VALUED FIELD GRAPH AND SOME RELATED PARAMETERS

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ABSTRACT. Let $K$ be any finite field. For any prime $p$, the $p$-adic valuation map is given by $\psi_p : K/\{0\} \to \mathbb{R}^+ \cup \{0\}$ is given by $\psi_p(r) = n$ where $r = p^n \frac{a}{b}$, where $p, a, b$ are relatively prime. The field $K$ together with a valuation is called valued field. Also, any field $K$ has the trivial valuation determined by $\psi(K) = \{0, 1\}$. Throughout the paper $K$ represents $\mathbb{Z}_q$. In this paper, we construct the graph corresponding to the valuation map called the valued field graph, denoted by $VFG_p(\mathbb{Z}_q)$ whose vertex set is $\{v_0, v_1, v_2, \ldots, v_{q-1}\}$ where two vertices $v_i$ and $v_j$ are adjacent if $\psi_p(i) = j$ or $\psi_p(j) = i$. Here, we tried to characterize the valued field graph in $\mathbb{Z}_q$. Also we analyse various graph theoretical parameters such as diameter, independence number etc.

1. INTRODUCTION

The theory of valuations propounded in 1912 by the Hungarian Mathematician Josepf Kurschak, also see [1–4]. It plays a fundamental role in the study of algebraic function field. In this paper, using the concept of p-adic valuations in a finite field, we construct a graph and study its characteristics.

Throughout the paper, $\beta_0$ represents independence number, $\beta_1$ represents the matching number and $\chi$ represents chromatic number of a graph.

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2. Preliminaries and Basic Results

**Definition 2.1.** Let \( K \) be a field. A mapping \( \psi : K \to R^+ \cup \{0\} \) is called a valuation of \( K \) if it satisfies the following properties:

1. \( \psi(x) = 0 \iff x = 0 \)
2. \( \psi(xy) = \psi(x)\psi(y) \forall x, y \in K \)
3. \( \psi(x + y) \leq \psi(x) + \psi(y) \forall x, y \in K \)

where \( R^+ \) is the set of all non-negative real numbers.

**Definition 2.2.** Let \( \psi : K \to R^+ \cup \{0\} \) be a valuation map defined by \( \psi(K) = \{0, 1\} \), then \( \psi \) is said to be a trivial valuation.

**Definition 2.3.** The field \( K \) together with a valuation is called valued field. That is, let \( K \) be a field and \( \psi \) be a valuation, then the pair \( (K, \psi) \) is called a valued field.

**Definition 2.4.** For every prime \( p \) and a non-zero element \( x \), \( \psi_p(x) \) is the exponent of \( p \) in the factorization of \( x \) into the product of prime powers, then \( \psi_p(x) \) is called the \( p \)-adic valuation of a finite field \( K \).

**Definition 2.5.** A graph \( G \) is a pair \( G = (V, E) \) consisting of a finite set \( V \) and a set \( E \) of 2-element subsets of \( V \). The set \( V \) is called the vertex set of \( G \) and \( E \) as the edge set of \( G \). If \( \{u, v\} \) is a member of \( E \) then we say \( u \) and \( v \) are adjacent and this edge is denoted as \( uv \). Two edges are said to be adjacent if they have a common vertex. If the edge set is empty it is called null (empty or void) graph.

**Definition 2.6.** The distance between two vertices \( u \) and \( v \) is the length of a shortest path joining them, and is denoted by \( d(u, v) \). The diameter of a graph \( G \), denoted by \( d(G) \) is defined by \( d(G) = \max_{u, v \in V(G)} d(u, v) \).

**Definition 2.7.** Let \( G \) be a simple graph. \( M \subseteq E(G) \) is said to be a matching in \( G \) if no two elements of \( M \) are adjacent in \( G \). \( M \) is said to be a maximum matching if there exists no matching \( M' \) of \( G \) with \( |M'| > |M| \). Number of edges in a maximum matching is known as matching number or edge independent number which is denoted by \( \beta_1 \).

**Definition 2.8.** A set \( S \subseteq V(G) \) is an independent set of \( G \) if no two vertices of \( S \) are adjacent in \( G \). An independent set is maximum, if \( G \) has no independent set \( S' \) with \( |S'| > |S| \). The number of vertices in a maximum independent set of \( G \) is called an independent number of \( G \) and is denoted by \( \beta_0 \).
Definition 2.9. Let $G$ be a loopless graph. A $k$-colouring of $G$ is an assignment of $k$ colours to the vertices of $G$ in such a way that adjacent vertices are received different colours. If $G$ has a $k$-colouring, then $G$ is said to be $k$-colourable. The chromatic number $\chi(G)$ of a graph $G$ is the smallest number $k$ for which $G$ is $k$-colourable.

