FUZZY QUASI- IDEALS IN NEAR- SUBTRACTION SEMIGROUPS

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Abstract: Dheena discussed and derived some properties of near subtraction semigroups. The concept of fuzzy set was first initiated by Zadeh. In this paper, we introduced the notation of fuzzy quasi- ideals in near- subtraction semigroups.

Keywords: Fuzzy sub-near subtraction semigroups, fuzzy two sided X-Subalgebra, fuzzy two-sided ideal, fuzzy bi-ideal and fuzzy quasi- ideal.

1. Introduction

properties. Murugadas et al. [8] studied interval-valued Q-fuzzy ideals generated by an interval-valued Q-fuzzy subset in ordered semi-groups and investigated some of its properties.

2. Preliminaries

Definition 2.1
A nonempty set X together with two binary operation – and . is called near subtraction algebra if it satisfying the following:
(i) \( x-(y-x) = x \)
(ii) \( x-(x-y) = y-(y-x) \)
(iii) \( (x-y)-z = (x-z)-y \)

Definition 2.2
A nonempty set X together with two binary operation – and . is said to be subtraction semigroup if it satisfying the following:
(i) \( (x,-) \) is a subtraction algebra.
(ii) \( (x,.) \) is a semigroup.
(iii) \( x(y-z) = xy-xz \) and \( (x-y)z = xz-yz \) \( \forall \) \( x,y,z \in X \)

Definition 2.3
A near-subtraction semigroup X is called zero-symmetric, if \( x0 = 0 \), for all \( x \) in X.

Definition 2.4
A nonempty subset S of a subtraction algebra X is said to be a subalgebra of X, if \( x-y \in S \).

Note: Let X be a near-subtraction semigroup. Given two subsets A and B of X,
\( A\ast B = \{ab/a \in A, b \in B\} \). Also we define another operation “*”
\( A \ast B = \{ab\ast (a\ast b) / a,a \in A, b \in B\} \).

Definition 2.5
A function A from a non-empty set X to the unit interval \([0,1]\) is called a fuzzy subset of X.

Definition 2.6
A subalgebra B of X is called bi-ideal if \( BXB \cap B \ast B \subseteq B \). In case of zero symmetric, \( BXB \subseteq B \).

Notation: Let A and B be two fuzzy subsets of a semigroup X. We define the relation \( \subseteq \) between A and B, the union, intersection and product of A and B, respectively as follows:
1. \( A \subseteq B \) if A(x)\( \leq \) B(x), for all \( x \in X \),
2. \( (A \cup B) (x) = \max \{A(x), B(x)\} \), for all \( x \in X \),
3. \( (A \cap B) (x) = \min \{A(x), B(x)\} \), for all \( x \in X \).
4. \( (A \ast B) (x) = \begin{cases} \sup_{x=yz} \min \{A(y), B(z)\}, & \text{if } x = yz \text{ for all } x,y \in X \\ 0, & \text{Otherwise} \end{cases} \)
5. \( (A \ast B) (x) = \begin{cases} \inf_{x=yz} \max \{A(y), B(z)\}, & \text{if } x = yz \text{ for all } x,y \in X \\ 0, & \text{Otherwise} \end{cases} \)

Definition 2.7
A Fuzzy subalgebra A of X is called fuzzy bi-ideal of X, if \( (AXA) \cap (AX \ast A) \subseteq A \). In case of zero symmetric if AX A \( \subseteq A \).
Definition 2.8
Let \((X, -, \cdot)\) be a near subtraction semigroup. A non-empty subset \(I\) of \(X\) is called

(i). A **Left ideal** if \(I\) is a subalgebra of \((X, -)\) and \(xi - (y - i) \in I\) for every \(x, y \in X, i \in I\)

(ii). A **right ideal** if \(I\) is a subalgebra of \((X, -)\) and \(IX \subseteq I\).

(iii). An **Ideal** of \(X\) if \(I\) is both a left and right.

Definition 2.9
A Fuzzy subset \(A\) of \(X\) is called **fuzzy ideal** if it satisfying the following conditions:

(i) \(A(x - y) \geq \min\{ A(x), A(y)\}\).

(ii) \(A(xi - x(y - i)) \geq A(i)\).

(iii) \(A(xy) \geq A(x)\), for every \(x, y \in X\).

