ON UNIQUENESS OF L-SHARING OF DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS

NINTU MANDAL\(^1\) AND ABHIJIT SHAW

ABSTRACT. In this paper, we shall study the uniqueness problems of differential polynomials of meromorphic functions sharing 1 value. Here we prove two uniqueness theorems which extend and improve recent results of H.P. Waghmore and N.H. Sannappala [10].

1. INTRODUCTION

Throughout the article we study the uniqueness of differential polynomials of \( f \) and \( g \), where \( f \) and \( g \) are non-constant meromorphic functions in whole complex plane. Here we use the standard definitions, theorems and notations of Nevanlinna's value distribution theory (see [3]). The Nevanlinna characteristic function is denoted by \( T(r, f) \) and \( S(r, f) \) is small quantity define by \( o(T(r, f)) = S(r, f) \), as \( r \to \infty \) and \( r \notin E \) where \( E \subseteq \mathbb{R}^+ \) and measure of \( E \) is finite. The greatest common divisor of positive real number shall be denoted by \( GCD(q_1, q_2, \ldots, q_p) \) where \( q_1, q_2, \ldots, q_p \) are positive integers. Let \( a \in \mathbb{C} \setminus \{0\} \) and we say that \( f \) and \( g \) share value \( a \) CM(Counting Multiplicities) if \( f - a \) and \( g - a \) have same zeros with same multiplicities. We say that \( f \) and \( g \) share \( a \) IM (Ignoring Multiplicities) if \( f - a \) and \( g - a \) have the same zeros. Now we define, \( \Theta(a, f) = 1 - \lim_{r \to \infty} sup \frac{N(r, a; f)}{T(r, f)} \). where \( a \in \mathbb{C} \cup \{\infty\} \). \( E_l(a, f) \) is the set of all \( a \)-points of \( f \) where an \( a \)-point with multiplicities \( m \) is counted \( m \) times if

\(^{1}\)corresponding author

2010 Mathematics Subject Classification. 30D35.

Key words and phrases. Non-constant meromorphic function, l-Sharing, Uniqueness.
Let \( f \) be a nonconstant meromorphic function and \( a \in \mathbb{C} \cup \{\infty\} \), the counting function of \( a \)-points of \( f \) with multiplicities at least \( p (\in \mathbb{Z}^+) \) is denoted by \( N(r,a;f \mid \geq p) \) and \( \overline{N}(r,a;f \mid \geq p) \) is the corresponding reduced counting function. Similarly, we can define \( N(r,a;f \mid \leq p) \) and \( \overline{N}(r,a;f \mid \leq p) \).

**Definition 1.2.** [4] The counting function of \( a \)-points of \( f \), where an \( a \)-point of multiplicities \( m \) is counted \( m \) times if \( m \leq p \) and \( p \) times if \( m > p \), where \( p \) is an integer.

**Definition 1.3.** [4] Let \( f \) and \( g \) be two nonconstant meromorphic functions those share the value \( 1 \) IM. Let \( z_0 \) be a 1-point of \( f \) and \( g \) with multiplicity \( p \) and \( q \) respectively. The counting function of those 1-points of \( f \) and \( g \), where \( p > q \) is denoted by \( \overline{N}_L(r,1;f) \), and the counting function of those 1-points of \( f \) and \( g \), where \( p = q \geq k \) is denoted by \( \overline{N}_E^{(k)}(r,1;f) \) (\( k \geq 2 \) is an integer), where each point in those counting functions is counted only once. Similarly, we can define \( \overline{N}_L(r,1;g) \) and \( \overline{N}_E^{(k)}(r,1;g) \).

**Definition 1.4.** [4] Let \( f \) and \( g \) be two nonconstant meromorphic functions those share the value \( a \) IM. The reduced counting function of those \( a \)-points of \( f \) whose multiplicities differ from the multiplicities of corresponding \( a \)-points of \( g \) is denoted by \( \overline{N}_s(r,a;f,g) \). So we claim that \( \overline{N}_s(r,a;f,g) = \overline{N}_s(r,a;g,f) \) and \( \overline{N}_s(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g) \).

In 2001, Fang and Hong [2] deduce the following theorem,

**Theorem 1.1.** [2] Let \( f \) and \( g \) be two transcendental entire functions and \( n (\geq 11) \) be an integer. If \( f^n(f-1) f' \) and \( g^n(g-1) g' \) share 1 CM then \( f \equiv g \).

In 2006, Lahiri and Pal proved the following result:

**Theorem 1.2.** [6] Let \( f \) and \( g \) be two nonconstant meromorphic functions and \( n (\geq 14) \) be an integer. Let \( f^{\alpha} = f^n(f^3-1) f' \) and \( G^{\alpha} = g^n(g^3-1) g' \). If \( E_3(1, F^{\alpha}) = E_3(1, G^{\alpha}) \) then \( f \equiv g \).
In 2015, Chao Meng established the following result:

**Theorem 1.3.** [7] Let $f$ and $g$ be two non-constant meromorphic functions, $n \geq 12$ a positive integer. If $E_2(1, f^n(f^3 - 1)f') = E_2(1, g^n(g^3 - 1)g')$ and $f$ and $g$ share $\infty$ IM, then $f \equiv g$.

Let $Q(z) = \sum_{i=0}^{p} a_i z^i$, where $a_0(\neq 0), a_1, a_2, ..., a_{p-1}, a_p(\neq 0)$ are complex constants and $i \in \mathbb{Z}^+$.

We assume that $F = [f^nQ(f)]^{(k)}$ and $G = [g^nQ(g)]^{(k)}$.

In 2018, H.P. Waghamore and N.H. Sannappala [10] proved the results:

**Theorem 1.4.** [10] Let $f$ and $g$ be two non-constant meromorphic functions whose zeros and poles are multiplicities at least $m$ and $n > p + k + \frac{1}{m}(3k + 4)$, where $m, n, p$ are positive integers, and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$. If $F$ and $G$ share $(1,2)$ and $f$ and $g$ share $\infty$ IM, then one of the following two cases holds:

i) $f = tg$ for a constant $t$ such that $t^{\chi} = 1$, where $\chi = GCD(n + p, ..., n + p - i, ..., n + 1, n)$ and $a_{p-i} \neq 0$ for some $i = 0, 1, ..., p$.

ii) $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(f, g) = f^nQ(f) - g^nQ(g)$.

**Theorem 1.5.** [10] Let $f$ and $g$ be two non-constant meromorphic functions whose zeros and poles are multiplicities at least $m$ and $n > p + k + \frac{1}{m}(3k + 4)$, where $m, n, p$ are positive integers. If $f^nQ(f)f'$ and $g^nQ(g)g'$ share $(1,2)$ and $f$ and $g$ share $\infty$ IM, then one of the following two cases holds:

i) $f = tg$ for a constant $t$ such that $f^{\chi} = 1$, where $\chi = GCD(n + p, ..., n + p - i, ..., n + 1, n)$ and $a_{p-i} \neq 0$ for some $i = 0, 1, ..., p$.

ii) $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(f, g) = f^{n+1} \sum_{i=0}^{p} a_{p-i}f^{m-1} - g^{n+1} \sum_{i=0}^{p} a_{p-i}g^{m-1}$.

In their paper [10], the authors posed the open problems that:

i) $n$ can be still reduced, and

ii) $(1,2)$ sharing can be replaced by $(1,l),(l \geq 0)$ sharing?

