MODULAR COLORINGS OF CORONA PRODUCT OF $C_m$ WITH $C_n$

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Abstract. For $\ell \geq 2$, a modular $\ell$-coloring of a graph $G$ without isolated vertices is a coloring of the vertices of $G$ with the elements in $\mathbb{Z}_\ell$ having the property that for every two adjacent vertices of $G$, the sums of the colors of their neighbors are different in $\mathbb{Z}_\ell$. The minimum $\ell$ for which $G$ has a modular $\ell$-coloring is the modular chromatic number of $G$. In this paper, we determine the modular chromatic number of corona product of cycles.

1. INTRODUCTION

For graph-theoretical terminology and notation, we in general follow [1]. For a vertex $v$ of a graph $G$, let $N_G(v)$, the neighborhood of $v$, denote the set of vertices adjacent to $v$ in $G$. For a graph $G$ without isolated vertices, let $c : V(G) \to \mathbb{Z}_\ell$, $\ell \geq 2$, be a vertex coloring of $G$ where adjacent vertices may be colored the same. The color sum $S(v) = \sum_{u \in N_G(v)} c(u)$ of a vertex $v$ of $G$ is the sum of the colors of the vertices in $N_G(v)$. The coloring $c$ is called a modular $\ell$-coloring of $G$ if $S(x) \neq S(y)$ in $\mathbb{Z}_\ell$ for all pairs $x, y$ of adjacent vertices in $G$. The modular chromatic number $Mc(G)$ of $G$ is the minimum $\ell$ for which $G$ has a modular $\ell$-coloring. This concept was introduced by Zhang et. al. [2].

Okamoto, Salehi and Zhang proved, in [2], they proved that: every nontrivial connected graph $G$ has a modular $\ell$-coloring for some integer $\ell \geq 2$ and $Mc(G) \geq \chi(G)$, where $\chi(G)$ denotes the chromatic number of $G$; for the cycle $C_n$ of length $n$, $Mc(C_n)$ is 2 if $n \equiv 0 \mod 4$ and it is 3 otherwise; every nontrivial

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tree has modular chromatic number 2 or 3; for the complete multipartite graph \( G \), \( Mc(G) = \chi(G) \); for the cartesian product \( G = K_r \square K_2 \), \( Mc(G) \) is \( r \) if \( r \equiv 2 \mod 4 \) and it is \( r+1 \) otherwise; for the wheel \( W_n = C_n \vee K_1 \), \( n \geq 3 \), \( Mc(W_n) = \chi(W_n) \), where \( \vee \) denotes the join of two graphs; for \( n \geq 3 \), \( Mc(C_n \vee K_2^r) = \chi(C_n \vee K_2^r) \), where \( G^c \) denotes the complement of \( G \); and for \( n \geq 2 \), \( Mc(P_n \vee K_2) = \chi(P_n \vee K_2) \), where \( P_n \) denotes the path of length \( n-1 \); and in [3] proved that: for \( m, n \geq 2 \), \( Mc(P_m \square P_n) = 2 \).

Paramaguru and Sampthkumar proved, in [5], that: \( Mc(C_3 \square P_2) = 4 \); except some special cases, for \( m \geq 3 \) and \( n \geq 2 \), \( Mc(C_m \square P_n) = \chi(C_m \square P_n) \); if \( m \equiv 2 \mod 4 \) and \( n \equiv 1 \mod 4 \), then \( Mc(C_m \square P_n) \leq 3 \); if \( n \equiv 1 \mod 4 \), then \( Mc(C_6 \square P_n) = 3 \). In [6], they proved that: if \( m \geq 4 \) and \( n \geq 4 \) are even integers and at least one of \( m, n \) is congruent to \( 0 \mod 4 \), then \( Mc(C_m \square C_n) = \chi(C_m \square C_n) \); if \( n \geq 3 \) is an integer, then \( Mc(C_3 \square C_n) = \chi(C_3 \square C_n) \); if at least one of \( m, n \) is congruent to \( 0 \mod 2 \), except some special cases, \( m \geq 4 \), \( n \geq 4 \), then \( Mc(C_m \square C_n) = \chi(C_m \square C_n) \); if \( n \equiv 2 \mod 4 \), and \( n \geq 6 \), then \( Mc(C_6 \square C_n) = 3 \), where \( \square \) denotes the Cartesian product of two graphs.

