BIMAGIC OF GRAPHS MERGING FROM PATH AND STAR

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Abstract. The graphs one point union of a finite copies of \(P_n\), \(P_n \ast 2S_k\) of category-I, caterpillar graph of category of level-I, one point union of stars and switching each edge in a path are graceful with bi-magic labeling. In this article, bi-magic labelings are obtained with suitable examples due to certain rules in the proofs of all results for one point union of paths of same size, finite copies of stars, two stars merging with pendent vertices of an path, caterpillar, roots of finite copies stars merging with all pendent vertices of a star, and switching each edge in a path.

1. INTRODUCTION

Choudum and Kishore [1999] found graceful labeling of the union of paths and cycles, Bhat-Nayak and Selvam [2003] got graceful labeling for \(n\)-cone \(C_m \lor K_n\). Barrientos [2005] obtained the graceful labeling for unions of cycles and complete bipartite graphs. Guo [1994, 1995] investigated graceful labelings for bipartite graph \(B(m, n)\) and \(B(m, n, p)\). Liu [995] proved that the star graph with top sides is graceful. Seoud and Youssef [2000] showed that some classes of families in terms of disconnected from paths and cycles are graceful. Xu et.al. [2008] verified that the graphs \(C_{13(t)}\) are graceful where \(t \equiv 0, 1 \pmod{4}\).

The labeling of graphs found an increasingly useful in a variety of scientific applications. Gallian dynamic survey of graph labeling provided the focus on

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variety of graph labeling. The graph labeling is magic if it obey the condition $G(V, E)$ with $p$ vertices and $q$ edges is a bijection $f$ from the set of vertices and edges to such that for every edge $uv$ in $E$, $f(u) + f(uv) + f(v)$ is a constant $k$. Furthermore the graph is bi-magic if it possess two constants $k_1$ and $k_2$ with $f(u) + f(v) + f(uv) = k_1$ or $k_2$ [Harary, 1988].

In our previous work we reported the super magic and arithmetic labelings of the graphs, $P_3 \times P_n$ and $C_3 \times C_n$. Bi-magic labelings of the graphs, $W_n \ast S_k, W_n \ast 2S_k$ and $W_n \ast nP_2$ were reported. In this article, the $n$ copies of stars graph was graceful with magic labeling and the graphs one point union of a finite copies of $P_n, P_n \ast 2S_k$ of category-I, caterpillar graph of category of level-I, one point union of stars and switching each edge in a path were graceful with bi-magic labeling defined by two constants $k_1$ and $k_2$.

2. Basic definitions

Definition 2.1. A graph $G$ with $p$ vertices and $q$ edges is magic if there is a bijection $f : V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ such that $f(u) + f(uv) + f(v) = k = \text{constant for every edge } uv \text{ in } E$.

Definition 2.2. A graph $G$ with $p$ vertices and $q$ edges is edge bi-magic total if there exists a bijection $f : V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ such that there are two constants $k_1$ and $k_2$ such that $f(u) + f(v) + f(uv) = k_1$ or $k_2$ for any edge $uv \in E$ [5, 6].

Definition 2.3. One point union of a finite copies of a path $P_n$ is a graph whose vertex set is $\{V_1, V_2, \ldots V_{Kn}\}$ and edge set is $\{V_iV_{i+1} ; i = 1 \text{ to } k_n ; i \neq 0(\mod n)\}$ and \{V_0V_i ; i = 1, 2, \ldots, n+1, (2n+1), \ldots, (k−1)n+1\}.

3. Some bi-magic labelings of graphs

Theorem 3.1. One point union of a finite copies of a path $P_n$ is bi-magic.

Proof. One of the arbitrary labelings for vertices is given below.
Define \( f : V(G) \to \{1, 2, \ldots, p\} \) by \( f(V_0) = 1 \); and let for \( i = 1 \) to \( n \),
\[
\begin{align*}
f(V_i) &= 2 + k(i - 1); \quad f(V_{n+i}) = 3 + k(i - 1); \\
f(V_{2n+i}) &= 4 + k(i - 1); \quad f(V_{(k-1)n+i}) = (k + 1) + k(i - 1).
\end{align*}
\]

Let \( j = 1, 2, \ldots, k \). Define \( f : E(G) \to \{p + 1, p + 2, \ldots, p + q\} \) by
\[
\begin{align*}
f(V_{(j-1)n+1}V_{(j-1)n+2}) &= k(2n - 1) - 2(j - 1); \\
f(V_{(j-1)n+2}V_{(j-1)n+3}) &= k(2n - 1) - 2(k - 1) - 2(j - 1); \\
f(V_{(j-1)n+3}V_{(j-1)n+4}) &= k(2n - 1) - 4(k - 1) - 2(j - 1).
\end{align*}
\]

Case (1): \( n = k \).