The solution of this problems is given in the section Main results, theorem 3.1 and theorem 3.2.
2. LEMMAS

To prove our results following lemmas will be needed. Let \( F_1 \) and \( G_1 \) be two non-constant meromorphic functions defined in \( \mathbb{C} \). We denote by \( H_1 \) the function as follows:

\[
H_1 = \left( \frac{F''_1}{F'_1} - 2F'_1 \right) - \left( \frac{G''_1}{G'_1} - \frac{2G'_1}{G_1 - 1} \right).
\]

Lemma 2.1. [12] Let \( f \) be a non-constant meromorphic function, where \( a_0, a_1, a_2, ..., a_n(\neq 0) \) are complex constants and \( i \in \mathbb{Z}^+ \). Then

\[
T(r, \sum_{i=0}^{n} a_i f^i) = nT(r, f) + S(r, f).
\]

Lemma 2.2. [13] Let \( f \) be a non-constant meromorphic function, and \( p, k \in \mathbb{Z}^+ \). Then:

\[
N(r, 0; f^{(k)}) \leq k\overline{N}(r, \infty; f) + N(r, 0; f) + S(r, f).
\]

Lemma 2.3. [14] Let \( f \) be a nonconstant meromorphic function, and \( p, k \in \mathbb{Z}^+ \). Then,

\[
N(r, 0; f^{(k)|p}) \leq k\overline{N}(r, \infty; f) + N(r, 0; f|p + k) + S(r, f).
\]

Lemma 2.4. [1] Let \( F_1 \) and \( G_1 \) be two non-constant meromorphic functions sharing \((1, 2), (\infty, 0)\) and \( H_1 \neq 0 \), then,

\[
T(r, F_1) \leq N(r, 0; F_1|2) + N(r, 0; G_1|2) + \overline{N}(r, \infty; F_1) + \overline{N}(r, \infty; G_1) + \overline{N}(r, \infty; F_1; G_1) - N_1^* \left( r, 1; F_1 \right) - \overline{N}_L(r, 1; G_1) + S(r, F_1) + S(r, G_1).
\]

We can deduce same result for \( T(r, G) \).

Lemma 2.5. [11] Let \( F_1 \) and \( G_1 \) be two non-constant meromorphic functions sharing \((1, 1), (\infty, 0)\) and \( H_1 \neq 0 \), then

\[
T(r, F_1) \leq N(r, 0; F_1|2) + N(r, 0; G_1|2) + \frac{3}{2}N(r, \infty; F_1) + N(r, \infty; G_1) + N_1^*(r, \infty; F_1; G_1) + \frac{1}{2}\overline{N}(r, 0; F_1) + S(r, F_1) + S(r, G_1).
\]

We can deduce same result for \( T(r, G) \).

Lemma 2.6. [11] Let \( F_1 \) and \( G_1 \) be two non-constant meromorphic functions sharing \((1, 0), (\infty, 0)\) and \( H_1 \neq 0 \), then

\[
T(r, F_1) \leq N(r, 0; F_1|2) + N(r, 0; G_1|2) + 3N(r, \infty; F_1) + 2N(r, \infty; G_1) + N_1^*(r, \infty; F_1; G_1) + 2\overline{N}(r, 0; F_1) + \overline{N}(r, 0; G_1) + S(r, F_1) + S(r, G_1).
\]

We can deduce same result for \( T(r, G) \).

Lemma 2.7. [5] Let \( f \) and \( g \) be two non-constant meromorphic functions and \( \Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n} \) for all integers \( n \geq 3 \). Now if \( f^n(af + b) = g^n(ag + b) \) then \( f = g \), where \( a \) and \( b \) are two finite non-zero complex constants.

Lemma 2.8. [10] Let \( F_1 \) and \( G_1 \) be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least \( m \), where \( m \) is positive integer. Let \( n, k \in \mathbb{Z}^+ \) and if there exists two non-zero constants \( \alpha \) and \( \beta \) such
that $\overline{N}(r, 0; F_1) = \overline{N}(r, 0; G_1 - \alpha)$ and $\overline{N}(r, 0; G_1) = \overline{N}(r, 0; F_1 - \beta)$ then $n > p + \frac{2}{m}(k + 1)$.

**Lemma 2.9.** [9] Let $f$ and $g$ be two non-constant meromorphic functions and let $n(\geq 1), k(\geq 1)$, and $m(\geq 1)$ be integers. Then $FG \neq 1$, where $F, G$ are as define above.

**Lemma 2.10.** [8] Let $f$ and $g$ be two nonconstant meromorphic functions and $n + p \geq 6$ is a positive integer then $f^nQ(f)f'g^nQ(g)g' \neq 1$.

3. Main results

**Theorem 3.1.** Let $f$ and $g$ be two non-constant meromorphic functions whose zeros and poles are multiplicities at least $m$, where $m, n, p, k$ are positive integers, and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$. If $E_i(1, F) = E_i(1, G)$ and $f$ and $g$ share $\infty$ IM. Then for the one of the following conditions:

i) $l \geq 2$; a) $m \geq 2$ and $n > p + k + \frac{1}{m}(3k + 7)$, b) $m = 1$ and $n > p + 4k + 6$;

ii) $l = 1$; a) $m \geq 2$ and $n > \frac{3p}{2} + k + \frac{1}{m}(4k + 8)$, b) $m = 1$ and $n > \frac{3p}{2} + 5k + 7$;

iii) $l = 0$; a) $m \geq 2$ and $n > 4p + k + \frac{1}{m}(9k + 13)$, b) $m = 1$ and $n > 4p + 10k + 12$;

one of the following results hold:

i) $f = tg$ for a constant $t$ such that $t^x = 1$, where $\chi = \text{GCD}(n + p, \ldots, n + p - i, \ldots, n + 1, n)$ and $a_{p-i} \neq 0$ for some $i = 0, 1, \ldots, p$.

ii) $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(f, g) = f^nQ(f) - g^nQ(g)$.

**Proof.** First we defined two functions $F^* = \sum_{i=0}^{p} \frac{a_{p-i}(n+p-i)!}{(n-k+1+p-i)!} f^{n-k+1+p-i}$ and $G^* = \sum_{i=0}^{p} \frac{a_{p-i}(n+p-i)!}{(n-k+1+p-i)!} g^{n-k+1+p-i}$. Then from lemma 2.1 we have

\begin{equation}
T(r, F^*) = (n - k + 1 + p)T(r, f) + S(r, f).
\end{equation}
Since $(F^*)' = F$, we deduce, $m(r, \frac{1}{F}) = m(r, \frac{1}{F}) + S(r, f)$. By Nevanlinna’s first fundamental theorem, we get

\[
T(r, F^*) \leq \overline{N}(r, \infty; F) + N(r, 0; F^*) - N(r, 0; F) + S(r, f)
\]

\[
\leq T(r, F) + N(r, 0; f) + \sum_{i=0}^{p} N(r, 0; f - \mu_i) - \sum_{i=0}^{p} N(r, 0; f - \nu_i)
\]

\[
(3.2)
\]

where $\mu_i$ and $\nu_i$ ($i = 1, 2, ..., p$) are roots of algebraic equations

\[
\sum_{i=0}^{p} \frac{a_{p-i}(n+p-i)!}{(n-k+1+p-i)!} z^{p-i} = 0 \quad \text{and} \quad \sum_{i=0}^{p} a_{p-i} z^{p-i} = 0
\]

respectively. Also we use the result for $m(\geq 2), m(= 1)$

\[
(3.3)
\]

\[
\overline{N}_*(r, \infty; f; g) \leq \overline{N}(r, \infty; f),
\]

\[
(3.4)
\]

\[
\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}_*(r, \infty; f; g) \leq N(r, \infty; f) + N(r, \infty; g),
\]

respectively. As we assume that zeros and poles of $f$ and $g$ are of multiplicities at least $m(\geq 2)$, then

\[
(3.5)
\]

\[
\overline{N}(r, \infty; f) \leq \frac{1}{m} N(r, \infty; f) \leq \frac{1}{m} T(r, f),
\]

\[
(3.6)
\]

\[
\overline{N}(r, 0; f) \leq \frac{1}{m} N(r, 0; f) \leq \frac{1}{m} T(r, f),
\]

\[
(3.7)
\]

\[
\overline{N}(r, \infty; f) \leq \frac{1}{m} N(r, \infty; f) \leq \frac{1}{m} T(r, f),
\]

\[
(3.8)
\]

\[
\overline{N}(r, 0; f) \leq \frac{1}{m} N(r, 0; f) \leq \frac{1}{m} T(r, f).
\]

Now $F$ and $G$ are transcendental meromorphic functions that share $(1,l)$ and $f, g$ share $(\infty, 0)$. We discuss the following two cases separately.

