FUZZY TOTALLY SOMEWHAT COMPLETELY IRRESOLUTE MAPPINGS

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ABSTRACT. This article is to study the concept of fuzzy totally somewhat completely irresolute and fuzzy totally somewhat completely continuous mappings. Further, some interesting properties of those mappings are given.

1. INTRODUCTION

After the introduction of fuzzy sets by L. A. Zadeh in his classical paper, C. L. Chang [3] first introduced the concept of fuzzy topology in 1968. The notions of fuzzy totally continuous functions and fuzzy totally semicontinuous functions were investigated by Anjan Mukherjee in [1]. G. Thangaraj and G. Balasubramanian introduced the concepts of somewhat fuzzy continuous and somewhat fuzzy semicontinuous functions. The notions of fuzzy totally somewhat continuous mapping and fuzzy totally somewhat semicontinuous mapping were investigated by A. Vadivel and A. Swaminathan.

In this paper we introduce the ideas of fuzzy totally somewhat completely irresolute and fuzzy totally somewhat completely continuous mappings and since some examples and relationships between these new classes with other classes of fuzzy functions are obtained. Finally, in section 3, some preservation of these mappings are discussed.

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The known definitions which are used in this paper are seen in [1,2,7,9].

2. FUZZY TOTALLY SOMEWHAT COMPLETELY IRRESOLUTE MAPPINGS

In this section, we introduce fuzzy totally somewhat completely irresolute and fuzzy totally somewhat completely continuous mappings and we characterize these mappings.

**Definition 2.1.** A mapping \( f : X \to Y \) is called fuzzy totally somewhat completely irresolute if there exists a fuzzy regular clopen set \( \mu \neq 0_X \) on \( X \) such that \( \mu \leq f^{-1}(\nu) \neq 0_X \) for any fuzzy semiopen set \( \nu \neq 0_Y \) on \( Y \).

**Definition 2.2.** A mapping \( f : X \to Y \) is called fuzzy totally somewhat completely continuous if there exists a fuzzy regular clopen set \( \mu \neq 0_X \) on \( X \) such that \( \mu \leq f^{-1}(\nu) \neq 0_X \) for any fuzzy open set \( \nu \neq 0_Y \) on \( Y \).

From the above definitions the following reverse implications are false:

(i) Every fuzzy totally completely irresolute is a fuzzy totally somewhat completely irresolute.

(ii) Every fuzzy totally somewhat completely irresolute is a fuzzy totally somewhat completely continuous.

(iii) Every fuzzy totally completely continuous is a fuzzy totally somewhat completely continuous.

**Example 1.** Let \( K_1(x) \) and \( K_2(x) \) be fuzzy sets on \( I = [0, 1] \) defined as follows:

\[
K_1(x) = \begin{cases} 
\frac{1}{4}, & 0 \leq x \leq \frac{1}{2}, \\
\frac{1-x}{2}, & \frac{1}{2} \leq x \leq 1.
\end{cases}
\]

\[
K_2(x) = \begin{cases} 
\frac{1}{2}, & 0 \leq x \leq \frac{1}{2}, \\
1-x, & \frac{1}{2} \leq x \leq 1.
\end{cases}
\]

\[
K_3(x) = \begin{cases} 
0, & 0 \leq x \leq \frac{1}{2}, \\
2x-1, & \frac{1}{2} \leq x \leq 1.
\end{cases}
\]

\[
K_4(x) = \begin{cases} 
1, & 0 \leq x \leq \frac{1}{4}, \\
-4x+2, & \frac{1}{4} \leq x \leq \frac{1}{2}, \\
0, & \frac{1}{2} \leq x \leq 1.
\end{cases}
\]