A fuzzy subset with (i) and (ii) is called a fuzzy left ideal of \(X\), Whereas a fuzzy subset with (i) and (iii)

is called a fuzzy right ideal of \(X\).

Definition 2.10
A Fuzzy subset \(A\) of \(X\) is called **fuzzy X subalgebra** if it satisfying the following

conditions:

(i) \(A\) is a fuzzy subalgebra of \((X, -)\).

(ii) \(A(xy) \geq A(x)\).

(iii) \(A(xy) \geq A(y)\), for every \(x, y \in X\).

A fuzzy subset with (i) and (ii) is called a fuzzy right \(X\)-subalgebra of \(X\), Whereas a fuzzy subset

with (i) and (iii) is called a fuzzy left \(X\)-subalgebra of \(X\).

3. Fuzzy quasi-ideals in near subtraction semigroups

Definition 3.1
A fuzzy subalgebra \(A\) of \(X\) is called a **fuzzy quasi-ideal** of \(X\) if \((A \cdot X) \cap (X \cdot A) \cap (A^* X) \subseteq A\). In case of zero symmetric of \(X\) if \((A \cdot X) \cap (X \cdot A) \subseteq A\).

Example 3.2
Let \(X = \{0, a, b, c\}\) be a near-subtraction semigroup with two binary operations ‘-’ and ‘*’ is defined as follows.

\[
\begin{array}{ccc}
- & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & 0 & 0 \\
b & b & b & 0 & 0 \\
c & c & b & a & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
\circ & 0 & A & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & A & b & a \\
b & 0 & 0 & b & b \\
c & a & A & c & c \\
\end{array}
\]

Define a fuzzy subset \(A: X \rightarrow (0, 1)\) defined by \(A(0) = 0.2, A(a) = 0.7, A(b) = 0.3\) Then
\(A \cdot X \cap (X \cdot A) \cap (A^* X) \cap (A \cdot X) \cap (X \cdot A) \cap (A^* X) = A\).

Theorem 3.3
Let \(X\) be a zero symmetric near subtraction semigroup and if fuzzy subalgebra \(A\) of \(X\) is a fuzzy quasi-ideal of \(X\) then \((A \cdot X) \cap (X \cdot A) \subseteq A\).

**Proof:** Assume that \(A\) is a fuzzy quasi-ideal of \(X\) Then \((A \cdot X) \cap (X \cdot A) \subseteq A\). Since \(A\) is a fuzzy subalgebra of \(X\), \(A(0) \geq A(x)\) for all \(x \in X\). We have:

\((A \cdot X)(0) \geq (A \cdot X)(x)\) for all \(x \in X\)

Since \(X\) is zero symmetric, \((A \cdot X) \cap (X \cdot A) \subseteq (A^* X) \subseteq A\). Therefore, \(A \cdot X \cap (X \cdot A) \subseteq A\).
**Theorem 3.4**

Every fuzzy quasi-ideal of A is a fuzzy bi-ideal of X.

**Proof:** Let A be a fuzzy quasi-ideal of X. Now,

\[ A(ab) \geq (A \circ X)(ab) \cap X \circ A(ab) \]

Therefore, \(A(ab)\) is a fuzzy bi-ideal of X.

**Note:** Every fuzzy quasi-ideal is a fuzzy bi-ideal, but the converse is not true.

**Example 3.5**

Let \(X = \{0, a, b, c\}\) be a near-subtraction semigroup, and two binary operations ‘-’ and ‘•’ are defined as follows.

\[
\begin{array}{cccc}
- & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & 0 \\
b & b & b & 0 & 0 \\
c & c & c & a & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & b & B & c \\
b & b & 0 & b & 0 \\
c & c & 0 & c & B & a \\
\end{array}
\]

Let \(X = \{0, a, b, c\}\) be a near-subtraction semigroup with two binary operations ‘-’ and ‘•’ defined as follows.

**Theorem 3.6**

Let X be a left permutable near subtraction semigroup. Then every fuzzy quasi-ideal A of X is a fuzzy strong bi-ideal of X.