Case 1. We assume that $H_1 \neq 0$. Now we study the following subcases.

Subcase 1.1 If $l \geq 2$, then, using Lemma 2.4, we obtain

\[
T(r, F) \leq N(r, 0; F|2) + N(r, 0; G|2) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G)
\]

\[
+ \overline{N}_*(r, \infty; F; G) + S(r, F) + S(r, G).
\]

\[
(3.9)
\]
Now from equation (3.2) and (3.9) we have:

\[
T(r, F^*) \leq N(r, 0; F|2) + N(r, 0; G|2) + N_*(r, \infty; F) + N(r, \infty; G) \\
+ N_*(r, \infty; F; G) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) \\
- \sum_{i=1}^{p} N(r, 0; f - \nu_i) - kN(r, \infty; f) + S(r, f) + S(r, g) \\
\leq N(r, 0; [f^nQ(f)]^{(k)}|2) + N(r, 0; [g^nQ(g)]^{(k)}|2) + N(r, \infty; f) \\
+ N(r, \infty; g) + N_*(r, \infty; f; g) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) \\
- \sum_{i=1}^{p} N(r, 0; f - \nu_i) - kN(r, \infty; f) + S(r, f) + S(r, g) \\
\leq (k + 2)N(r, 0; f) + kN(r, \infty; g) + (k + 2)N(r, 0; g) \\
+ \sum_{i=1}^{p} N(r, 0; g - \nu_i) + N(r, \infty; f) + N(r, \infty; g) + N_*(r, \infty; f; g) \\
+ N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) + S(r, f) + S(r, g).
\]

(3.10)

Subsubcase 1.1.1 If \( m \geq 2 \), then we deduce with help of inequalities (3.1),(3.3),(3.5) - (3.8) and (3.10) that,

\[
(\frac{n - k + 1 + p}{m})T(r, f) \leq \frac{k + 2}{m}T(r, f) + \frac{k}{m}T(r, g) + \frac{k + 2}{m}T(r, f) + pT(r, f) + \frac{2}{m}T(r, f) \\
+ \frac{1}{m}T(r, g) + T(r, f) + pT(r, f) + S(r, f) + S(r, g) \\
\leq [p + 1 + \frac{1}{m}(k + 4)]T(r, f) + [p + \frac{1}{m}(2k + 3)]T(r, g) \\
+ S(r, f) + S(r, g).
\]

(3.11)
Similarly we can show that:

\[
(n - k + 1 + p)T(r, g) \\
\leq \left[ p + 1 + \frac{1}{m}(k + 4) \right] T(r, g) + \left[ p + \frac{1}{m}(2k + 3) \right] T(r, f) \\
+ S(r, f) + S(r, g).
\]

(3.12)

Adding (3.11) and (3.12) we have: 

\[
(n - k + 1 + p)\left[ T(r, f) + T(r, g) \right] \leq \left[ 2p + 1 + \frac{1}{m}(3k + 7) \right] \left[ T(r, f) + T(r, g) \right] + S(r, f) + S(r, g),
\]

which implies that \( n \leq p + k + \frac{1}{m}(3k + 7) \), but \( n > p + k + \frac{1}{m}(3k + 7) \), a contradiction.

Subsubcase 1.1.2 If \( m = 1 \), then using inequalities (3.1), (3.4) and (3.10), we have:

\[
(n - k + 1 + p)T(r, f) \\
\leq (k + 2)T(r, f) + kT(r, g) + (k + 2)T(r, f) + pT(r, f) + T(r, f) \\
+ T(r, g) + T(r, f) + pT(r, f) + S(r, f) + S(r, g) \\
\leq (p + k + 4)T(r, f) + (p + 2k + 3)T(r, g) + S(r, f) + S(r, g).
\]

(3.13)

Adding (3.13) and (3.14) we can deduce that \( n \leq p + 4k + 6 \) which is contradiction as \( n > p + 4k + 6 \).

Subcase 1.2 If \( l = 1 \), then, we obtain from lemma 2.5

\[
T(r, F) \leq N(r, 0; F|2) + N(r, 0; G|2) + \frac{3}{2} \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) \\
+ \overline{N}_*(r, \infty; F; G) + \frac{1}{2} \overline{N}(r, 0; F) + S(r, F) + S(r, G).
\]

(3.15)

Now using (3.2) and (3.15) we have:

\[
T(r, F^*) \\
\leq T(r, F) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) - \sum_{i=1}^{p} N(r, 0; f - \nu_i) \\
- k \overline{N}(r, \infty; f) + S(r, f)
\]
ON UNIQUENESS OF L-SHARING OF DIFFERENTIAL... 127

\[ \begin{align*}
\leq &\ N(r, 0; F|2) + N(r, 0; G|2) + \frac{3}{2} N(r, \infty; F) + N(r, \infty; G) \\
&+ \ N_s(r, \infty; F; G) + \frac{1}{2} N(r, 0; F) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) \\
&- \sum_{i=1}^{p} N(r, 0; f - \nu_i) - kN(r, \infty; f) + S(r, f) + S(r, g) \\
\leq &\ N(r, 0; [f^nQ(f)]^{(k)}|2) + N(r, 0; [g^nQ(g)]^{(k)}|2) + \frac{3}{2} N(r, \infty; F) \\
&+ \ N(r, \infty; G) + N_s(r, \infty; F; G) + \frac{1}{2} N(r, 0; [f^nQ(f)]^{(k)}) + N(r, 0; f) \\
&+ \sum_{i=1}^{p} N(r, 0; f - \mu_i) - \sum_{i=1}^{p} N(r, 0; f - \nu_i) - kN(r, \infty; f) \\
&+ \ S(r, f) + S(r, g) \\
\leq &\ (k + 2)N(r, 0; f) + kN(r, \infty; g) + (k + 2)N(r, 0; g) \\
&+ \sum_{i=1}^{p} N(r, 0; g - \nu_i) + \frac{3}{2} N(r, \infty; f) + N(r, \infty; g) + N_s(r, \infty; f; g) \\
&+ \frac{1}{2} kN(r, \infty; f) + \frac{1}{2} (k + 1)N(r, 0; f) + \frac{1}{2} \sum_{i=1}^{p} N(r, 0; f - \nu_i) \\
&+ \ N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) + S(r, f) + S(r, g),
\end{align*} \]

(3.16)

Subsubcase 1.2.1 If \( m \geq 2 \), then we deduce with help of inequalities (3.1),(3.3),(3.5) - (3.8) and (3.16) that

\[ \begin{align*}
(n - k + 1 + p)T(r, f) \\
\leq &\ [\frac{3p}{2} + 1 + \frac{1}{2m}(4k + 10)]T(r, f) + [p + \frac{1}{m}(2k + 3)]T(r, g) \\
&+ \ S(r, f) + S(r, g).
\end{align*} \]

(3.17)

Similarly we can show,

\[ \begin{align*}
(n - k + 1 + p)T(r, g) \\
\leq &\ [\frac{3p}{2} + 1 + \frac{1}{2m}(4k + 10)]T(r, g) + [p + \frac{1}{m}(2k + 3)]T(r, f) \\
&+ \ S(r, f) + S(r, g).
\end{align*} \]

(3.18)
Adding (3.17) and (3.18), we have \((n - k + 1 + p)[T(r, f) + T(r, g)] \leq \left[ \frac{3p}{2} + 1 + \frac{1}{m}(4k + 8) \right][T(r, f) + T(r, g)] + S(r, f) + S(r, g)\), which implies that \(n \leq \frac{3p}{2} + k + \frac{1}{m}(4k + 8)\), but \(n > \frac{3p}{2} + k + \frac{1}{m}(4k + 8)\), a contradiction.