Let \( \mathcal{J}_1 = \{0, K_1, K_1^c, K_2, 1\} \) be a fuzzy topology on \( I \). Let \( f : (I, \mathcal{J}_1) \to (I, \mathcal{J}_1) \) defined by \( f(x) = x \) for each \( x \in I \). It is observed that, for the fuzzy regular clopen set \( K_1 \) on \( (I, \mathcal{J}_1) \), \( K_1 \leq f^{-1}(K_1) = K_1 \), \( K_1 \leq f^{-1}(K_1^c) = K_1^c \) and \( K_1 \leq f^{-1}(K_2) = K_2 \). Therefore, \( f \) is fuzzy totally somewhat completely irresolute mapping. But
for a fuzzy semiopen set $K_2$ on $(I, J_2)$, $f^{-1}(K_2) = K_2$ which is not fuzzy regular clopen set on $(I, J_1)$. Hence $f$ is not fuzzy totally completely irresolute mapping.

**Example 2.** Consider the fuzzy topologies $J_3 = \{0, K_3, K_3^c, K_3 \lor K_3^c, K_3 \land K_3^c, 1\}$ and $J_4 = \{0, K_3, K_4, K_3 \lor K_4, 1\}$ a mapping $f : (I, J_3) \to (I, J_4)$ defined by $f(x) = \frac{x}{2}$ for each $x \in I$. It is observed that, for the fuzzy open sets $K_3$, $K_4$ and $K_3 \lor K_4$ on $(I, J_4)$, $f^{-1}(K_3) = 0$ and $K_3^c \leq f^{-1}(K_3 \lor K_4) = K_3^c = f^{-1}(K_4)$. Therefore, $f$ is fuzzy totally somewhat completely continuous mapping. But for a fuzzy semiopen set $K_5$ on $(I, J_4)$,

$$f^{-1}(K_5) = K_5 f\left(\frac{x}{2}\right) = M(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2} \\ \frac{1}{3}(2x - 1), & \frac{1}{2} \leq x \leq 1 \end{cases},$$

for each $x \in I$. Hence there is no non-zero fuzzy regular clopen set smaller than $f^{-1}(K_5) = K_5 f\left(\frac{x}{2}\right) = M(x)$ on $(I, J_3)$. Therefore $f$ is not fuzzy totally somewhat completely irresolute mapping.

**Example 3.** As described in Example 3.1, consider the fuzzy topology $J_3 = \{0, K_1, K_2, 1\}$ and a mapping $f : (I, J_3) \to (I, J_3)$ defined by $f(x) = x$ for each $x \in I$. It is observed that, for the fuzzy open sets $K_1$ and $K_2$ on $(I, J_3)$, $K_1 \leq f^{-1}(K_1) = K_1$ and $K_1 \leq f^{-1}(K_2) = K_2$. Therefore, $f$ is fuzzy totally somewhat completely continuous mapping. But for a fuzzy open set $K_2$ on $(I, J_3)$, $f^{-1}(K_2) = K_2$ which is fuzzy regular open but not regular closed on $(I, J_1)$. Hence $f$ is not fuzzy totally completely continuous mapping.

**Theorem 2.1.** Let $X_1$ be product related to $X_2$ and let $Y_1$ be product related to $Y_2$. If $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ is fuzzy totally somewhat completely irresolute mappings, then the product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is also fuzzy totally somewhat completely irresolute mappings.