**Proof:** Let A be a fuzzy quasi-ideal of X. Then by the above theorem A is a fuzzy bi-ideal of X.

\[ A(axb) \geq (A \circ X)(axb) \cap X \circ A(axb) \]

\[ = \sup_{ab=xy} \min\{A(y); X(x)\} \cap \sup_{ab=xy} \min\{X(y), A(x)\} \geq \min\{A(ax), A(b)\} \]

Therefore, \(A(axb)\) is a fuzzy bi-ideal of X.

\[ \min\{A(x), X(ab)\} \cap \min\{X(ax), A(b)\} = A(x) \cap A(b) = \min\{A(x), A(b)\} \]

**Theorem 3.7**

Let X be a regular near subtraction semigroup, and A be any fuzzy quasi-ideal of X. Then A = (A \circ X) \cap X \circ A.

**Proof:** Let X be regular near subtraction semigroup, and A be any fuzzy quasi-ideal of X. There exists a \(x \in X\) such that \(x = \sup \{A(x); X(q)\} \geq \min\{A(x), X(ax)\} = A(x)\)

\[ (A \circ X)(x) = \sup_{xp} \min\{A(p); X(q)\} \geq \min\{A(x), X(ax)\} = A(x) \]

\[ (X \circ A)(x) = \sup_{xp} \min\{X(p); A(q)\} \geq \min\{X(ax), A(x)\} = A(x) \]

\[ A \circ X \cap X \circ A \subseteq A \]

Therefore, A = A \circ X \cap X \circ A.

**Theorem 3.8**

Let A be a fuzzy near subtraction semigroup of X. Then A is a fuzzy quasi-ideal of X if and only if upper level cut \(U(A;t)\) of X is a fuzzy quasi-ideal of X, for each \(t \in [0, 1]\).

**Proof:** Assume that A is a fuzzy quasi-ideal of X and \(U(A;t)\) is a nonempty upper level subset of X. Let \(x, y \in U(A;t)\). Then \(A(x) \geq t\) and \(A(y) \geq t\). Now \(A(x-y) \geq \min\{A(x), A(y)\} \geq t\). This implies that: \(x-y \in U(A;t)\). Hence \(U(A;t)\) is a subalgebra of X.

To prove \(U(A;t)\) is a fuzzy quasi-ideal of X. Let \(a \in U(A;t) \cap X \cap U(A;t)\), then...
\[\Rightarrow a \in U(A;t) \cap X \text{ and } a \in X \cap U(A;t) \text{ then } a = xy \text{ and } a = wz, \text{ where } x, z \in X \text{ and } y, w \in U(A;t)\]

This implies that \(A(y) \geq t\) and \(A(w) \geq t\),

\[A(a) \geq (A \cap X)(a) \cap (X \cap A)(a) = \min \{\sup_{p=q} \min[A(p); X(q)]; a=p,q=xy,\min[X(p); A(q)]\}
\]

\[2 \min \{A(x), A(y)\} \geq t.\]

Thus \(a \in U(A;t)\). Hence \(U(A;t) \cap X \cap U(A;t) \subseteq U(A;t)\). Thus \(U(A;t)\) is a fuzzy quasi-ideal of \(X\).

Conversely, \(U(A;t)\) is a fuzzy quasi-ideal of \(X\). Let \(a \in X\) such that \(a \in U(A;t)\) and \((A \cap X)(a) \cap (X \cap A)(a)\) for some \(t \in (0,1)\). \((A \cap X)(a) \geq t_1\) and \((X \cap A)(a) \geq t_2\) but \(a < t_1\).

\[\Rightarrow a \in U(A;t)\] \(\text{which is contradiction. Therefore,}\)

\[A(a) \geq (A \cap X)(a) \cap (X \cap A)(a).\]

Hence \(A\) is a fuzzy quasi-ideal of \(X\).

**Theorem 3.9**

Suppose \(A\) is a fuzzy quasi-ideal of \(X\) then the set \(A_0 = \{a \in A; A(a) > 0\}\) is a quasi-ideal of \(X\).

**Proof:** To show that \(A_0\) is a quasi-ideal of \(X\). We need only to show that \(X \cap A_0 \cap A_0 \subseteq A_0\)

Let \(a \in X\) and \(a \in A_0\) implies that \(a \in X\) and \(a \in A_0\). So \(a = rx\) and \(a = ys\) for \(r, s \in X\) and \(x, y \in A_0\) then \(A(x) > 0\) and \(A(y) > 0\). Now, \(A(a) \geq (A \cap X)(a) \cap (X \cap A)(a)\). Since:

\[(A \cap X)(a) = \sup_{p=q} \min[A(p); X(q)] \geq A(y) \text{ because } a = ys\]

\[(X \cap A)(a) = \sup_{p=q} \min[X(p); A(q)] \geq A(x) \text{ because } a = rx.\]

Hence, \((A \cap X)(a) \geq A(y)\) and \((X \cap A)(a) \geq A(x)\).

\[A(a) \geq (A \cap X)(a) \cap (X \cap A)(a) \cap A(y) \cap A(x) > 0\] because \(A(x) > 0\) and \(A(y) > 0\). Therefore, \(A_0 \cap X \cap A_0 \subseteq A_0\), thus \(a \in A_0\). Hence \(A_0\) is a quasi-ideal of \(X\).