3. Main Results

Definition 3.1. Let $\mathbb{Z}_q$ be a finite field. We define a new graph associated with a $p$-adic valuation $\psi_p$ of a field denoted by $VFG_p(\mathbb{Z}_q)$ where $p$ and $q$ are prime numbers, is called valued field graph. The vertices of $VFG_p(\mathbb{Z}_q)$ are the elements of $\mathbb{Z}_q$. i.e, $v_i$ denotes the vertex corresponds to the element $i \in \mathbb{Z}_q$. where two vertices $v_i$ and $v_j$ are adjacent if $\psi_p(i) = j$ or $\psi_p(j) = i$.

Example 1. Consider the field $(\mathbb{Z}_5, +, \cdot)$. Let $\psi_2$ be a valuation from $\mathbb{Z}_5/\{0\} \rightarrow R^+ \cup \{0\}$. Since $1 = 2^0, 1$, $\psi_2(1) = 0$. Therefore the vertices $v_0$ and $v_1$ are adjacent in $VFG_2(\mathbb{Z}_5)$. Also $2 = 2^1, 1$, $\psi_2(2) = 1$. Therefore, the vertices $v_1$ and $v_2$ are adjacent. Similarly the other pair of vertices and the corresponding graph $VFG_2(\mathbb{Z}_5)$ is shown on Figure 1:

![Figure 1](image)

Example 2. Consider the field $(\mathbb{Z}_{19}, +, \cdot)$. Let $\psi_p$ be a valuation from $\mathbb{Z}_{19} \rightarrow R^+ \cup \{0\}$. The two vertices $v_i$ and $v_j$ are adjacent in $VFG_p(\mathbb{Z}_{19})$ if $\psi_p(i) = j$. The graph $VFG_p(\mathbb{Z}_{19})$ is shown on Figure 2 for a prime $p = 3$:

![Figure 2](image)

Observation 1. (1) The only complete valued field graph is $VFG_p(\mathbb{Z}_2)$
(2) For $p < q$, $VFG_p(\mathbb{Z}_q)$ is a tree.
(3) For $p = q$, $VFG_p(\mathbb{Z}_q)$ is a star graph.
(4) For any prime $p$, $VFG_p(\mathbb{Z}_3)$ is the star graph $K_{1,2}$.
Theorem 3.1. Let \( p \) be a fixed prime. Then for any two primes \( q_1 \) and \( q_2 \) such that \( q_1 < q_2 \), \( VFG_p(\mathbb{Z}_{q_2}) \) contains a subgraph isomorphic to \( VFG_p(\mathbb{Z}_{q_1}) \).

Definition 3.2. The graph which corresponds to the trivial valuation in which all the vertices are adjacent to either \( v_0 \) or \( v_1 \) is called trivial valued field graph. Otherwise, it is called non-trivial valued field graph.

Example 3. \( VFG_5(\mathbb{Z}_{11}) \), is a trivial valued field graph.

Theorem 3.2. In \( VFG_p(\mathbb{Z}_q) \), all graphs are non-trivial valued field graphs if and only if \( p \leq \lfloor \sqrt{q} \rfloor \), the greatest integer function of \( \sqrt{q} \).

Proof. For \( p \leq \lfloor \sqrt{q} \rfloor \), suppose on the contrary let us assume that \( VFG_p(\mathbb{Z}_q) \) is a trivial valued field graphs. Therefore all vertices except \( v_0 \) and \( v_1 \) are adjacent to only \( v_0 \) and \( v_1 \). And the maximum level of a trivial valued field graph is
2. But for $p \leq \lfloor \sqrt{q} \rfloor$, if level of a graph is 2, then its branches increase. For example, the vertex $v_2$ is adjacent to $v_{p^2}$, $v_3$ is adjacent to $v_{p^3}$ etc. If level of a graph is greater than 2, then clearly the vertex $v_i$ is adjacent to $v_j$ for $i, j \neq 0$ and $1$. Hence $p \leq \lfloor \sqrt{q} \rfloor$, all the valued field graphs are non trivial valued field graph.

**Corollary 3.1.** The total number of non-trivial valued field graph in $VFG_p(Z_q)$ is $\phi(\lfloor \sqrt{q} \rfloor)$, where $\phi$ represents the Euler-Totient function.

**Proposition 3.1.** For $p < q$ and $q > 3$, the centre of a trivial valued field graph is $\{v_0, v_1\}$ with diameter 3 and radius 2.