**Proof.** Let $\lambda = \bigvee_{i,j} (\mu_i \land \nu_j)$ be a fuzzy semiopen set on $Y_1 \times Y_2$ where $\mu_i \neq 0_{Y_1}$ and $\nu_j \neq 0_{Y_2}$ are fuzzy semiopen sets on $Y_1$ and $Y_2$ respectively. Then $(f_1 \times f_2)^{-1}(\lambda) = \bigvee_{i,j} (f_1^{-1}(\mu_i) \times f_2^{-1}(\nu_j))$. Since $f_1$ is fuzzy totally somewhat completely irresolute, there exists a fuzzy regular clopen set $\delta_i \neq 0_{X_1}$ such that $\delta_i \leq f_1^{-1}(\mu_i) \neq 0_{X_1}$. And, since $f_2$ is fuzzy totally somewhat completely irresolute, there exists a fuzzy regular clopen set $\eta_j \neq 0_{X_2}$ such that $\eta_j \leq f_2^{-1}(\nu_j) \neq 0_{X_2}$. Now $\delta_i \times \eta_j \leq f_1^{-1}(\mu_i) \times f_2^{-1}(\nu_j) = (f_1 \times f_2)^{-1}(\mu_i \times \nu_j)$ and $\delta_i \times \eta_j \neq 0_{X_1} \times 0_{X_2}$ is a fuzzy regular clopen set on $X_1 \times X_2$. Hence $\bigvee_{i,j} (\delta_i \times \eta_j) \neq 0_{X_1} \times 0_{X_2}$ is a fuzzy regular
clopen set on $X_1 \times X_2$ such that $\bigvee_{i,j}(\delta_i \times \eta_j) \leq \bigvee_{i,j}(f_1^{-1}(\mu_i) \times f_2^{-1}(\nu_j)) = (f_1 \times f_2)^{-1}(\bigvee_{i,j}(\mu_i \times \nu_j)) = 0_{X_1 \times X_2}$. Therefore, $f_1 \times f_2$ is fuzzy totally somewhat completely irresolute.

**Definition 2.3.** A fuzzy topological space $(X, F)$ is said to be fuzzy super connected [4] if it has no regular open subsets.

**Proposition 2.1.** If $f$ is fuzzy totally somewhat fuzzy irresolute mapping from a fuzzy super connected space $X$ into any fuzzy topological space $Y$, then $Y$ is indiscrete fuzzy topological space.

**Proof.** If possible suppose $Y$ is not indiscrete. Then $Y$ has a proper ($\neq 0$ and $\neq 1$) fuzzy open set $\lambda$ (say). Then by hypothesis on $f$, $f^{-1}(\lambda)$ is a proper fuzzy regular clopen set of $X$, which is a contradiction to the assumption that $X$ is fuzzy super connected. Hence the proposition.

**Definition 2.4.** Let $(X, F)$ be any fuzzy topological space. $X$ is called fuzzy semi $T_0$ [5] (fuzzy almost $T_0$ [8]) if for any pair of distinct fuzzy points $x_t$ and $y_s$, there is a fuzzy semiopen (regular open) set $\lambda$ such that $x_t \in \lambda$, $y_s \notin \lambda$ or $x_t \notin \lambda$, $y_s \in \lambda$.

**Definition 2.5.** Let $(X, F)$ be any fuzzy topological space. $(X, T)$ is called fuzzy semi $T_2$ [5] if for any pair of distinct fuzzy points $x_t$ and $y_s$ there exist fuzzy semiopen sets $\lambda$ and $\mu$ such that $x_t \in \lambda$, $y_s \in \mu$ and $sCl\lambda \leq 1 - sCl\mu$.

**Definition 2.6.** Let $(X, F)$ be any fuzzy topological space. $(X, T)$ is called fuzzy almost $T_2$ [8] if for any pair of distinct fuzzy points $x_t$ and $y_s$ there exist fuzzy regular open sets $\lambda$ and $\mu$ such that $x_t \in \lambda$, $y_s \in \mu$ and $rCl\lambda \leq 1 - rCl\mu$.

**Proposition 2.2.** Let $f : (X, F) \to (Y, H)$ be an injective fuzzy totally somewhat completely irresolute mapping. If $Y$ is fuzzy semi $T_0$, then $X$ is fuzzy almost $T_2$.