Subsubcase 1.2.2 If \(m = 1\), then using inequalities (3.1), (3.4) and (3.16), we have

\[
\begin{align*}
(n - k + 1 + p)T(r, f) & \leq (k + 2)T(r, f) + kT(r, g) + (k + 2)T(r, g) + pT(r, g) + \frac{1}{2}T(r, f) \\
& + T(r, f) + T(r, g) + \frac{1}{2}kT(r, f) + \frac{1}{2}(k + 1)T(r, f) + \frac{1}{2}pT(r, f) \\
& + T(r, f) + pT(r, f) + S(r, f) + S(r, g) \\
\end{align*}
\]

\[\leq \left(\frac{3p}{2} + 2k + 5\right)T(r, f) + (p + 2k + 3)T(r, g) + S(r, f) + S(r, g).\]  

(3.19)

Similarly we can show,

\[
\begin{align*}
(n - k + 1 + p)T(r, g) & \leq \left(\frac{3p}{2} + 2k + 5\right)T(r, g) + (p + 2k + 3)T(r, f) \\
& + S(r, f) + S(r, g). \\
\end{align*}
\]

(3.20)

Adding (3.19) and (3.20) we can deduce that \(n \leq \frac{3p}{2} + 5k + 7\) which is contradiction as \(n > \frac{3p}{2} + 5k + 7\).

Subcase 1.3 If \(l = 0\), then using lemma 2.6, we obtain

\[
\begin{align*}
T(r, F) & \leq N(r, 0; F|2) + N(r, 0; G|2) + 3\overline{N}(r, \infty; F) + 2\overline{N}(r, \infty; G) \\
& + \overline{N}_s(r, \infty; F; G) + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + S(r, F) + S(r, G). \\
\end{align*}
\]

(3.21)

Now using (3.2) and (3.21)

\[
\begin{align*}
T(r, F^*) & \leq T(r, F) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) - \sum_{i=1}^{p} N(r, 0; f - \nu_i) \\
& - k\overline{N}(r, \infty; f) + S(r, f) \\
\end{align*}
\]
ON UNIQUENESS OF L-SHARING OF DIFFERENTIAL... 129

\[ \begin{align*}
& \leq N(r, 0; F|2) + N(r, 0; G|2) + 3N(r, \infty; F) + 2N(r, \infty; G) \\
& + N_*(r, \infty; F; G) + 2N(r, 0; F) + N(r, 0; G) + N(r, 0; f) \\
& + \sum_{i=1}^{p} N(r, 0; f - \mu_i) - \sum_{i=1}^{p} N(r, 0; f - \nu_i) - kN_*(r, \infty; f) + S(r, f) + S(r, g) \\
& \leq N(r, 0; [f^n Q(f)]^{(k)}|2) + N(r, 0; [g^n Q(g)]^{(k)}|2) + 3N(r, \infty; F) \\
& + 2N(r, \infty; G) + N_*(r, \infty; F; G) + 2N(r, 0; [f^n Q(f)]^{(k)}) \\
& + N(r, 0; [g^n Q(g)]^{(k)}) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) \\
& - \sum_{i=1}^{p} N(r, 0; f - \nu_i) - kN_*(r, \infty; f) + S(r, f) + S(r, g) \\
& \leq (k + 2)N(r, 0; f) + kN_*(r, \infty; g) + (k + 2)N(r, 0; g) + \sum_{i=1}^{p} N(r, 0; g - \nu_i) \\
& + 3N(r, \infty; f) + 2N(r, \infty; g) + N_*(r, \infty; f; g) + 2kN_*(r, \infty; f) \\
& + 2(k + 1)N(r, 0; f) + 2 \sum_{i=1}^{p} N(r, 0; f - \nu_i) + kN_*(r, \infty; g) \\
& + (k + 1)N(r, 0; g) + \sum_{i=1}^{p} N(r, 0; g - \nu_i) + N(r, 0; f)
\end{align*} \]

\[(3.22) \quad + \sum_{i=1}^{p} N(r, 0; f - \mu_i) + S(r, f) + S(r, g).\]

Subsubcase 1.3.1 If \( m \geq 2 \), then we deduce with help of inequalities (3.1), (3.3), (3.5) - (3.8) and (3.22) that

\[(n - k + 1 + p)T(r, f) \leq [3p + 1 + \frac{1}{m}(5k + 8)]T(r, f) + [2p + \frac{1}{m}(4k + 5)]T(r, g) \]

\[(3.23) \quad + S(r, f) + S(r, g).\]

Similarly we can show,

\[(n - k + 1 + p)T(r, g) \leq [3p + 1 + \frac{1}{m}(5k + 8)]T(r, g) + [2p + \frac{1}{m}(4k + 5)]T(r, f) \]

\[(3.24) \quad + S(r, f) + S(r, g).\]
Adding (3.23) and (3.24) we have, \((n - k + 1 + p)[T(r, f) + T(r, g)] \leq [5p + 1 + \frac{1}{m}(9k + 13)][T(r, f) + T(r, g)] + S(r, f) + S(r, g)\). Which implies that \(n \leq 4p + k + \frac{1}{m}(9k + 13)\), but \(n > 4p + k + \frac{1}{m}(9k + 13)\), a contradiction.

Subsubcase 1.3.2 If \(m = 1\), then using inequalities (3.1), (3.4) and (3.22), we have

\[
(n - k + 1 + p)T(r, f) + 2T(r, f) + T(r, g) + 2kT(r, f) + (k + 2)T(r, f) + 2pT(r, f) + kT(r, f) + (k + 1)T(r, g) + pT(r, g) + T(r, f) + pT(r, f) + S(r, f) + S(r, g)
\]

\[
\leq (3p + 5k + 8)T(r, f) + (2p + 4k + 5)T(r, g) + S(r, f) + S(r, g).
\]

Similarly we can show,

\[
(n - k + 1 + p)T(r, g) \leq (3p + 5k + 8)T(r, f) + (2p + 4k + 5)T(r, f)
\]

Adding (3.25) and (3.26) we can deduce that \(n \leq 4p + 10k + 12\) which is contradiction as \(n > 4p + 10k + 12\).

Case 2 We assume that \(H_1 \equiv 0\). Then we can write for our functions \(F\) and \(G\),

\[
\left(\frac{F''}{F} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G} - \frac{2G'}{G-1}\right) = 0,
\]

Now after two times integration of the equation, we have

\[
\frac{1}{F-1} = \frac{C}{G-1} + D,
\]

where \(C\) and \(D\) are complex constants. Now we can say from (3.27) that \(F\) and \(G\) share 1 CM, that is \(F\) and \(G\) share 1 with weight \(l(\geq 2)\). Now we study the following subcases.