Nicholas and Sanma discussed in [4], that: the modular chromatic number of Fan, Helm graph, Friendship graph and gear graph.

The corona of two graphs \( G \) and \( H \) is the graph \( G \circ H \) formed from one copy of \( G \) and \( |V(G)| \) copies of \( H \), where the \( i \)th vertex of \( G \) is adjacent to every vertex in the \( i \)th copy of \( H \). Such type of graph products was introduced by Frucht and Harary in 1970.

2. CORONA OF \( C_m \) WITH \( C_n \)

Define \( V(C_m) = \{u_1, u_2, u_3, \ldots, u_m\} \); \( V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\} \); \( E(C_m) = \{u_1u_2, u_2u_3, u_3u_4, \ldots, u_{m-1}u_m, u_mu_1\} \); \( E(C_n) = \{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n, v_nv_1\} \);
\[
\begin{align*}
V(C_m \circ C_n) &= V(C_m) \cup \{v_j^i : i \in \{1, 2, 3, \ldots, m\} \text{ and } j \in \{1, 2, 3, \ldots, n\}\}; \\
E(C_m \circ C_n) &= E(C_m) \cup \{v_j^i v_{j+1}^i : i \in \{1, 2, 3, \ldots, m\} \text{ and } j \in \{1, 2, 3, \ldots, n-1\}\} \cup \{u_i v_j^i : i \in \{1, 2, 3, \ldots, m\} \text{ and } j \in \{1, 2, 3, \ldots, n\}\} \cup \{v_j^m v_1^i : i \in \{1, 2, 3, \ldots, m\}\}. 
\end{align*}
\]

**Theorem 2.1.** For \( m \) even and \( n \) even, \( m \geq 4 \), \( n \geq 4 \), \( Mc(C_m \circ C_n) = 3 \).

**Proof.** Let \( c : V(C_m \circ C_n) \rightarrow \mathbb{Z}_3 \).

**Case 1.** \( n \equiv 4 \mod 6 \).

Define \( c \) as follows: \( c(u_i) = 0 \) if \( i \) is even; \( c(u_i) = 1 \) if \( i \) is odd; \( c(v_j^i) = 0 \) if
i \in \{1, 2, 3, \ldots, m\}, j \text{ is even}; \ c(v^i_j) = 1 \text{ if } i \in \{1, 2, 3, \ldots, m\}, j \text{ is odd}; \text{ then }
S(u_i) = 1 \text{ if } i \text{ is even}; \ S(u_i) = 2 \text{ if } i \text{ is odd}; \ S(v^i_j) = 1 \text{ if } i, j \text{ odd}; \ S(v^i_j) = 2 \text{ if } i, j \text{ even}; \ S(v^i_j) = 0 \text{ if } i \text{ is odd, } j \text{ is even}; \ S(v^i_j) = 0 \text{ if } i \text{ is even, } j \text{ is odd}.

\textbf{Case 2.} n \equiv 2 \mod 6.
Define c as follows: \ c(u_i) = 0 \text{ if } i \in \{1, 2, 3, \ldots, m\}; \ c(v^i_j) = 0 \text{ if } i \in \{1, 2, 3, \ldots, m\}, j \text{ is even}; \ c(v^i_j) = 1 \text{ if } i, j \text{ odd}; \ c(v^i_j) = 2 \text{ if } i \text{ is even, } j \text{ is odd}; \text{ then } S(u_i) = 1 \text{ if } i \text{ is even}; \ S(v^i_j) = 1 \text{ if } i \text{ is odd, } j \text{ is even}; \ S(v^i_j) = 2 \text{ if } i \text{ is odd, } j \text{ is even}; \ S(v^i_j) = 0 \text{ if } i \in \{1, 2, 3, \ldots, m\}, j \text{ is odd}.