**Proof.** Let $x_t$ and $y_s$ be any two distinct fuzzy points of $X$. Then $f(x_t) \neq f(y_s)$. That is $(f(x))_t \neq (f(y))_s$. Since $Y$ is fuzzy semi $T_0$, there exists a fuzzy semiopen set say $\lambda \neq 0$ in $Y$ such that $f(x_t) \in \lambda$ and $f(y_s) \notin \lambda$. This means $x_t \in f^{-1}(\lambda)$ and $y_s \notin f^{-1}(\lambda)$. Since $f$ is fuzzy totally somewhat completely irresolute, there exists fuzzy regular open set $\mu \neq 0$ in $X$ such that $\mu \leq f^{-1}(\lambda)$ is fuzzy regular open set of $X$. 

Also $x_t \in f^{-1}(\lambda)$ and $y_s \in 1 - f^{-1}(\lambda)$. Now put $\mu = 1 - f^{-1}(\lambda)$. Then $f^{-1}(\lambda) = r\text{Cl}(\lambda)$ and $r\text{Cl}(1 - f^{-1}(\lambda)) = r\text{Cl}\mu = 1 - f^{-1}(\lambda)$ and $r\text{Cl}(\lambda) = f^{-1}(\lambda) = 1 - r\text{Cl}(\mu) \leq 1 - r\text{Cl}(\mu)$. Hence the Proposition. □

**Proposition 2.3.** Let $(X, \mathcal{F})$ be any fuzzy super connected space. Then every fuzzy totally somewhat completely irresolute mapping from a space $X$ onto any fuzzy semi $T_0$-space $Y$ is constant.

**Proof.** Given that $(X, \mathcal{F})$ is fuzzy super connected. Suppose $f : X \to Y$ be any fuzzy totally somewhat completely irresolute mapping and we assume that $Y$ is a fuzzy semi $T_0$ space. Then by Proposition 2.1, $Y$ should be an indiscrete space. But an indiscrete fuzzy topological space containing two or more points cannot be fuzzy semi $T_0$. Therefore, $Y$ must be singleton and this proves that $f$ must be a constant function. □

**Definition 2.7.** A fuzzy space $X$ is called fuzzy almost regular [8] if for each fuzzy regular open set $\lambda$ and each fuzzy point $x_t \notin \lambda$ there exist disjoint fuzzy open sets $\sigma$ and $\delta$ such that $\lambda \leq \sigma$ and $x_t \in \delta$.

**Definition 2.8.** A fuzzy space $X$ is called fuzzy almost normal [8] if for every pair of disjoint fuzzy sets sets $\lambda_1$ and $\lambda_2$ in $X$, where $\lambda_1$ is fuzzy open and $\lambda_2$ is fuzzy regular open, there exist disjoint fuzzy open sets $\sigma$ and $\delta$ such that $\lambda_1 \leq \sigma$ and $\lambda_2 \leq \delta$.

**Theorem 2.2.** If $f : (X, \mathcal{F}) \to (Y, \mathcal{H})$ be an injective fuzzy totally somewhat completely continuous mapping and $X$ is a fuzzy almost regular space, then $Y$ is fuzzy regular.

**Proof.** Let $\lambda$ be a fuzzy open set on $Y$ and a fuzzy point $y_{\beta} \notin \lambda$. Take $y_{\beta} = f(x_{\alpha})$. Since $f$ is fuzzy totally somewhat completely continuous such that $f^{-1}(\lambda)$ is a fuzzy regular open set of $X$. Take $\mu = f^{-1}(\lambda)$. We have $x_{\alpha} \notin \mu$. Since $X$ is fuzzy almost regular, there exist disjoint fuzzy open sets $\eta$ and $\delta$ in $X$ such that $\mu \leq \eta$ and $x_{\alpha} \in \delta$. We obtain that $\lambda = f(\mu) \leq f(\eta)$ and $y_{\beta} = f(x_{\alpha}) \in \delta$ such that $f(\eta)$ and $f(\delta)$ are disjoint fuzzy semiopen sets of $Y$. This shows that $Y$ is fuzzy regular. □