Subcase 2.1 Let \(D \neq 0\) and \(C = D\). Then from (3.27) we have

\[
\frac{1}{F-1} = \frac{DG}{G-1}.
\]

If \(D = -1\), then from (3.28), we obtain \(FG = 1\). Then by lemma 2.9, we get a contradiction. If \(D \neq -1\), we have, \(\frac{1}{G} = \frac{1}{D(F - \frac{D}{D-1})}\); and then \(\overline{N}(r, \frac{D-1}{D}; F) = \ldots\)
Now using the second fundamental theorem of Nevanlinna

\[ T(r, F) \leq \overline{N}(r, 0; F) + \overline{N}(r, D^{-1}; F) + \overline{N}(r, \infty; F) + S(r, F) \]

(3.29) \[ \leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, F) + S(r, G), \]

and using (3.3) and (3.29) we have

\[ T(r, F^*) \leq T(r, F) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) - \sum_{i=1}^{p} N(r, 0; f - \nu_i) \]

\[ - k\overline{N}(r, \infty; f) + S(r, f) \]

\[ \leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) \]

\[ - \sum_{i=1}^{p} N(r, 0; f - \nu_i) - k\overline{N}(r, \infty; f) + S(r, f) + S(r, g) \]

\[ \leq (k + 1)\overline{N}(r, 0; f) + k\overline{N}(r, \infty; g) + (k + 1)\overline{N}(r, 0; g) + \sum_{i=1}^{p} N(r, 0; g - \nu_i) \]

(3.30) \[ + \overline{N}(r, \infty; f) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) + S(r, f) + S(r, g). \]

Subsubcase 2.1.1 If \( m \geq 2 \), then we deduce with help of inequalities (3.1),(3.3),(3.5) - (3.8) and (3.30) that

\[ (n - k + 1 + p)T(r, f) \leq [p + 1 + \frac{1}{m}(k + 2)]T(r, f) \]

(3.31) \[ + [p + \frac{1}{m}(2k + 1)]T(r, g) + S(r, f) + S(r, g). \]

Similarly we can show,

\[ (n - k + 1 + p)T(r, g) \leq [p + 1 + \frac{1}{m}(k + 2)]T(r, g) \]

(3.32) \[ + [p + \frac{1}{m}(2k + 1)]T(r, f) + S(r, f) + S(r, g). \]

Adding (3.31) and (3.32) we have \((n - k + 1 + p)[T(r, f) + T(r, g)] \leq [2p + 1 + \frac{1}{m}(3k + 3)][T(r, f) + T(r, g)] + S(r, f) + S(r, g)\), Which implies that \( n \leq p + k + \frac{1}{m}(3k + 3) \), but \( n > p + k + \frac{1}{m}(3k + 7) \), a contradiction.
Subsubcase 2.1.2 If $m = 1$, then using inequalities (3.1), (3.4) and (3.32), we have:

\[
(n - k + 1 + p)T(r, f) \leq (p + k + 3)T(r, f) + (p + 2k + 1)T(r, g) + S(r, f) + S(r, g).
\]  
(3.33)

Similarly we can show,

\[
(n - k + 1 + p)T(r, g) \leq (p + k + 3)T(r, g) + (p + 2k + 1)T(r, f) + S(r, f) + S(r, g).
\]  
(3.34)

Adding (3.33) and (3.34) we can deduce that $n \leq p + 4k + 3$ which is contradiction as $n > p + 4k + 6$.

Subcase 2.2 Let $D \neq 0$ and $C \neq D$, then from (3.27), $G = \frac{C + D + 1 - F}{p + 1 - F}$. So, $N(r, C^+D + 1; F) = N(r, 0; G)$ and proceeding similarly as case 2.1, we obtain a contradiction.

Subcase 2.3 Let $D = 0$ and $C \neq 0$. Then $F = \frac{G + C - 1}{C}$ and $G = CF - (C - 1)$. If $C \neq 1$, then we have $N(r, C^{-1}; F) = N(r, 0; G)$ and $N(r, 1 - C; F) = N(r, 0; F)$ and proceeding similarly as case 2.1 we attain a contradiction. Thus $C = 1$, which implies $F = G$ i.e

\[
[f^nQ(f)]^{(k)} = [g^nQ(g)]^{(k)}.
\]  
(3.35)

Now integrating equation (3.35) we have:

\[
[f^nQ(f)]^{(k-1)} = [g^nQ(g)]^{(k-1)} + b_{k-1},
\]

where $b_{k-1}$ is constant. If $b_{k-1} \neq 0$, then by lemma 2.8, we have $n < p + \frac{1}{m}(3k + 3)$ which is contradiction for both the cases $m \geq 2$ or $m = 1$ as $n > p + k + \frac{1}{m}(3k + 7)$ or $n > p + k + 6$ respectively. Now repeating the process up to k-times we have

\[
[f^nQ(f)] = [g^nQ(g)].
\]  
(3.37)

Now if $p = 1$, then from equation (3.37) and lemma 2.7 we have $f = g$.

Suppose $p \geq 2$ and let $h = \frac{L}{g}$. If $h$ is constant then putting $f = hg$ in equation (3.37) we get

\[
\sum_{i=0}^{p} a_{p-i}g^{n+p-i}(h^{n+p-i} - 1) = 0,
\]  
(3.38)
which implies \( h^x = 1 \), where \( \chi = \text{GCD}(n + p, n + p - 1, \ldots, n + p - i, \ldots, n + 1, n) \), \( i = 0, 1, \ldots, p \). If \( h \) is not constant, then we can show that \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) = 0 \) and from (3.38), we have, \( R(f, g) = f^nQ(f) - g^nQ(g) \).

This complete the proof of theorem.

**Remark 3.1.** It is observed at the theorem 3.1 that the value of \( n \) is continuously decreasing for the increasing value of \( m \) when \( k \) is fixed, in any case according to \( l(\geq 0) \).

**Theorem 3.2.** Let \( f \) and \( g \) be two non-constant meromorphic functions whose zeros and poles are multiplicities at least \( m \), where \( m, n, p, k \) are positive integers. If \( E_l(1, f^nQ(f)f') = E_l(1, g^nQ(g)g') \) and \( f \) and \( g \) share \( \infty \) IM. Then for one of the following conditions:

i) \( l \geq 2; \) \( a) \) \( m \geq 2 \) and \( n > p + 1 + \frac{8}{m} \), \( b) \) \( m = 1 \) and \( n > p + 8 \)

ii) \( l = 1; \) \( a) \) \( m \geq 2 \) and \( n > \frac{3p}{2} + k + \frac{1}{m}(4k + 8) \), \( b) \) \( m = 1 \) and \( n > \frac{3p}{2} + 5k + 7 \)

iii) \( l = 0; \) \( a) \) \( m \geq 2 \) and \( n > 4p + k + \frac{1}{m}(9k + 13) \), \( b) \) \( m = 1 \) and \( n > 4p + 10k + 12 \)

one of the following results hold

i) \( f = tg \) for a constant \( t \) such that \( t^x = 1 \), where \( \chi = \text{GCD}(n + p, \ldots, n + p - i, \ldots, n + 1, n) \) and \( a_{p-i} \neq 0 \) for some \( i = 0, 1, \ldots, p \).

ii) \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) = 0 \), where \( R(\phi, \psi) = \phi^{n+1} \sum_{i=0}^{p} \frac{a_{p-i}}{n+p+1-i} \phi^{m-i} - \psi^{n+1} \sum_{i=0}^{p} \frac{a_{p-i}}{n+p+1-i} \psi^{m-i} \).

