VARIETY OF RATIONAL RESOLVING SETS OF CORONA PRODUCT OF GRAPHS

M. JAYALAKSHMI AND M. M. PADMA

ABSTRACT. A subset $S$ of $V(G)$ is called a resolving set if for each pair of vertices $u, v$ in $V - S$, there exists a vertex $w \in S$ such that $d(u, w) \neq d(v, w)$ and is called a rational resolving set of $G$, if for each pair $u, v \in V - S$, there is a vertex $w \in S$ such that $d(u/w) \neq d(v/w)$, where $d(x/w)$ denotes the mean of the distances from the vertex $w$ to all those $y \in N[x]$. A rational resolving set is called minimal rational resolving set if no proper subset of it is a rational resolving set. In this paper we study the varieties of minimal rational resolving sets of graph $G$ and compute their minimum and maximum cardinality for certain classes of corona product of graphs.

1. INTRODUCTION

Many networks are represented by a graph, in which vertex play an important role and it depends on its neighbors. In the development of smart cities one of the most important task is to identify every building uniquely and to be given a unique ID, which helps to avail numerous public digital services quickly. For example a fire engine or a police vehicle or an ambulance can reach the destination building quickly with the help of a unique ID. This concept helps to find and optimize the number of source places required for the unique representation of

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1 corresponding author

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the destination with the help of graph theory, since many networks are represented by graphs. To obtain the position of a vertex in the network, we need to select the landmarks in such a way that the distance of the vertex from the landmark and the distance of its neighborhood vertices from the landmark are considered. This problem is equivalent to finding a minimal rational resolving set of the constructed graph $G$.

A subset $S$ of $V(G)$ is called a resolving set if for each pair of vertices $u, v$ in $V - S$ there exists a vertex $w$ in $S$ such that $d(u, w) \neq d(v, w)$. The minimum cardinality of such a set $S$ is called metric dimension of a graph $G$, denoted by $\beta(G)$. Metric dimension was defined by F. Harary et al. [2] and Peter J. Slater [6]. B. Sooryanarayana [7] and various authors in articles [8, 9] have obtained many results on metric dimension. Rational metric dimension of graphs were originally proposed by A. Raghavendra et al. [10]. M. M. Padma and M. Jayalakshmi [4, 5] introduced and developed the concept of $r_r$, $r_r^*$, $R_r$, $R_r^*$ sets of graphs. We use the standard terminology, the terms not defined here may be found in [1,3].

2. $r_r$, $r_r^*$, $R_r$, $R_r^*$ Sets of a Graph

Let $G(V, E)$ be a simple, connected, non-trivial, finite, undirected graph. Let $N(u) = \{w : uw \in E(G)\}$ be the open neighbourhood and $N[u] = N(u) \cup \{u\}$ be the closed neighbourhood of a vertex $u$. For the vertex $u$ of $G$ and an ordered subset $S = \{s_1, s_2, ..., s_k\}$ of $V$, associate a vector denoted and defined by $\Gamma(u/S) = (d(u/s_1), d(u/s_2), ..., d(u/s_k))$, where $d(u/v) = \sum_{u_i \in N[u]} \frac{d(u_i, v)}{deg(u)+1}$. Then $S$ is said to be a rational resolving set if $\Gamma(x/S) \neq \Gamma(y/S) \forall x, y \in V - S$ and the minimum cardinality of such a set $S$ is called the rational metric dimension or the lower $r_r$ number, denoted by $\text{rmd}(G)$ or $l_{r_r}(G)$. The associated set is called the rational metric basis or an rmb set. The maximum cardinality of a minimal $r_r$ set of graph $G$ is said to be an upper $r_r$ number of $G$, denoted by $u_{r_r}(G)$.

A subset $S$ of $V(G)$ is said to be an $r_r^*$ set if $S$ is an $r_r$ set and $\bar{S} = V - S$ is also an $r_r$ set. The minimum and maximum cardinality of a minimal $r_r^*$ set of graph $G$ are called respectively, the lower $r_r^*$ number and the upper $r_r^*$ number of $G$ and are denoted by $l_{r_r^*}(G)$ and $u_{r_r^*}(G)$.

A subset $S$ of $V(G)$ is said to be an $R_r$ set if $S$ an $r_r$ set and $\bar{S} = V - S$ is not an $r_r$ set. The minimum and maximum cardinality of minimal $R_r$ sets of $G$ are
called respectively, the lower \( R_r \) number and the upper \( R_r \) number of \( G \), denoted by \( l_{R_r}(G) \) and \( u_{R_r}(G) \).