**Proposition 3.2.** $\chi(VFG_p(Z_q)) = 2$

**Proof.** Since $VFG_p(Z_q)$ is a tree for all values of $p$ and $q$, and the chromatic number of a tree is 2, the proof follows immediately. □

**Proposition 3.3.** In a trivial valued field graph

$$\beta_0(VFG_p(Z_q)) = \begin{cases} 
q - 2 & \text{for } p < q, \\
q - 1 & \text{for } p = q.
\end{cases}$$

**Proof.** For $p = q$, $VFG_p(Z_q)$ is a star graph. The independence number of a star graph with $q$ vertices is $q - 1$.

From the definition of a trivial valued field graph, all the vertices except $v_0$ and $v_1$ are adjacent to either $v_0$ or $v_1$. Therefore, excluding the vertices $v_0$ and $v_1$, the collection of other vertices forms maximum independent set. Hence, in trivial valued field graph with $q$ vertices, $q - 2$ vertices form a maximum independent set. Therefore, independence number of trivial valued field graph is $q - 2$. □

**Proposition 3.4.** In a trivial valued field graph

$$\beta_1(VFG_p(Z_q)) = \begin{cases} 
2 & \text{for } p < q, \\
1 & \text{for } p = q.
\end{cases}$$

**Proof.** For $p = q$, $VFG_p(Z_q)$ is a star graph. Hence the matching number is 1.

By Proposition 3.1 for $p < q$, the diameter of trivial valued field graph is 3. Hence the matching number of trivial valued field graph is 2. □
Proposition 3.5. For any prime \( p \), the vertices in \( VFG_p(\mathbb{Z}_q) \) are parent vertices if and only if the elements corresponding to those vertices belong to the range set of \( \psi_p(\mathbb{Z}_q/\{0\}) \).

Proof. Case 1. Suppose \( VFG_p(\mathbb{Z}_q) \) is a trivial valued field graph. Then all the vertices are adjacent to \( v_0 \) or \( v_1 \) only. Hence all other vertices are pendant vertices. And by Theorem 3.2, we know that the total number of levels of a trivial valued field graph is 2. Since all vertices except \( v_0 \) and \( v_1 \) are pendant, the only parent vertices in \( VFG_p(\mathbb{Z}_q) \) are \( v_0 \) and \( v_1 \) where \( v_0 \) and \( v_1 \) correspond to the elements 0 and 1 in \( \psi_p(\mathbb{Z}_q/\{0\}) \).

Case 2. Suppose \( VFG_p(\mathbb{Z}_q) \) is a non-trivial valued field graph. Assume that for some \( j \), \( v_j \) is a parent vertex. Then by definition 3.2, we have for \( i, j \neq 0, 1 \), some vertex \( v_i \) is adjacent to \( v_j \). This is possible only when \( \psi_p(i) = j \), i.e., \( j \in \psi_p(\mathbb{Z}_q/\{0\}) \), where \( v_j \) is the vertex corresponding to the element \( j \) in \( \psi_p(\mathbb{Z}_q/\{0\}) \). Hence the proof. \( \square \)

Theorem 3.3. If two fields are isomorphic then the corresponding valued field graphs are isomorphic. But the converse need not be true.

Proof. Suppose \( K_1 \) and \( K_2 \) are two fields which are isomorphic. Then they are structurally same. Therefore the corresponding valued field graphs of these fields are same. Hence \( VFG_p(K_1) \cong VFG_p(K_2) \)

Conversely, consider two valued field graphs are isomorphic. For example, we have \( VFG_p(\mathbb{Z}_q) \cong VFG_p(\mathbb{Z}_q^*) \), where \( \mathbb{Z}_q^* \) is the multiplicative group of \( \mathbb{Z}_q \). Since the domain of the valuation map from \( \mathbb{Z}_q \) and \( \mathbb{Z}_q^* \) are same, the corresponding vertices in \( VFG_p(\mathbb{Z}_q) \) and \( VFG_p(\mathbb{Z}_q^*) \) are same. Hence the two vertices in \( VFG_p(\mathbb{Z}_q) \) is also adjacent in \( VFG_p(\mathbb{Z}_q^*) \). That is if \( v_0 \) and \( v_1 \) are adjacent vertices in \( VFG_p(\mathbb{Z}_q) \) then they are also adjacent in \( VFG_p(\mathbb{Z}_q^*) \), and the degree of vertices in both the graphs are same. Therefore, \( VFG_p(\mathbb{Z}_q) \cong VFG_p(\mathbb{Z}_q^*) \).

But we have, \( \mathbb{Z}_q \) is not isomorphic to \( \mathbb{Z}_q^* \).

Hence the valued field graphs of two fields are isomorphic does not imply the fields are isomorphic. \( \square \)

References


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