**Theorem 3.10**

Suppose \(A\) is a non empty subset of \(X\), then \(A\) is a quasi-ideal of \(X\) if and only if \(K\) and the characteristic function of \(A\) is a fuzzy quasi-ideal of \(X\).

**Proof:** Suppose \(A\) is a quasi-ideal of \(X\) and \(K\) and \(\Lambda\) is the characteristic function of \(A\). Let \(a \in X\), then \(a \in A\). Then \(a \in X \cap A\) or \(a \in A \cap X\). Thus \((X \cap K)(a) = 0\) and \((K \cap X)(a) = 0\), and so \((K \cap A)(a) = 0\).

If \(a \in K\), then \((K \cap X)(a) = 1\) and \((K \cap A)(a) = 1\). Hence \(K\) is a fuzzy quasi-ideal of \(X\).

Conversely, assume that \(K\) is a fuzzy quasi-ideal of \(X\). Let \(a \in A \cap X \cap A\). Then there exists \(y, z \in X\) and \(b, c \in A\) such that \(a = by\) and \(a = cz\). Thus:

\[K \cap X(a) = \sup_{p=q} \min[K(p); X(q)] = \min[K(a); X(a)] = \min[K(by); X(by)] \geq K(b); X(y) = 1 \cap 1 = 1\]

So \(K \cap X(a) = 1\). Similarly \((X \cap K)(a) = 1\). Since \(K(a) \geq (K \cap X)(a) \cap (X \cap K)(a)\), thus \(K(a) = 1\), which implies that \(a \in A\). Hence \(A \cap X \cap A \subseteq A\).

Therefore \(A\) is a quasi-ideal of \(X\).

**Theorem 3.11**

Let \(A\) and \(B\) be any two fuzzy quasi-ideal of \(X\) then \(A \cap B\) is also a fuzzy quasi-ideal of \(X\).

**Proof:** \((A \cap B)(x-y) = \min \{A(x-y); B(x-y)\} \geq \min \{\min \{\min \{A(x), A(y)\}, \min \{\min \{B(x), B(y)\}\}\}\}, \min \{\min \{A(x), A(y)\}, \min \{\min \{B(x), B(y)\}\}\}\}\}

\[\min \{\min \{A(x), A(y)\}, \min \{\min \{B(x), B(y)\}\}\}\} = \sup \{\sup \{A(x-y); B(x-y)\} \cap \sup \{\sup \{A(x); B(x)\} \cap \sup \{\sup \{A(y); B(y)\}\}\}\}\}

Thus \(\min \{\sup \{A(x-y); B(x-y)\} \cap \sup \{\sup \{A(x); B(x)\} \cap \sup \{\sup \{A(y); B(y)\}\}\}\}\} \leq A(x-y).

Similarly, \(\min \{\sup \{A(x-y); B(x-y)\} \cap \sup \{\sup \{A(x); B(x)\} \cap \sup \{\sup \{A(y); B(y)\}\}\}\}\} \leq B(x-y).

\[((A \cap B) \cap (X \cap (A \cap B)) \cap (A \cap B) = \min \{\min \{\min \{A \cap B\}; X \cap (A \cap B)\}\} \cap \sup \{\sup \{A \cap B\}; X \cap (A \cap B)\}\} \leq (A \cap B)\]
\[
\leq \min \{ \min \{ \sup_{x=ab} A(a), \sup_{x=ab} A(b), \sup_{x=zy=\bar{z}(c-y)} A(c) \}, \\
\min \{ \sup_{x=ab} B(a), \sup_{x=ab} B(b), \sup_{x=zy=\bar{z}(c-y)} B(c) \} \}
\]
\[
\leq \min \{ A(x), B(x) \} = (A \cap B)(x).
\]

So \((A \cap B)\) is a fuzzy quasi-ideal of \(X\).

References