\textbf{Case 3.} n \equiv 0 \mod 6.
Define c as follows:
\ c(u_i) = 0 \text{ if } i \text{ is odd}; \ c(u_i) = 2 \text{ if } i \text{ is even}; \ c(v^i_j) = 0 \text{ if } i \in \{1, 2, 3, \ldots, m\}, j \text{ is even}; \ c(v^i_j) = 1 \text{ if } i \in \{1, 2, 3, \ldots, m\}, j \text{ is odd}; \text{ then } S(u_i) = 1 \text{ if } i \text{ is odd}; \ S(u_i) = 0 \text{ if } i \text{ is even}; \ S(v^i_j) = 1 \text{ if } i, j \text{ even}; \ S(v^i_j) = 2 \text{ if } i \text{ is odd, } j \text{ is even}; \ S(v^i_j) = 2 \text{ if } i \text{ is even, } j \text{ is odd}; \ S(v^i_j) = 0 \text{ if } i, j \text{ odd}. \text{ Clearly, } \chi(C_m \circ C_n) = 3. \text{ Hence, } Mc(C_m \circ C_n) = 3. \text{ This completes the proof.} \quad \square

\textbf{Theorem 2.2.} \textbf{For } m \text{ even and } n \text{ odd, } m \geq 4, n \geq 3, Mc(C_m \circ C_n) = 4.

\textbf{Proof.} \textbf{Let } c : V(C_m \circ C_n) \to \mathbb{Z}_4.

\textbf{Case 1.} n \equiv 1 \mod 8.
Define c as follows:
\ c(u_i) = 0 \text{ if } i \text{ is even}; \ c(u_i) = 1 \text{ if } i \text{ is odd}; \ c(v^i_j) = 0 \text{ if } i \in \{1, 2, 3, \ldots, m\}, j \equiv 0, 2, 3 \mod 4; \ c(v^i_j) = c(v^i_{n-4}) = 1 \text{ if } i \text{ is odd}; \ c(v^i_n) = 1 \text{ if } i \text{ is even}; \ c(v^i_j) = 2 \text{ if } i \text{ is odd, } j \equiv 1 \mod 4; \ c(v^i_j) = 2 \text{ if } i \text{ is odd, } j \equiv 1 \mod 4, j \neq n; \text{ then } S(u_i) = 0 \text{ if } i \text{ is odd}; \ S(u_i) = 3 \text{ if } i \text{ is even}; \ S(v^i_j) = 0 \text{ if } i \text{ is even, } j \in \{3, 5, 7, \ldots, n - 2\}; \ S(v^i_j) = 1 \text{ if } i \text{ is odd, } j \in \{3, 5, 7, \ldots, n - 2\}; \ S(v^i_j) = 1 \text{ if } i \text{ is even, } j \in \{1, n - 1\}; \ S(v^i_j) = 2 \text{ if } i \text{ is odd, } j \in \{1, n - 1, n - 3, n - 5\}; \ S(v^i_j) = 2 \text{ if } i \text{ is even, } j \in \{2, 4, 6, \ldots, n - 3, n\}; \ S(v^i_j) = 3 \text{ if } i \text{ is odd, } j \in \{2, 4, 6, \ldots, n - 7, n\}.

\textbf{Case 2.} n \equiv 3 \mod 8 \text{ and } n \neq 3.
Define c as follows:
\ c(u_i) = 0 \text{ if } i \text{ is even}; \ c(u_i) = 1 \text{ if } i \text{ is odd}; \ c(v^i_j) = 0 \text{ if } i \in \{1, 2, 3, \ldots, m\}, j \equiv 0, 2, 3 \mod 4; \ c(v^i_n) = c(v^i_{n-2}) = 1 \text{ if } i \text{ is odd}; \ c(v^i_{n-2}) = 1 \text{ if } i \text{ is even}; \ c(v^i_j) = 2 \text{ if } i \text{ is odd, } j \equiv 1 \mod 4; \ c(v^i_j) = 2 \text{ if } i \text{ is odd, } j \equiv 1 \mod 4, j \neq n - 2; \text{ then } S(u_i) = 0 \text{ if } i \text{ is odd}; \ S(u_i) = 3 \text{ if } i \text{ is even}; \ S(v^i_j) = 0 \text{ if } i \text{ is even, } j \in \{1, 3, 5, \ldots, n - 2\}; \ S(v^i_j) = 1 \text{ if } i \text{ is odd, } j \in \{1, 3, 5, \ldots, n - 2\}; \ S(v^i_j) = 1 \text{ if } i \text{ is even, } j \in \{n - 1, n - 3\}; \ S(v^i_j) = 2 \text{ if } i \text{ is odd, } j \in \{n - 1, n - 3, n - 5, n - 7\}; \ S(v^i_j) = 2 \text{ if } i \text{ is even, } j \in \{2, 4, 6, \ldots, n - 5, n\};
Define \( c \) as follows: \( c(u_i) = 0 \) if \( i \in \{1, 2, 3, \ldots, m\} \); \( c(v_j^i) = 0 \) if \( j \in \{2, 4, 6, \ldots, n - 9, n\} \).