A subset \( S \) of \( V(G) \) is said to be an \( R_r^* \) set if both \( S \) and \( \bar{S} = V - S \) are not \( r_r \) sets. The minimum and maximum cardinality of minimal \( R_r^* \) sets of \( G \) are called respectively, the lower \( R_r^* \) number and the upper \( R_r^* \) number of \( G \), denoted by \( l_{R_r^*}(G) \) and \( u_{R_r^*}(G) \).

**Remark 2.1.** For any graph \( G(V, E) \), we use the convention that if \( r_r^* \), \( R_r \), \( R_r^* \) sets does not exist, then their cardinality is zero.

3. \( r_r \), \( r_r^* \), \( R_r \), \( R_r^* \) Sets of Corona Product of Graphs

The corona of two graphs \( G_1 \) and \( G_2 \) having the order \( n_1 \), \( n_2 \) and size \( m_1 \), \( m_2 \) respectively is the graph obtained by taking a copy of \( G_1 \) and \( n_1 \) copies of \( G_2 \), and joining \( i^{th} \) vertex of \( G_1 \) with an edge to every vertex in the \( i^{th} \) copy of \( G_2 \). It is denoted by \( G_1 \odot G_2 \).

Let \( P_n \) be a path with \( V(P_n) = \{v_i/1 \leq i \leq n\} \). The comb graph is defined as \( G = P_n \odot K_1 \) with \( V(G) = \{v_i/1 \leq i \leq n\} \) and \( E(G) = \{v_iu_i/1 \leq i \leq n\} \) or \( \{v_iv_{i+1}/1 \leq i \leq n-1\} \).

**Lemma 3.1.** For the comb graph \( G = P_n \odot K_1 \) with \( n \geq 3 \),

(i) The singleton sets \( \{v_1\}, \{v_i\}, \{v_2\} \) and by symmetry \( \{u_n\}, \{v_n\}, \{v_{n-1}\} \) are minimal \( r_r \) sets.

(ii) The singleton set \( \{v_i\} \) with \( 3 \leq i \leq n-2 \) and \( \{u_i\} \) with \( 2 \leq i \leq n-1 \) are non \( r_r \) sets.

(iii) Any two element subset of \( H = \{v_i \cup u_j/3 \leq i \leq n-2, 2 \leq j \leq n-1, i \neq j\} \) of a comb graph \( G \), is a minimal \( r_r \) set with maximum cardinality.

**Proof.** The proof follows, since for any integer \( i \) with \( 1 \leq i \leq n \), we have

\[
d(v_i/v_j) = \begin{cases} 
\frac{2}{3} & \text{if } i = j = 1, \\
\frac{3}{4} & \text{if } i = j \text{ and } 2 \leq i \leq n - 1, \\
|i - j| & \text{if } i = 1, n \text{ and } j \neq i, \\
|i - j| + \frac{1}{4} & \text{if } 2 \leq i, j \leq n - 1 \text{ and } i \neq j.
\end{cases}
\]

\[
d(u_i/u_j) = \begin{cases} 
\frac{1}{2} & \text{if } i = j, \\
\frac{5}{2} & \text{if } |i - j| = 1, \\
|i - j| + \frac{3}{2} & \text{if } |i - j| \geq 2.
\end{cases}
\]
Theorem 3.1. For a comb graph $G$ with $n \geq 3$,

(i) $l_r(G) = 1, u_r(G) = 2$.

(ii) $l_r^*(G) = 1, u_r^*(G) = 2$.

(iii) $l_{R_r}(G) = u_{R_r}(G) = 2n - 2$.

(iv) $l_{R_r^*}(G) = u_{R_r^*}(G) = 0$.

Proof. Let $G = P_n \odot K_1$.

(i) From Lemma 3.1, The singleton set $\{u_1\}$ of $V(G)$ is a minimal $r_r$ set with minimum cardinality and any two element subset of $H = \{v_i \cup u_j \mid 3 \leq i \leq n - 2, 2 \leq j \leq n - 1, i \neq j\}$ of $V(G)$, is a minimal $r_r$ set with maximum cardinality. Therefore, $l_{r_r}(G) = 1$ and $u_{r_r}(G) = 2$.

(ii) $S = \{u_1\}$ is an $r_r$ set and $\bar{S} = \{V - S\}$ is also an $r_r$ set as it contain the end vertex $u_n$. Hence, $S$ is an $r_r^*$ set with minimum cardinality. Therefore, $l_{r_r^*}(G) = 1$.