**Proof.** Let \( X = f^nQ(f)f' \) and \( Y = g^nQ(g)g' \). We define two functions: \( X = \sum_{i=0}^{p} \frac{a_{p-i}}{n+p+1-i} f^{n+p+1-i} \) and \( Y = \sum_{i=0}^{p} \frac{a_{p-i}}{n+p+1-i} g^{n+p+1-i} \). From lemma 2.1, we have

\[
T(r, X) = (n + p + 1)T(r, f) + S(r, f).
\]

Since \( (X')' = X \), we deduce \( m(r, \frac{1}{X}) = m(r, \frac{1}{X}) + S(r, f) \). By Nevanlinna’s first fundamental theorem, we get

\[
T(r, X) \leq N(r, \infty; X) + N(r, 0; X') - N(r, 0; X) + S(r, f)
\]

\[
\leq T(r, X) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) - \sum_{i=1}^{p} N(r, 0; f - \mu_i)
\]

\[
T(r, X) - N(r, 0; f') + S(r, f),
\]

(3.40)
where \( \mu_i \) and \( \nu_i (i = 1, 2, ..., p) \) are roots of algebraic equations
\[
\sum_{i=1}^{p} \frac{a_{p-i}}{(n+p+1-i)} z^{p-i} = 0 \quad \text{and} \quad \sum_{i=1}^{p} a_{p-i} z^{p-i} = 0
\]
respectively. Also we use the result for \( m (\geq 2), m (= 1) \)

\[
N_*(r, \infty; f; g) \leq N(r, \infty; f), \quad (3.41)
\]

\[
N(r, \infty; f) + N(r, \infty; g) + N_*(r, \infty; f; g) \leq N(r, \infty; f) + N(r, \infty; g), \quad (3.42)
\]

respectively. As we assume that zeros and poles of \( f \) and \( g \) are of multiplicities at least \( m (\geq 2) \), then

\[
N(r, \infty; f) \leq \frac{1}{m} N(r, \infty; f) \leq \frac{1}{m} T(r, f), \quad (3.43)
\]

\[
N(r, 0; f) \leq \frac{1}{m} N(r, 0; f) \leq \frac{1}{m} T(r, f), \quad (3.44)
\]

\[
N(r, \infty; f) \leq \frac{1}{m} N(r, \infty; f) \leq \frac{1}{m} T(r, f), \quad (3.45)
\]

\[
N(r, 0; f) \leq \frac{1}{m} N(r, 0; f) \leq \frac{1}{m} T(r, f), \quad (3.46)
\]

Now \( X \) and \( Y \) are transcendental meromorphic functions that share \((1, 1)\) and \( f, g \) share \((\infty, 0)\). We discuss the following two cases separately.

Case 1. We assume that \( H_1 \neq 0 \). Now we study the following subcases,

Subcase 1.1 If \( l \geq 2 \), then using lemma 2.4, we obtain

\[
T(r, X) \leq N(r, 0; X|2) + N(r, 0; Y|2) + \overline{N}(r, \infty; X) + \overline{N}(r, \infty; Y)
\]

\[
+ \overline{N}_*(r, \infty; X; Y) + S(r, X) + S(r, Y). \quad (3.47)
\]

Now from inequalities (3.40) and (3.47)and lemma 2.2 we have:

\[
T(r, X^*) \leq N(r, 0; X|2) + N(r, 0; Y|2) + \overline{N}(r, \infty; X) + \overline{N}(r, \infty; Y)
\]

\[
+ \overline{N}_*(r, \infty; X; Y) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) - \sum_{i=1}^{p} N(r, 0; f - \nu_i)
\]

\[
- N(r, 0; f') + S(r, f) + S(r, g) \]
\[ \begin{align*}
&\leq N(r, 0; |f^n Q(f) f'|) + N(r, 0; |g^n Q(g) g'|) + N(r, \infty; f) \\
&+ N(r, \infty; g) + N_*(r, \infty; f; g) + N(r, 0; f) + \sum_{i=1}^p N(r, 0; f - \mu_i) \\
&- \sum_{i=1}^p N(r, 0; f - \nu_i) - N(r, 0; f') + S(r, f) + S(r, g) \\
&\leq 2N(r, 0; f) + 2N(r, 0; g) + \sum_{i=1}^p N(r, 0; g - \nu_i) + N(r, 0; g) \\
&+ N(r, \infty; g) + N(r, \infty; f) + N_*(r, \infty; g; f) + N(r, 0; f) \\
&+ \sum_{i=1}^p N(r, 0; f - \mu_i) + S(r, f) + S(r, g). \\
\end{align*} \]

(3.48)

Subsubcase 1.1.1 If \( m \geq 2 \), then we deduce with help of inequalities (3.39), (3.41), (3.43) - (3.46) and (3.48) that

\[ \begin{align*}
(n + p + 1)T(r, f) \\
\leq \left( \frac{2}{m} + \frac{2}{m} + p + 1 \right)T(r, f) + \left( \frac{2}{m} + \frac{2}{m} + p + 1 \right)T(r, g) + S(r, f) + S(r, g) \\
\leq [p + 1 + \frac{4}{m}]T(r, f) + [p + 1 + \frac{4}{m}]T(r, g) + S(r, f) + S(r, g). \\
\end{align*} \]

(3.49)

Similarly we can show that

\[ \begin{align*}
(n + p + 1)T(r, g) &\leq [p + 1 + \frac{4}{m}]T(r, f) + [p + 1 + \frac{4}{m}]T(r, g) \\
&+ S(r, f) + S(r, g). \\
\end{align*} \]

(3.50)

Adding (3.49) and (3.50) we have \((n + p + 1)[T(r, f) + T(r, g)] \leq 2[p + 1 + \frac{4}{m}][T(r, f) + T(r, g)] + S(r, f) + S(r, g)\). Which implies that \( n \leq p + \frac{8}{m} + 1 \), but \( n > p + \frac{8}{m} + 1 \), a contradiction.

Subsubcase 1.1.2 If \( m = 1 \), then using inequalities (3.39), (3.42) and (3.48), we have

\[ \begin{align*}
(n + p + 1)T(r, f) &\leq 2T(r, f) + 2T(r, g) + pT(r, g) + T(r, g) + T(r, g) + T(r, g) \\
&+ T(r, f) + T(r, g) + pT(r, f) + S(r, f) + S(r, g) \\
&\leq (p + 1)T(r, f) + (p + 5)T(r, g) + S(r, f) + S(r, g). \\
\end{align*} \]

(3.51)

Similarly we can show,

\[ \begin{align*}
(n + p + 1)T(r, g) &\leq (p + 4)T(r, g) + (p + 5)T(r, f) + S(r, f) + S(r, g). \\
\end{align*} \]
Adding (3.51) and (3.52) we can deduce that \( n \leq p + 8 \) which is contradiction as \( n > p + 8 \).

Subcase 1.2 If \( l = 1 \), then we obtain from lemma 2.5

\[
T(r, X) \leq N(r, 0; X|2) + N(r, 0; Y|2) + \frac{3}{2}N(r, \infty; X) + N(r, \infty; Y)
\]

(3.53)

\[
+ \overline{N}_*(r, \infty; X; Y) + \frac{1}{2}N(r, 0; X) + S(r, X) + S(r, Y).
\]

Now using (3.40) and (3.53) and lemma 2.2

\[
T(r, X^*)
\]

\[
\leq T(r, X) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) - \sum_{i=1}^{p} N(r, 0; f - \nu_i)
\]

\[- N(r, 0; f') + S(r, f)
\]

\[
\leq N(r, 0; X|2) + N(r, 0; Y|2) + \frac{3}{2}N(r, \infty; X) + N(r, \infty; Y)
\]

\[
+ \overline{N}_*(r, \infty; X; Y) + \frac{1}{2}N(r, 0; X) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i)
\]

\[- \sum_{i=1}^{p} N(r, 0; f - \nu_i) - N(r, 0; f') + S(r, f) + S(r, g)
\]

\[
\leq N(r, 0; [f^nQ(f)f']|2) + N(r, 0; [g^nQ(g)g']|2) + \frac{3}{2}N(r, \infty; X)
\]

\[
+ \overline{N}(r, \infty; Y) + \overline{N}_*(r, \infty; X; Y) + \frac{1}{2}N(r, 0; [f^nQ(f)f'])
\]

\[
+ N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) - \sum_{i=1}^{p} N(r, 0; f - \nu_i)
\]

\[- N(r, 0; f') + S(r, f) + S(r, g)
\]

\[
\leq 2\overline{N}(r, 0; f) + 2\overline{N}(r, 0; g) + \sum_{i=1}^{p} N(r, 0; g - \nu_i) + N(r, 0; g) + \overline{N}(r, \infty; g)
\]

\[
+ \frac{3}{2}N(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}_*(r, \infty; f; g) + \frac{1}{2}N(r, 0; f)
\]

\[
+ \frac{1}{2} \sum_{i=1}^{p} N(r, 0; f - \nu_i) + \frac{1}{2}N(r, 0; f) + \frac{1}{2}N(r, \infty; f) + N(r, 0; f)
\]
(3.54) \[ + \sum_{i=1}^{p} N(r, 0; f - \mu_i) + S(r, f) + S(r, g). \]