**Case 3.** \( n \equiv 5 \mod 8 \).

Define \( c \) as follows: \( c(u_i) = 0 \) if \( i \in \{1, 2, 3, \ldots, m\} \); \( c(v_j^i) = 0 \) if \( j \in \{1, 2, 3, \ldots, m\} \), \( j \equiv 0, 2, 3 \mod 4 \); \( c(v_j^i) = 1 \) if \( i \) is even; \( c(v_j^i) = 3 \) if \( i \) is odd; \( c(v_j^i) = 2 \) if \( i \) is even; \( S(v_j^i) = 0 \) if \( i \in \{1, 2, 3, \ldots, m\} \), \( j \in \{3, 5, 7, \ldots, n - 2\} \); \( S(v_j^i) = 1 \) if \( i \) is even, \( j \in \{n - 1\} \); \( S(v_j^i) = 3 \) if \( i \) is odd, \( j \in \{n - 1\} \); \( S(v_j^i) = 2 \) if \( i \in \{1, 2, 3, \ldots, m\} \), \( j \in \{2, 4, 6, \ldots, n - 3, n\} \).

**Case 4.** \( n \equiv 7 \mod 8 \).

Define \( c \) as follows: \( c(u_i) = 0 \) if \( i \in \{1, 2, 3, \ldots, m\} \); \( c(v_j^i) = 0 \) if \( i \in \{1, 2, 3, \ldots, m\} \), \( j \equiv 0, 2, 3 \mod 4 \); \( c(v_{n-2}^i) = 1 \) if \( i \) is even; \( c(v_{n-2}^i) = 3 \) if \( i \) is odd; \( c(v_j^i) = 2 \) if \( i \) is even; \( S(v_j^i) = 0 \) if \( i \in \{1, 2, 3, \ldots, m\} \), \( j \in \{1, 3, 5, \ldots, n - 2\} \); \( S(v_j^i) = 1 \) if \( i \) is even, \( j \in \{n - 1, n - 3\} \); \( S(v_j^i) = 3 \) if \( i \) is odd, \( j \in \{n - 1, n - 3\} \); \( S(v_j^i) = 2 \) if \( i \in \{1, 2, 3, \ldots, m\} \), \( j \in \{2, 4, 6, \ldots, n - 5, n\} \).

**Case 5.** \( n \equiv 3 \).

**Subcase 5.1.** \( m \equiv 0 \mod 4 \).

Define \( c \) as follows: \( c(u_i) = 0 \) if \( i \equiv 0, 2, 3 \mod 4 \); \( c(u_i) = 1 \) if \( i \equiv 1 \mod 4 \); \( c(v_1^i) = 0 \) if \( i \not\equiv 1 \mod 4 \); \( c(v_2^i) = 2 \) if \( i \equiv 1 \mod 4 \); \( c(v_3^i) = 3 \) if \( i \not\equiv 1 \mod 4 \); \( c(v_4^i) = 0 \) if \( i \) is even; \( c(v_4^i) = 1 \) if \( i \) is even; \( c(v_4^i) = 2 \) if \( i \) is even; \( c(v_4^i) = 1 \) if \( i \equiv 3 \mod 4 \); \( c(v_4^i) = 2 \) if \( i \equiv 3 \mod 4 \); \( c(v_4^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(u_i) = 1 \) if \( i \equiv 1 \mod 4 \); \( S(v_1^i) = 0 \) if \( i \equiv 0, 2, \mod 4 \); \( S(u_i) = 2 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 2 \) if \( i \equiv 1 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 1 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \); \( S(v_1^i) = 3 \) if \( i \equiv 3 \mod 4 \);
$S(v_i) = 1$ if $i \in \{2, 4, 6, \ldots, m-2\}$; $S(v_1^m) = 0$; $S(v_2^m) = 3$; $S(v_3^m) = 1$. Clearly, $Mc(C_m \circ C_n) \geq \chi(C_m \circ C_n) = 4$. Hence, $Mc(C_m \circ C_n) = 4$. This completes the proof. □

3. Conclusion

For some graphs $G$ and $H$ considered in this paper, we have seen that $Mc(G \circ H) = \chi(G \circ H)$. Except the case: For $m \geq 1, n \geq 1$, $Mc(C_{2m+1} \circ C_{2n+1})$.

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