From Lemma 3.1, for any two element subset of $S = \{v_i \cup u_j \mid 3 \leq i \leq n - 2, 2 \leq j \leq n - 1, i \neq j\}$ of $V(G)$, $\bar{S}$ contain an end vertex and Hence, is an $r_r$ set. Therefore, $S$ is an $r_r^*$ set with maximum cardinality and $u_{r_r^*}(G) = 2$.

(iii) From Lemma 3.1, every two element subset $\{v_i \cup u_j \mid 3 \leq i \leq n - 2\}$ is a non $r_r$ set and every $k$-element subset of $V(G)$ for $k \geq 3$ is an $r_r$ set.
Hence, \( S \subseteq V(G) \) is a minimal \( R_r \) set, only if \( S = \{v_i \cup u_i \mid 3 \leq i \leq n-2\} \). Therefore, \( l_{R_r}(G) = u_{R_r}(G) = 2n - 2 \).

(iv) For any \( k \)-element subset \( S \) of \( V(G) \) with \( 1 \leq k < n - 1 \), either \( S \) or \( V - S \) contain atleast one end vertex which imply either \( S \) or \( V - S \) is always an \( R_r \) set. Therefore, there exists no \( R_r^* \) set for \( G \) and Hence, \( l_{R_r^*}(G) = u_{R_r^*}(G) = 0 \).

Hence the proof. \( \square \)

Let \( C_n \) be the cycle with vertex set \( V(C_n) = \{v_i/1 \leq i \leq n\} \). The sun(sunlet) graph is defined as \( C_n \odot K_1 \). Let \( G = C_n \odot K_1, n \geq 3 \) with \( V(C_n \odot K_1) = \{v_i \cup u_i/1 \leq i \leq n\} \), \( E(C_n \odot K_1) = E(C_n) \cup \{v_iu_i/1 \leq i \leq n\} \).

**Lemma 3.2.** For the sunlet graph \( G \), \( rmd(G) = 2 \).

**Proof.** From the vertex \( v_i \) or \( u_i \) with \( 1 \leq i \leq n \), the rational metric distance of \( u_j \) (or \( v_j \)) which are at equidistant is the same. Hence, minimum of two vertices are required to rational resolve and except diagonally opposite or \( \{u_i, v_i\} \), rational resolves \( G \). Therefore, \( rmd(G) = 2 \), which leads to the following theorem. \( \square \)

**Theorem 3.2.** For the sunlet graph \( G \),

(i) \( l_{r_r}(G) = u_{r_r}(G) = 2 \).

(ii) \( l_{r_r^*}(G) = u_{r_r^*}(G) = 2 \).

(iii) \( l_{R_r}(G) = u_{R_r}(G) = \begin{cases} 2n - 2 & \text{if } n \text{ is odd or } n = 4. \\ 2n - 4 & \text{if } n \text{ is even} \end{cases} \)

(iv) \( l_{R_r^*}(G) = u_{R_r^*}(G) = \begin{cases} 4 & \text{if } n = 4. \\ 0 & \text{if } n \neq 4. \end{cases} \)

**Lemma 3.3.** For \( G = K_n \odot K_1, n \geq 4 \), \( rmd(G) = n - 1 \).

**Proof.** Let \( V = \{v_i \cup u_i/1 \leq i \leq n\} \) be the vertex set and \( E(G) = E(K_n) \cup \{v_iu_i/1 \leq i \leq n\} \) be the edge set of \( G \). With respect to \( u_1, d(v_i/u_1) \) remains same and \( d(u_i/u_1) \) remains same for every \( i \) with \( 2 \leq i \leq n \). Also with respect to \( v_1, d(v_i/v_1) \) remains same and \( d(u_i/v_1) \) remains same for every \( i \) with \( 2 \leq i \leq n \). Hence, minimum \( n - 1 \) vertices are required to rational resolve \( G \). Therefore, \( rmd(G) = n - 1 \), which leads to the following theorem. \( \square \)

**Theorem 3.3.** For \( G = K_n \odot K_1 \),

(i) \( l_{r_r}(G) = u_{r_r}(G) = n - 1 \).
Proof. Consider the corona product

\[ G \odot K_m \]

pendent edges and let

\[ \ell^*_r(G) = u_r^*(G) = n - 1. \]

\[ \ell_R_r(G) = u_{R_r}(G) = n + 2. \]

\[ \ell_{R_r^*}(G) = u_{R_r^*}(G) = 0. \]

Lemma 3.4. For any graph \( G \) of order \( n \), the corona product \( G \odot K_m \) with \( n \geq 2, m > 1 \), \( \text{rmd}(G \odot K_m) = n(m - 1) \).