Subsubcase 1.2.1 If \( m \geq 2 \), then we deduce with help of inequalities (3.39), (3.41), (3.43) - (3.46) and (3.54) that

\[ (n + p + 1)T(r, f) \leq \left[ \frac{3p}{2} + 1 + \frac{11}{2m} + \frac{3}{2} \right] T(r, f) \]

\[ + [p + \frac{4}{m} + 1] T(r, g) + S(r, f) + S(r, g). \]

Subsubcase 1.2.2 If \( m = 1 \), then using inequalities (3.39), (3.42) and (3.54),

\[ (n + p + 1)T(r, g) \leq \left[ \frac{3p}{2} + 1 + \frac{11}{2m} + \frac{3}{2} \right] T(r, g) \]

\[ + [p + \frac{4}{m} + 1] T(r, f) + S(r, f) + S(r, g). \]

Adding (3.55) and (3.56) we have \((n + p + 1)[T(r, f) + T(r, g)] \leq \frac{3p}{2} + \frac{19}{2m} + \frac{5}{2} [T(r, f) + T(r, g)] + S(r, f) + S(r, g).\) Which implies that \( n \leq \frac{3p}{2} + \frac{19}{2m} + \frac{3}{2} \), but \( n > \frac{3p}{2} + \frac{19}{2m} + \frac{3}{2} \), a contradiction.

Subcase 1.3 If \( l = 0 \), then using lemma 2.6, we obtain

\[ T(r, X) \leq N(r, 0; X|2) + N(r, 0; Y|2) + 3N(r, \infty; X) + 2N(r, \infty; Y) \]

\[ + N_{\ast}(r, \infty; X; Y) + 2N(r, 0; X) + N(r, 0; Y) + S(r, X) + S(r, Y). \]

Now using (3.40), (3.59) and lemma 2.2

\[ T(r, X^*) \]

\[ \leq T(r, X) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) - \sum_{i=1}^{p} N(r, 0; f - \nu_i) \]

\[ - N(r, 0; f') + S(r, f) \]
\[
\begin{align*}
&\leq N(r, 0; X[2] + N(r, 0; Y[2] + 3N(r, \infty; X) + 2N(r, \infty; Y) \\
&+ N_*(r, \infty; X; Y) + 2N(r, 0; X) + N(r, 0; f)) \\
&+ \sum_{i=1}^{p} N(r, 0; f - \mu_i) - \sum_{i=1}^{p} N(r, 0; f - \nu_i) - N(r, 0; f') \\
&+ S(r, f) + S(r, g) \\
&\leq N(r, 0; [f^nQ(f)f'][2] + N(r, 0; [g^nQ(g)g'][2] + 3N(r, \infty; X) \\
&+ 2N(r, \infty; Y) + N_*(r, \infty; X; Y) + 2N(r, 0; [f^nQ(f)f']) \\
&+ N(r, 0; [g^nQ(g)g']) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) \\
&- \sum_{i=1}^{p} N(r, 0; f - \nu_i) - N(r, 0; f') + S(r, f) + S(r, g) \\
&\leq 2N(r, 0; f) + 2N(r, 0; g) + \sum_{i=1}^{p} N(r, 0; g - \nu_i) + N(r, 0; g) \\
&+ N(r, \infty; g) + 3N(r, \infty; f) + 2N(r, \infty; g) + N_*(r, \infty; f; g) \\
&+ 2N(r, 0; f) + 2\sum_{i=1}^{p} N(r, 0; f - \nu_i) + 2N(r, 0; f) + 2N(r, \infty; f) \\
&+ N(r, 0; g) + \sum_{i=1}^{p} N(r, 0; f - \nu_i) + N(r, 0; g) \\
&+ N(r, \infty; g) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) + S(r, f) + S(r, g).
\end{align*}
\]

(3.60)

Subsubcase 1.3.1 If \(m \geq 2\), then we deduce with help of inequalities (3.39),(3.41),(3.43) - (3.46) and (3.60) that

\[
(n + p + 1)T(r, f) \leq [3p + \frac{10}{m} + 3]T(r, f) + [2p + \frac{7}{m} + 2]T(r, g) + + S(r, f) + S(r, g).
\]

(3.61)

Similarly we can show,

\[
(n + p + 1)T(r, g) \leq [3p + \frac{10}{m} + 3]T(r, g) + [2p + \frac{7}{m} + 2]T(r, f) + + S(r, f) + S(r, g).
\]

(3.62)
Adding (3.61) and (3.62) we have \((n + p + 1)[T(r, f) + T(r, g)] \leq [5p + \frac{17}{m} + 5][T(r, f) + T(r, g)] + S(r, f) + S(r, g)\). Which implies that \(n \leq 4p + \frac{17}{m} + 4\), but \(n > 4p + \frac{17}{m} + 4\), a contradiction.

Subsubcase 1.3.2 If \(m = 1\), then using inequalities (3.39),(3.42) and (3.60), we have

\[(n + p + 1)T(r, f)\]
\[\leq 2T(r, f) + 2T(r, g) + pT(r, g) + T(r, g) + T(r, g) + 2T(r, f) + T(r, g)\]
\[+ T(r, f) + T(r, g) + 2T(r, f) + 2pT(r, f) + 2T(r, f) + 2T(r, f)\]
\[+ T(r, g) + pT(r, g) + T(r, g) + T(r, g) + T(r, f) + pT(r, f)\]
\[+ S(r, f) + S(r, g)\]
(3.63) \[\leq (3p + 12)T(r, f) + (2p + 9)T(r, g) + S(r, f) + S(r, g)\].

Similarly we can show,

(3.64) \[(n + p + 1)T(r, g) \leq (3p + 12)T(r, g) + (2p + 9)T(r, f) + S(r, f) + S(r, g)\].

Adding (3.63) and (3.64) we can deduce that \(n \leq 4p + 20\) which is contradiction as \(n > 4p + 20\).

Case 2 We assume that \(H_1 \equiv 0\). Then we can write for our functions \(X\) and \(Y\),
\[(\frac{X''}{X'} - \frac{2X''}{X-1}) - (\frac{Y''}{Y'} - \frac{2Y''}{Y-1}) = 0,\]
Now after two times integration of the equation, we have

(3.65) \[\frac{1}{X-1} = \frac{C}{Y-1} + D,\]

Where \(C\) and \(D\) are complex constants. Now we can say from (3.65) that \(X\) and \(Y\) share 1 CM, that is \(X\) and \(Y\) share 1 with weight \(l(\geq 2)\). Now we study the following subcases.