**Theorem 2.3.** If $f : (X, \mathcal{F}) \to (Y, \mathcal{H})$ be an injective fuzzy totally somewhat completely continuous and $X$ is a fuzzy almost normal space, then $Y$ is fuzzy normal.
Proof. Let $\lambda_1$ and $\lambda_2$ be disjoint fuzzy open set in $Y$. Since $f$ is fuzzy totally somewhat completely continuous, $f^{-1}(\lambda_1)$ and $f^{-1}(\lambda_2)$ are fuzzy regular open sets in $X$. Let $\alpha = f^{-1}(\lambda_1)$ and $\beta = f^{-1}(\lambda_2)$. We have $\alpha \land \beta = 0$. Since $X$ is fuzzy almost normal, there exist disjoint fuzzy open sets $\gamma$ and $\delta$ such that $\alpha \leq \gamma$ and $\beta \leq \delta$. We obtain that $\lambda_1 = f(\alpha) \leq f(\gamma)$ and $\lambda_2 = f(\beta) \leq f(\delta)$ such that $f(\gamma)$ and $f(\delta)$ are disjoint fuzzy open sets. Thus, $Y$ is fuzzy semi normal.

3. SOME PRESERVATION RESULTS

In this section by means of fuzzy totally somewhat completely irresolute and fuzzy totally somewhat completely continuous mapping preservation of some fuzzy topological structures are discussed.

Definition 3.1. A fuzzy topological space $(X, \mathcal{F})$ is called

(i) fuzzy semi-compact [6] if every fuzzy semiopen cover has a finite subcover.

(ii) fuzzy $r$-closed [9] if every fuzzy regular clopen cover has a finite subcover.

Theorem 3.1. Every surjective fuzzy totally somewhat completely irresolute image of a fuzzy $r$-closed space is fuzzy semi-compact.

Proof. Let $f : X \to Y$ be a fuzzy totally somewhat completely irresolute mapping of a fuzzy $r$-closed space $(X, \mathcal{F}_1)$ onto a fuzzy space $(Y, \mathcal{F}_2)$. Let $\{\beta_a : a \in A\}$ be any fuzzy semiopen cover of $Y$. Since $f$ is fuzzy totally somewhat completely irresolute, $\{f^{-1}(\beta_a) : a \in A\}$ is a fuzzy regular clopen cover of $X$. Since $X$ is a fuzzy $r$-closed space, then there exists a finite subfamily $\{f^{-1}(\beta_{a_i}) : i = 1, \ldots, n\}$ of $\{f^{-1}(\beta)\}$ which covers $X$. It implies that $\{\beta_{a_i} : i = 1, \ldots, n\}$ is a finite subcover of $\{\beta_a : a \in A\}$ which covers $Y$. Hence $f(X) = Y$ is fuzzy semi-compact.

Theorem 3.2. Every surjective fuzzy totally somewhat completely continuous image of a fuzzy $r$-closed space is fuzzy compact.

Proof. Let $f : X \to Y$ be a fuzzy totally somewhat completely continuous mapping of a fuzzy $r$-closed space $(X, \mathcal{F}_1)$ onto a fuzzy space $(Y, \mathcal{F}_2)$. Let $\{\beta_a : a \in A\}$ be any fuzzy open cover of $Y$. Since $f$ is fuzzy totally somewhat completely continuous, $\{f^{-1}(\beta_a) : a \in A\}$ is a fuzzy regular clopen cover of $X$. Since $X$ is a fuzzy $r$-closed space, then there exists a finite subfamily $\{f^{-1}(\beta_{a_i}) : i = 1, \ldots, n\}$ of $\{f^{-1}(\beta)\}$ which covers $X$. It implies that $\{\beta_{a_i} : i = 1, \ldots, n\}$ is a finite subcover of $\{\beta_a : a \in A\}$ which covers $Y$. Hence $f(X) = Y$ is fuzzy compact.
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