PRIME IDEALS OF $M\Gamma$-GROUPS

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ABSTRACT. In this paper we consider the algebraic system $M\Gamma$-group, a generalization of the concept module over a nearring. We define prime ideal of $M\Gamma$-group and obtain some equivalent conditions for a prime ideal of an $M\Gamma$-group. Some related fundamental results and examples are also presented.

1. INTRODUCTION

In this section we provide elementary definition and examples from Satyanarayana [11,13] for the sake of completeness.

Let $(M, +)$ be a group (not necessarily Abelian) and $\Gamma$ a non-empty set. Then $M$ is said to be a $\Gamma$-nearring if there exists a mapping $M \times \Gamma \times M \rightarrow M$ (denote the image of $(m_1, \alpha_1, m_2)$ by $m_1\alpha_1m_2$ for $m_1, m_2 \in M$ and $\alpha_1 \in \Gamma$) satisfying the following conditions:

1. $(m_1 + m_2)\alpha_1m_3 = m_1\alpha_1m_3 + m_2\alpha_1m_3$ and
2. $(m_1\alpha_1m_2)\alpha_2m_3 = m_1\alpha_1(m_2\alpha_2m_3)$

for all $m_1, m_2, m_3 \in M$ and for all $\alpha_1, \alpha_2 \in \Gamma$.

Furthermore, $M$ is said to be a zero-symmetric $\Gamma$-nearring if $ma0 = 0$ for all $m \in M$, $a \in \Gamma$ (where ‘0’ is the additive identity in $M$).

Consider an example, take $Z_8 = \{0, 1, 2, 3, \ldots, 7\}$, the group of integers modulo 8 and a set $X = \{a, b\}$. Write $M = \{f | f : X \rightarrow Z_8$ and $f(a) = 0\}$.

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Then \( M = \{ f_0, f_1, f_2, \ldots, f_7 \} \) where \( f_i \) is defined by \( f_i(a) = 0 \) and \( f_i(b) = i \) for \( 0 \leq i \leq 7 \). Now define two mappings \( g_0, g_1 : \mathbb{Z}_8 \rightarrow X \) by setting \( g_0(i) = a \) for all \( i \in \mathbb{Z}_8 \) and \( g_1(i) = a \) if \( i \notin \{3, 7\} \), \( g_1(i) = b \) if \( i \in \{3, 7\} \). Write \( \Gamma = \{ g_0, g_1 \} \), \( \Gamma^* = \{ g_0 \} \). Then \( M \) is a \( \Gamma \)-nearring and a \( \Gamma^* \)-nearring.

Let \( M \) be a \( \Gamma \)-nearring. An additive group \( G \) is said to be an \( M\Gamma \)-group if there exists a mapping \( G \times \Gamma \times G \rightarrow G \) (denote the image of \( (m, \alpha, g) \) by \( m\alpha g \) for \( m \in M, \alpha \in \Gamma, g \in G \)) satisfying the conditions:

1. \( (m_1 + m_2)\alpha_1 g = m_1\alpha_1 g + m_2\alpha_1 g \) and
2. \( (m_1\alpha_1 m_2)\alpha_2 g = m_1\alpha_1 (m_2\alpha_2 g) \)

for all \( m_1, m_2 \in M, \alpha_1, \alpha_2 \in \Gamma \) and \( g \in G \).

Satyanarayana [6, 7, 12] introduced and studied the concepts like \( f \)-prime ideals and corresponding \( f \)-prime radical in \( \Gamma \)-near-rings. Further Satyanarayana [13] generalized the notion of module over nearring to module over gamma nearrings and established fundamental structure theorems. Radical of gamma nearrings also studied by Booth [1–3]. The concept of equi-prime ideal of a gamma nearring is a generalization of equi-prime ideal of a nearring which was studied in Booth and Groenewald [4]. Satyanarayana and Syam Prasad [8, 15] studied fuzzy aspects of gamma nearrings.

For standard notations, elementary definitions and results on nearrings, we refer Pilz [5], Satyanarayana and Syam Prasad [9]. Throughout, we denote \( M \) for a \( \Gamma \)-nearring and \( G \) for an \( M\Gamma \)-group.

