \in z(R) \Rightarrow d(x) \in z(R) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (25)$

So, by (24) and (25) we get that:

$$d(x) \in z(R) \forall x \in R \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (26)$$

(i.e) $d(R) \subset z(R)$

Then by Lemma 1.1. $R$ is a commutative ring

For (ii) we assume that it holds.
\[ D(x \cdot y) = -x^k (x^m \cdot y)x^l \forall x, y \in R \]  \hspace{1cm} (27)  \\

Replace y by \( yx \) in (27)  \\
\[ D(x \cdot y) = -x^k (x^m \cdot y)x^l \forall x, y \in R \]  \hspace{1cm} (28)  \\

Since \( (x \cdot y)x = (x \cdot y)x, \forall x, y \in R \), (28) becomes  \\
\[ D((x \cdot y)x) = -x^k (x^m \cdot y)x^{l+1} \forall x, y \in R \]  \hspace{1cm} (29)  \\

By definition we have:  \\
\[ D((x \cdot y)x) = D(x \cdot y)x + (x \cdot y)d(x) \]  \\
\[ -x^k (x^m \cdot y)x^{l+1} = -x^k (x^m \cdot y)x^l + (x \cdot y)d(x) \]  \hspace{1cm} (using (27))  \\
\[ -x^k (x^m \cdot y)x^{l+1} = -x^k (x^m \cdot y)x^l + (x \cdot y)d(x) \]  \\
\[ \Rightarrow (x \cdot y)d(x) = 0 \hspace{1cm} \forall x, y \in R \]  \hspace{1cm} (30)  \\

Replacing y by \( yz \) we have:  \\
\[ 0 = (x \cdot yz)d(x) = (z \cdot y) + (x \cdot y)y)d(x) \]  \\
\[ = z(x \cdot y)d(x) + (x \cdot z)y)d(x) \]  \hspace{1cm} (using (30))  \\
This implies \( (x \cdot z)Rd(x) = 0 \hspace{1cm} \forall x, z \in R \) \hspace{1cm} (31)  \\

Since \( R \) is prime \( (32) \) yields that for each:  \\
\[ x \in Rd(x) = 0 \hspace{1cm} (or) \hspace{1cm} (x, z) = 0 \hspace{1cm} \forall z \in R \]  \hspace{1cm} (33)  \\

Now \( x \in z(R) \) then \( d(x) \in z(R) \) \hspace{1cm} (34)  \\
So, by (33) and (34) we get:  \\
\[ d(x) \in z(R) \forall x \in R, (ie) \hspace{1cm} d(R) \subset z(R) \]  \\
Then by Lemma 1.1. \( R \) is a commutative ring.  \\

**Remark 2.8** If \( m = 1 \), we get Theorem 2\[15\]  \\

**Corollary 2.9**  
Let \( R \) be a prime near-ring. If the prime graph \( G(R) \) is a star and there exist \( k, l, m \in N \) such that \( N \) admits a generalized derivation \( D \) associated with a non-zero derivation \( d \) satisfying either (i) (or) (ii) in Theorm 2.8, then the zero divisor graph of \( R \) is sub graph of \( G(R) \).  \\

**Remark 2.10** The condition \( R \) is prime in Theorem 2.7 is necessary even in the case of arbitrary rings as seen in the following example.  \\

**Example 2.11**  
Let \( S \) be a non-commutative ring in which the square of each elements is zero.  
Let \( R = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \bigg| x, y, z \in S \right\} \). Defined: \( R \to R \) as: \( d \left( \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \right) = \left( \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \right) \)  

Then \( d \) is a derivation on \( R \). Define \( D \left( \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \right) = \left( \begin{pmatrix} 0 & y + z \\ 0 & 0 \end{pmatrix} \right) \)  

Then \( D \) is a generalized derivation with association derivation \( d \). As already stated \( R \) is not prime. For any \( x, y \in S \), we have  
\[ 0 = (x + y)^2 = x^2 + xy + yx + y^2 = xy + yx \]  
So, \( \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \cdot \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = \begin{pmatrix} 0 & xv + yw + uy + vz \\ 0 & 0 \end{pmatrix} \) for all \( x, y, z, u, v, w \in S \)  

Consequently \( D(A \cdot B) = (A \cdot B) \) for all \( A, B \in R \). But \( R \) is non-commutative ring.  

**References**  