Suppose \( n \) is odd, \( k \) is even; \( j = 1, 2, \ldots, k \).
\[
\begin{align*}
f(V_{(j-1)n+(n/2+1)}V_{(j-1)n+(n/2+2)}) \quad [ \text{n is even}] &= (2kn + 1 - k) - 2(j - 1); \\
f(V_{(j-1)n+(n+1)/2}V_{(j-1)n+[(n+1)/2]+j}) \quad [ \text{n is odd}] &= (2kn + 1 - k) - 2(j - 1);
\end{align*}
\]

Case (2). \( n \neq k \).

Suppose \( n \) is odd and \( k \) is even. Magic constant = \( n(2k + 1) + 1 \); Second magic constant = \( n \).

\[\square\]

**Example 1.** One point of four copies of \( P_6 \) is bimagic with magic constants 44 and 61.
Definition 3.1. $n$ copies of stars is defined a finite of stars with different sizes whose vertex set is \{\text{\textit{V}}}^1, \text{\textit{V}}^2, \ldots, \text{\textit{V}}^{(k_1+1)}, \ldots, \text{\textit{V}}^{(n_2+1)}, \ldots, \text{\textit{V}}^{(n_3+1)}, \ldots, \text{\textit{V}}^{(n_l+1)}, \ldots, \text{\textit{V}}^{(n_l+n)}\} and edge set = \{\text{\textit{V}}_{n_l+1}\text{\textit{V}}_i; i = 1 \text{ to } k_1\} \cup \{\text{\textit{V}}_{n_l+2}\text{\textit{V}}_{k_1+i}; i = 1 \text{ to } k_2\} \cup \{\text{\textit{V}}_{n_1+3}\text{\textit{V}}_{n_2+i}; i = 1 \text{ to } k_3\} \cup \cdots \cup \{\text{\textit{V}}_{n_l+n}\text{\textit{V}}_{(l-1)+i}; i = 1 \ldots n_l\} where, \(n_1 = k_1; n_2 = k_1 + k_2; n_3 = k_1 + k_2 + k_3; n_l = k_1 + k_2 + \cdots + k_l\).

Theorem 3.2. $n$ copies $S_{k_1} \ast S_{k_2} \ast \cdots \ast S_{k_l}$ of stars are magic.

Proof. One of the arbitrary labelings for vertices of $S_{k_1} \ast S_{k_2} \ast \cdots \ast S_{k_l}$ is given below.

Define $f : V(G) \rightarrow \{1, 2, \ldots, p\}$ by $f(V_{n_l+i}) = i; i = 1 \text{ to } n; f(V_i) = n + i; i = 1 \ldots n_l$, and define $f : E(G) \rightarrow \{p + 1, p + 2, \ldots, p + q\}$ by $f(V_{n_l+1}V_i) = (2n + 2n - 1) + (i - 1); i = 1, 2, \ldots, n_l$.

Magic constant = $(2n_l + 2n - 1) + n = 2n_l + 3n + 1$. □

Example 2. $S_{k_5} \ast S_{k_5} \ast S_{k_5} \ast S_{k_6}$ is magic with magic number 49.
Theorem 3.3. \( P_n \ast 2S_k \) of category I is bi-magic.

Proof. One of the arbitrary labelings for vertices is mentioned below.

Define \( f : V(G) \rightarrow \{1, 2, \ldots, p\} \) by
\[
f(V_{k+s+i}) = i, \quad i = 1 \text{ to } n; \quad f(V_i) = n + i, \quad i = 1 \text{ to } k;
\]
\[
f(V_{k+i}) = n + k + i, \quad i = 1 \text{ to } s.
\]

Also, define \( f : E(G) \rightarrow \{p + 1, p + 2, \ldots, p + q\} \) by
\[
f(V_{k+s+n}V_{k+s+n-i}) = (k + s + n) + i; \quad i = 1 \text{ to } s;
\]
\[
f(V_{k+s+1}V_{k+1-i}) = k + 2s + n + i; \quad i = 1 \text{ to } k.
\]
\[
f(V_{k+s+i}) = 2(k + s + n) + 1 - 2i; \quad i = 1 \text{ to } n/2 \text{ or } (n + 1)/2.
\]

\[
f(V_{k+s+i}) = \begin{cases} 
2(k + s + n) - 2 - 2[i - (n/2)], & i = (n/2) \text{ to } n \text{ if } n \text{ is even} \\
2(k + s + n) - 2 - 2[i - (n + 3/2)], & i = (n + 3/2) \text{ to } n \text{ if } n \text{ is odd}
\end{cases}
\]

Magic constant = \( 2(k + s + n + 1) \); Second magic constant = \( k + s \), \( \square \)

Example 3. \( P_4 \ast 2S_{11} \) of category I is bimagic with magic numbers 54 and 57.
Definition 3.2. Caterpillar graph of category of level I is a graph whose vertex set is \( \{V_1, V_2, \ldots, V_{2n}\} \) and edge set is \( \{V_iV_{i+1}; i = 1 \text{ to } n-1\} \cup \{V_iV_{n+i}; i = 1, 2, \ldots n\} \).

Theorem 3.4. Caterpillar graph \( C_n \) with chords of category of level I is bimagic.