2. Subgroups and ideals of \( M\Gamma \)-group:

**Definition 1** (Satyanarayana [11, 13]). An additive subgroup \( H \) of \( G \) is said to be \( M\Gamma \)-subgroup if \( m\alpha h \in H \) for all \( m \in M, \alpha \in \Gamma \) and \( h \in H \). Note that \( (0) \) and \( G \) are the trivial \( M\Gamma \)-subgroups. A normal subgroup \( H \) of \( G \) is said to be an ideal of \( G \) if \( m\alpha (g + h) - m\alpha g \in H \) for \( m \in M, \alpha \in \Gamma, g \in G \) and \( h \in H \). Moreover, a subgroup \( A \) of \( M \) is said to be an \( M\Gamma \)-subgroup of \( M \) if \( MTA \subseteq A \).

**Note 1.** If \( M \) is zero-symmetric then every ideal is a \( M\Gamma \)-subgroup. However, the converse need not be true. Consider the following example.

**Example 1.** Let \( G = \mathbb{Z}_4 = \{0, 1, 2, 3\} \), the ring of integers modulo 4 and \( X = \{a, b\} \). Write \( M = \{g \mid g : X \rightarrow G, g(a) = 0\} = \{g_0, g_1, g_2, g_3\} \), where \( g_0(a) = 0 \), \( g_i(b) = i \) for \( 0 \leq i \leq 3 \). Let \( \Gamma = \{f_1, f_2, f_3, f_4\} \) where each \( f_i : G \rightarrow X \) defined by
Let $P$ be an ideal of $G$. Suppose that for any $M$-subgroup $H$ of $G$ such that $P \subset H$, we have $(P : \Gamma G) = (P : \Gamma H)$. Then for all $a \in M$ and $b \in G$, $a\Gamma[b]_M \subseteq P$ implies $a\Gamma G \subseteq P$ or $b \in P$.

**Proof.** Take $a \in M$, $b \in G$ such that $a\Gamma[b]_M \subseteq P$. Suppose $b \notin P$. Then we have the following cases.
Case 1: $P \not\subseteq [b]_M$. Now $a\Gamma[b]_M \subseteq P$. This implies $a \in (P : \Gamma[b]_M) = (P : \Gamma G)$ (by hypothesis) = $(P : \Gamma G)$ (we considered with respect to $M$). This implies $a\Gamma G \subseteq P$.

Case 2: $[b]_M \not\subseteq P$. Then there exists $x \in P$ such that $x /\not\in [b]_M$. This implies $P \not\subseteq (P + [b]_M)$. By hypothesis $(P : \Gamma M) = (P : \Gamma (P + [b]_M))$. Now $a\Gamma[b]_M \subseteq P \implies a \in (P : \Gamma[b]_M) = (P : \Gamma G)$. This implies $a\Gamma G \subseteq P$.

□

Note 2. Let $G$ be an $M\Gamma$-group. Then a subgroup of $G$ need not be an $M\Gamma$-group, in general.

Consider the following example:

Example 2. Take $M = \{0, a, b, c\}$, $\Gamma = \{\gamma_1, \gamma_2\}$ and $G = M$ with the following binary operations.

\[
\begin{array}{c|cccc}
+ & 0 & a & b & c \\
\hline
0 & 0 & a & b & c \\
a & a & 0 & c & b \\
b & b & c & 0 & a \\
c & c & b & a & 0 \\
\end{array}
\quad
\begin{array}{c|cccc}
\gamma_1 & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & 0 & b & 0 & b \\
b & 0 & 0 & 0 & 0 \\
c & 0 & b & 0 & b \\
\end{array}
\quad
\begin{array}{c|cccc}
\gamma_2 & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & 0 & 0 \\
b & 0 & 0 & b & 0 \\
c & 0 & 0 & 0 & c \\
\end{array}
\]

Clearly $(M, +, \Gamma)$ is a $\Gamma$-nearring, and $G$ is an $M\Gamma$-group. Now $H = \{0, c\}$ is a subgroup of $G$. But it is not an $M\Gamma$-subgroup. For this, $c\gamma_1c = b \not\in \{0, c\} = H$. Therefore $M\Gamma H \not\subseteq H$. Hence $H$ is not an $M\Gamma$-subgroup of $G$.

3. Prime ideals of $M\Gamma$-groups.

**Definition 2.** Let $P$ be a proper ideal of $G$ such that $M\Gamma G \not\subseteq P$. Then $P$ is called prime if $A\Gamma B \subseteq P \implies A\Gamma G \subseteq P$ or $B \subseteq P$, for all ideals $A$ of $M$, $B$ of $G$.