Subcase 2.1 Let \(D \neq 0\) and \(C = D\). Then from (3.65) we have

(3.66) \[\frac{1}{X-1} = \frac{DY}{Y-1}.\]

If \(D = -1\), then from (3.66), we obtain \(XY = 1\). Then by lemma 2.10, we get a contradiction. If \(D \neq -1\), we have \(\frac{1}{Y} = \frac{1}{D(\frac{D-1}{D^2})}\) and then \(N(r, \frac{D-1}{D}; X) = \overline{N}(r, 0; Y)\). Using the second fundamental theorem of Nevanlinna

\[T(r, X) \leq \overline{N}(r, 0; X) + \overline{N}(r, \frac{D-1}{D}; X) + \overline{N}(r, \infty; X) + S(r, X)\]
\[\leq \overline{N}(r, 0; X) + \overline{N}(r, 0; Y) + \overline{N}(r, \infty; X) + S(r, X) + S(r, Y).\]
(3.67)
Using (3.40) and (3.67) we have
\[
T(r, X^*) \\
\leq T(r, X) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) - \sum_{i=1}^{p} N(r, 0; f - \nu_i) \\
- N(r, 0; f') + S(r, f) \\
\leq \overline{N}(r, 0; X) + \overline{N}(r, 0; Y) + \overline{N}(r, \infty; X) + N(r, 0; f) + \sum_{i=1}^{p} N(r, 0; f - \mu_i) \\
- \sum_{i=1}^{p} N(r, 0; f - \nu_i) - N(r, 0; f') + S(r, f) + S(r, g) \\
\leq \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \sum_{i=1}^{p} N(r, 0; g - \nu_i) + N(r, 0; g') + N(r, 0; f) \\
+ \sum_{i=1}^{p} N(r, 0; f - \mu_i) + S(r, f) + S(r, g).
\]
(3.68)

Subsubcase 2.1.1 If \(m \geq 2\), then we deduce with help of inequalities (3.39),(3.41),(3.43) - (3.46) and (3.68) that
\[
(n + p + 1)T(r, f) \\
\leq [p + \frac{1}{m} + 1]T(r, f) + [p + \frac{2}{m} + 1]T(r, g) + S(r, f) + S(r, g).
\]
(3.69)

Similarly we can show,
\[
(n + p + 1)T(r, g) \\
\leq [p + \frac{1}{m} + 1]T(r, g) + [p + \frac{2}{m} + 1]T(r, f) + S(r, f) + S(r, g).
\]
(3.70)

Adding (3.69) and (3.70) we have \((n + p + 1)[T(r, f) + T(r, g)] \leq [2p + \frac{3}{m} + 2][T(r, f) + T(r, g)] + S(r, f) + S(r, g). Which implies that \(n \leq p + \frac{3}{m} + 1\), but \(n > p + \frac{8}{m} + 1\), a contradiction.

Subsubcase 1.3.2 If \(m = 1\), then using inequalities (3.39),(3.42) and (3.68), we have
\[
(n + p + 1)T(r, f) \\
\leq (p + 2)T(r, f) + (p + 3)T(r, g) + S(r, f) + S(r, g).
\]
(3.71)
Similarly we can show,

\[(n + p + 1)T(r, g)\]

(3.72) \[\leq (p + 2)T(r, g) + (p + 3)T(r, f) + S(r, f) + S(r, g).\]

Adding (3.71) and (3.72) we can deduce that \[n \leq p + 4\] which is contradiction as \[n > p + 8\].

**Subcase 2.2** Let \( D \neq 0 \) and \( C \neq D \), then from (3.65), \( Y = \frac{C + D}{C - D} \). So,

\[N(r, \frac{C + D}{C - D}; X) = N(r, 0; Y)\]

and proceeding similarly as case 2.1, we attain a contradiction.

**Subcase 2.3** Let \( D = 0 \) and \( C \neq 0 \). Then \( X = Y + C^{-1} \) and \( Y = CX - (C - 1) \). If \( C \neq 1 \), then we have \[N(r, \frac{C + 1}{C - 1}; X) = N(r, 0; Y)\] and \[N(r, 1 - C; Y) = N(r, 0; X)\]

and proceeding similarly as subcase 2.1, we attain a contradiction. Thus \( C = 1 \), which implies \( X = Y \) i.e \([f^nQ(f)f'] = [g^nQ(g)g']\). Now we can write \( X^* = Y^* + c \), where \( c \) is constant, then it follows

\[(3.73)\]

\[T(r, f) = T(r, g) + S(r, g).\]

Suppose that \( c \neq 0 \). By the second fundamental theorem and lemma 2.10 we have

\[T(r, Y^*) = N(r, 0; Y^*) + N(r, 0; Y^* + c) + N(r, \infty; Y^*) + S(r, g)\]

\[\leq N(r, 0; Y^*) + N(r, 0; X^*) + N(r, \infty; Y^*) + S(r, g)\]

\[\leq N(r, 0; g) + N(r, 0; \sum_{i=0}^{p} \frac{a_{p-i}(n + p + 1)}{a_p(n + p + 1 - i)}g^{p-i}) + N(r, 0; f)\]

\[+ \sum_{i=0}^{p} \frac{a_{p-i}(n + p + 1)}{a_p(n + p + 1 - i)}f^{p-i}) + N(r, \infty; g)\]

\[(3.74)\]

\[+ S(r, f) + S(r, g).\]

**Subsubcase 2.3.1** If \( m \geq 2 \), then using (3.39), (3.43)-(3.46) and (3.74) we have

\[(3.75)\]

\[(n + p + 1)T(r, g) \leq (p + \frac{2}{m})T(r, g) + (p + \frac{1}{m})T(r, f) + S(r, f) + S(r, g).\]

Similarly we can show

\[(3.76)\]

\[(n + p + 1)T(r, f) \leq (p + \frac{2}{m})T(r, f) + (p + \frac{1}{m})T(r, g) + S(r, f) + S(r, g).\]
Adding (3.75) and (3.76) we have \((n + p + 1)(T(r, f) + T(r, g)) \leq (2p + \frac{3}{m})(T(r, f) + T(r, g)) S(r, f) + S(r, g)\). Which implies that \(n \leq p + \frac{3}{m} - 1\) which is contradiction as \(n > p + \frac{8}{m} + 1\).

Subsubcase 2.3.2 If \(m = 1\) then using (3.39) and (3.74) we can show \(n \leq p + 2\) which is contradiction as \(n > p + 8\). That is for all \(m\), we arrive at a contradiction. Now we claim that \(c = 0\). Therefore \(X^* = Y^*\), that is

\[
\sum_{i=0}^{p} a_{n+p+1-i} f^{p-i} = \sum_{i=0}^{p} a_{n+p+1-i} g^{p-i}.
\]

Let \(h = \frac{f}{g}\). If \(h\) is constant, then, substituting \(f = gh\) into (3.77), we deduce,

\[
\sum_{i=0}^{p} a_{p-i} h^{n+p+1-i} - 1 = 0,
\]

which implies \(h^\chi = 1\), where \(\chi = (n + p + 1, n + p, n + p - 1, ..., n + p - i, ..., n + 1)\) and \(a_{p-i} \neq 0\) for some \(i = 0, 1, ..., p\). Thus \(f = tg\), for a constant \(t\), such that \(t^\chi = 1\), where \(\chi\) is previously defined. If \(h\) is not constant, then by (3.78) \(f\) and \(g\) satisfy the algebraic equation \(R(f, g) = 0\), where,

\[
R(\phi, \psi) = \sum_{i=0}^{p} \frac{a_{p-i}\phi^{m-i}}{n+p+1-i} - \sum_{i=0}^{p} \frac{a_{p-i}\psi^{m-i}}{n+p+1-i}.
\]

This complete the proof of the theorem. \(\square\)

**Remark 3.2.** Let \(Q(f) = f^3 - 1\) and \(m = 1\) in theorem 3.2, then it will reduce to theorem 1.4.

**Remark 3.3.** In the theorem 3.2 for every case according to \(l(\geq 0)\), the value of \(n\) is continuously decreasing for the increasing value of \(m\).

**ACKNOWLEDGMENT**

The authors are grateful to referees for their careful reading and effective suggestions.

**REFERENCES**


ON UNIQUENESS OF L-SHARING OF DIFFERENTIAL...


DEPARTMENT OF MATHEMATICS
CHANDERNAGORE COLLEGE
CHANDERNAGORE, HOOGHLY, WEST BENGAL, INDIA-712136.
E-mail address: nintu311209@gmail.com

DEPARTMENT OF MATHEMATICS
BALAGARH HIGH SCHOOL
BALAGARH, HOOGHLY, WEST BENGAL, INDIA-712501.
E-mail address: ashaw2912@gmail.com