Proof.

Define \( f : V(G) \to \{1, 2, \ldots p\} \) by \( f(V_i) = i, i = 1 \text{ to } n; f(V_{n+i}) = n+i+1, i = 1 \) to \( (n-1); f(V_{2n}) = n+1, \) and define \( f : E(G) \to \{p + 1, p + 2, \ldots, p + q\} \) by \( f(V_iV_{i+1}) = 4n - 2i + 2; i = 1 \text{ to } n-2; f(V_{n-1}V_n) = 4n - 1; f(V_iV_{n+i}) = (4n - 1) - 2i; i = 1 \text{ to } (n-1); f(V_nV_{2n}) = 2n + 2. \)

Magic constant = \( 4n + 3; \) Second magic constant = \( 5n. \) \( \Box \)

Example 4. Caterpillar graph \( C_{14} \) with chords of category of level I is bi-magic with magic numbers 59 and 70.
Definition 3.3. Merging of stars with a star is a graph whose vertex set \( \{V_0, V_1 \ldots V_n\} \) and edge set is \( \{V_0V_i; i = 1 \ldots n\} \cup \{V_jV_{n+i}; i = 1 \ldots j; j = 1 \ldots n\} \).

Theorem 3.5. Merging of different size of stars \( S_{k_1} \ast S_{k_2} \ldots \ast S_{k_l} \) with a star is bi-magic.

Proof.

Define \( f : V(G) \to \{1, 2 \ldots p\} \) by
\[
\begin{align*}
  f(V_0) &= 6n + 1; \\
  f(V_i) &= 1 + 6(i - 1), i = 1 \ldots n; \\
  f(V_{n+i}) &= i + 1 + k, i = 5k + j, 0 \leq j \leq 4, i = n + 1 \text{ to } 5n.
\end{align*}
\]

Define \( f : E(G) \to \{p + 1, p + 2, \ldots, p + q\} \) by
\[
\begin{align*}
  f(V_0V_1) &= 6n + 2; \\
  f(V_0V_{n+1-i}) &= 6n + 2 + 6i, i = 1 \text{ to } (n - 1). \\
  f(V_jV_{n+i}) &= (12n + 2) + (j - 1)(-2n) + 2j - i, j = 1 \text{ to } (n + 1)/2; i = 1 \text{ to } 5.
\end{align*}
\]
\[
f(V_j V_{n+j}) = (11n + 2) + (j - (n + 1)/2)(-2n) + 2(j - (n + 1)/2),
\]
\[
j = (n + 3)/2, (n + 5)/2, \ldots n; i = 1 \text{ to } 5.
\]
Magic first constant = \(12n + 4\); Second magic constant = \(C_2 = 18n + 4\).

\textbf{Example 5.} Merging of different size of stars \(S_5 \ast S_5 \ast S_5 \ast S_5\) with a star \(S_5\) is bimagic with magic numbers 59 and 70.

\textbf{Definition 3.4.} Switching each edge in a path is a graph whose vertex set is \(\{V_1, V_2, \ldots V_{2n+1}\}\) and edge set is \(\text{Edge set} = \{V_i V_{n+i}; i = 1 \ldots n\} \cup \{V_i V_{n+1+i}; i = 1 \ldots n\} \cup \{V_{n+i} V_{n+1+i}; i = 1 \ldots 2n\}\).

\textbf{Theorem 3.6.} Switching each edge in a path is bimagic.

\textbf{Proof.} One of arbitrary labelings for vertices of given graph is as follows.

Define \(f : V(G) \to \{1, 2, \ldots, p\}\) by \(f(V_i) = i, i = 1 \text{ to } 2n + 1\), and define \(f : E(G) \to \{p+1, p+2, \ldots, p+q\}\) by \(f(V_i V_{n+i}) = 6n+3-2i, i = 1 \text{ to } n; f(V_i V_{n+1+i}) = 6n + 2 - 2i, i = 1 \ldots n,\)

\[
f(V_i V_{n+i}) = \begin{cases} 3n + 2 - 2i, & i = 1 \text{ to } n/2 \text{ if } n \text{ is even} \\ 3n + 2 - 2i, & i = 1 \text{ to } (n - 1)/2 \text{ if } n \text{ is odd} \end{cases}
\]
and

\[
\begin{align*}
\mathcal{F}(V_j V_{n+1}) &= \begin{cases} 
3n + 3 - 2\left[i - \frac{n + 2}{2}\right], & i = \frac{n + 2}{2} \text{ to } n \text{ if } n \text{ is even} \\
3n + 3 - 2\left[i - \frac{n - 1}{2}\right], & i = \frac{n + 1}{2} \text{ to } n \text{ if } n \text{ is odd}.
\end{cases}
\end{align*}
\]

Magic constant = 6n + 3; Second magic constant = 7n + 3. □

**Example 6.** Switching each edge in a path \(P_7\) is bimagic with magic numbers 59 and 70.

![Graph Diagram]

**References**


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