**Definition 3.** An $M\Gamma$- group $G$ is said to be 0-prime $M\Gamma$- group if $M\Gamma G \neq (0)$ and $(0)$ is a prime ideal of $G$.

**Example 3.** Consider $M = \{0, a, b, c\}$, $\Gamma = \{\gamma_1, \gamma_2\}$, $G = M$ and the following binary operations.
Then $M$ is a $\Gamma$-nearring, and $G$ is an $M\Gamma$-group. Since there are no ideals $A$, $B$ of $G$ such that $A\Gamma B = \{0\}$ we have that $(0)$ is prime ideal of $G$.

**Definition 4.** (Satyanarayana, MBV Rao, Syam Prasad [14]): A left ideal $P$ of a nearring $N$ is said to be a prime left ideal if $A$ and $B$ are left ideals of $N$ such that $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

**Example 4.** Let $N$ be a nearring and $P$ be a prime left ideal of $N$. Write $M = N$ and consider $M$ as the gamma nearring with $\Gamma = \{\cdot\}$ (here $\cdot$ deremarks the multiplication in $N$). Write $G = N$. Clearly $G$ is an $M\Gamma$-group. Then $P$ becomes a prime ideal of the $M\Gamma$-group $G$.

**Verification:** Let $A$ be an ideal of $M$ and $B$ be an ideal of $G$ such that $A\Gamma B \subseteq P$. This implies $AB \subseteq P$ (since $\Gamma = \{\cdot\}$). Since $A$, $B$ are left ideals in $N$, we have that $A \subseteq P$ or $B \subseteq P$. If $B \not\subseteq P$ then $A \subseteq P$. Since $A$ is two sided ideal in $N$, we have $AN \subseteq A$. In the case $A \subseteq P$, we have that $A\Gamma G = AG = AN \subseteq A \subseteq P$.

**Proposition 2.** Let $G$ be an $M\Gamma$-group. Suppose $M\Gamma G \neq (0)$. If $(0)$ is a prime ideal of $G$ then the following two conditions are equivalent.

1. $B \neq (0)$ (where $B$ is an ideal of $G$), and
2. $A\Gamma B = (0) \iff A \subseteq (0 : \Gamma G)$.

**Proof.** (1) $\implies$ (2) : Suppose (1) holds. That is $B \neq (0)$. To show $A\Gamma B = (0) \iff A \subseteq (0 : \Gamma G)$, suppose $A\Gamma B = (0)$. Since $0$ is prime and $B \neq (0)$, we have $AG = (0)$. This implies $A \subseteq (0 : \Gamma G)$. Conversely suppose that $A \subseteq (0 : \Gamma G)$. This means $A\Gamma G = \{0\}$. Now $A\Gamma B \subseteq A\Gamma G \subseteq \{0\}$. This implies $A\Gamma B = (0)$.

(2) $\implies$ (1) : Suppose that $A\Gamma B = (0) \iff A \subseteq (0 : \Gamma G)$ holds. In a contrary way suppose that $B = (0)$. Then $M\Gamma B = (0) \implies M \subseteq (0 : \Gamma G)$ (by converse hypothesis) $\implies M\Gamma G = (0)$, a contradiction.

**Proposition 3.** Let $G$ be an $M\Gamma$-group such that $(P : \Gamma G) \neq M$. If $P$ is a prime ideal of $G$ then the following two conditions are equivalent.
(1) B is an ideal of G and \( B \not\subseteq P \), and
(2) \( A\Gamma B \subseteq P \iff A \subseteq (P : \Gamma G) \).

**Proof.** (1) \( \implies \) (2) : Suppose B is an ideal of G and \( B \not\subseteq P \). To show \( A\Gamma B \subseteq P \iff A \subseteq (P : \Gamma G) \), suppose \( A\Gamma B \subseteq P \). Since P is prime and \( B \not\subseteq P \), we have \( A\Gamma G \subseteq P \). This implies \( A \subseteq (P : \Gamma G) \). Conversely suppose that \( A \subseteq (P : \Gamma G) \). This means \( A\Gamma G \subseteq P \). Now \( A\Gamma B \subseteq A\Gamma G \subseteq P \). This implies \( A\Gamma B \subseteq P \).

(2) \( \implies \) (1) Suppose that \( A\Gamma B \subseteq P \iff A \subseteq (P : \Gamma G) \) holds. In a contrary way suppose that \( B \subseteq P \). Then \( M\Gamma B \subseteq P \iff M \subseteq (P : \Gamma G) \) (by converse hypothesis) \( \implies M\Gamma G \subseteq P \), a contradiction. \( \Box \)

**Theorem 4.** Let G be an \( M\Gamma \)-group, P be an ideal of G, A and B be ideals of M then the following conditions (1) and (2) are equivalent.

(1) P is 0-prime.
(2) \( < a > \Gamma < b > \subseteq P \) implies that \( < a > \Gamma G \subseteq P \) or \( b \in P \).

Moreover, if M is a zero symmetric \( \Gamma \)-nearring then conditions (1) to (4) are equivalent.

(3) If M is zero symmetric, \( a\Gamma < b > \subseteq P \) implies that \( a\Gamma G \subseteq P \) or \( b \in P \).
(4) \( a\Gamma B \subseteq P \) implies that \( a\Gamma G \subseteq P \) or \( B \subseteq P \).

**Proof.** (1) \( \implies \) (2) : Suppose \( < a > \Gamma < b > \subseteq P \). Write \( A = < a > \) and \( B = < b > \). Then \( A\Gamma B \subseteq P \). This implies \( A\Gamma G \subseteq P \) or \( B \subseteq P \) (by (1)) \( \iff < a > \Gamma G \subseteq P \) or \( b \in P \). Hence (2).

(2) \( \implies \) (1) : Suppose (2).

In contrary way suppose that (1) is not true.

Then there exists an ideal A of M, an ideal B of G such that \( A\Gamma B \subseteq P \) but \( A\Gamma G \not\subseteq P \) and \( B \not\subseteq P \). This implies \( a\gamma g \notin P \) for some \( a \in A, \gamma \in \Gamma, g \in G \) and \( b \in B \setminus P \).

Now \( < a > \Gamma < b > \subseteq A\Gamma B \subseteq P \). By (2) we have that \( < a > \Gamma G \subseteq P \) or \( b \in P \). Since \( b \notin P \) we have \( < a > \Gamma G \not\subseteq P \).

Now \( a\gamma g \in < a > \Gamma G \subseteq P \) implies \( a\gamma g \in P \), a contradiction.

(2) \( \implies \) (3): Suppose \( a\Gamma < b > \subseteq P \). This implies \( a \in (P : \Gamma < b >) \iff < a > \subseteq (P : \Gamma < b >) \) (since \( P : \Gamma < b > \) is an ideal and M is zero symmetric) \( \iff < a > \Gamma < b > \subseteq P \iff < a > \Gamma G \subseteq P \) or \( b \in P \) (by (2))
\[ a \Gamma G \subseteq < a > \Gamma G \subseteq P \text{ or } b \in P. \] This proves (3).

(3) \implies (2): Suppose \( < a > \Gamma b > \subseteq P \). Then \( a \Gamma < b > \subseteq < a > \Gamma b > \subseteq P \).

This implies \( a \Gamma G \subseteq P \) or \( b \in P \) (by (3)) \implies \( a \in (P : \Gamma G) \) or \( b \in P \)
\implies \( < a > \subseteq (P : \Gamma G) \) or \( b \in P \) \implies \( < a > \Gamma G \subseteq P \).

(3) \implies (4): Suppose (3). In contrary way, suppose (4) is not true. Then \( a \Gamma B \subseteq P \), \( a \Gamma G \not\subseteq P \) and \( B \not\subseteq P \) for some \( a \in A \). So there exists \( \gamma \in \Gamma \), and \( g \in G \) such that \( a\gamma g \notin P \) and \( b \in B \setminus P \).

Now \( a \Gamma < b > \subseteq a \Gamma B \subseteq P \) \implies \( a \Gamma G \subseteq P \) or \( b \in P \) (by (3)) \implies \( a \Gamma G \subseteq P \)
(since \( b \notin B \setminus P \)) \implies \( a\gamma g \in a \Gamma G \subseteq P \).

(4) \implies (3): Suppose \( a \Gamma < b > \subseteq P \). Write \( B = < b > \). Now \( a \Gamma B \subseteq P \) \implies \( a \Gamma G \subseteq P \) or \( B \subseteq P \) (by (4)) \implies \( a\Gamma b \subseteq P \) or \( b \in < b > = B \subseteq P \). \( \square \)

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