Proof. Let \( G \) be a graph of order \( n \) with the vertex set \( V(G) = \{v_1, v_2, v_3, ..., v_n\} \). By the definition of corona product \( G \odot K_m \), a vertex \( v_i \) of \( G \) is attached with \( m \) pendent edges and let \( u_{i1}, u_{i2}, u_{i3}, ..., u_{im} \) be the associated pendent vertices for every \( i \) with \( 1 \leq i \leq n \). Then the subgraph induced by the vertex set \( \{v_1, u_{i1}, u_{i2}, ..., u_{im}\} \) is a star graph with \( v_i \) as the central vertex and hence, \( G \odot K_m \) has \( n \) copies of induced star graph \( K_{1,m} \). For a star graph \( K_{1,m} \), \( m - 1 \) pendent vertices are required to rational resolve and any \((m - 1)\) element subset \( S \) of \( V(K_{1,m}) \) containing only pendent vertices is a minimal \( r_r \) set. Hence, \( m - 1 \) vertices from each copy of \( K_{1,m} \) are required and also sufficient to rational resolve \( G \odot K_m \). Therefore, \( \text{rmd}(G \odot K_m) = n(m - 1) \).

Theorem 3.4. For \( G \odot K_m \) with \( n \geq 3, m > 1 \),

(i) \( \ell_r(G \odot K_m) = u_r(G \odot K_m) = n(m - 1) \).

(ii) \( \ell_r^*(G \odot K_m) = u_r^*(G \odot K_m) = 0. \)

(iii) \( \ell_{R_r}(G \odot K_m) = u_{R_r}(G \odot K_m) = n(m - 1). \)

(iv) \( \ell_{R_r^*}(G \odot K_m) = u_{R_r^*}(G \odot K_m) = 2. \)

Proof. Consider the corona product \( G \odot K_m \).

(i) From Lemma 3.4, set \( S \) containing \( n(m - 1) \) elements of \( V(G \odot K_m) \) is a minimal \( r_r \) set with minimum cardinality, which imply \( \ell_r(G \odot K_m) = m - 1 \). Also for induced star subgraph \( K_{1,m} \) of \( G \odot K_m \), there exists no minimal \( r_r \) set with cardinality greater than \( m - 1 \) and hence, there exists no minimal \( r_r \) set with cardinality greater than \( n(m - 1) \) for \( G \odot K_m \). Therefore, \( u_r(G \odot K_m) = n(m - 1) \).

(ii) From each induced star subgraph \( K_{1,m} \) of \( G \odot K_m \), any \( r_r \) set of \( K_{1,m} \) must contain minimum \( m - 1 \) elements. Equivalently, any \( r_r \) set \( S \) of \( V(G \odot K_m) \), must contain \( n(m - 1) \) elements. Hence, for any \( r_r \) set \( S \) of \( V(G \odot K_m) \), \( S \) can not contain \( n(m - 1) \) elements. Therefore, there exists no \( r_r^* \) set for \( G \odot K_m \) and hence, \( \ell_r^*(G \odot K_m) = u_r^*(G \odot K_m) = 0. \)
(iii) Any \( r_r \) set of \( V(G \odot \bar{K}_m) \) must contain minimum \( n(m - 1) \) elements and hence, for any \( r_r \) set \( S, \bar{S} \) is not an \( r_r \) set. Also any \( r_r \) set of \( V(G \odot \bar{K}_m) \) with cardinality greater than \( n(m - 1) \) is not minimal. Therefore, any \( r_r \) set of \( V(G \odot \bar{K}_m) \) with \( n(m - 1) \) elements is a minimal \( R_r \) set with minimum and maximum cardinality which imply \( l_{R_r}(G \odot \bar{K}_m) = u_{R_r}(G \odot \bar{K}_m) = n(m - 1) \).

(iv) Any \( r_r \) set \( S \) of \( V(G \odot \bar{K}_m) \), has to contain minimum \( (m - 1) \) elements from each induced star subgraph. Hence, any 2-element subset \( S \) of \( V(G \odot \bar{K}_m) \) containing two pendent vertices from any one of the induced star subgraph is not an \( r_r \) set of \( V(G \odot \bar{K}_m) \). Also for such \( S, \bar{S} \) is not an \( r_r \) set of \( V(G \odot \bar{K}_m) \) as one of its induced star subgraph contain \( (m - 2) \) elements and hence, \( S \) is a minimal \( R_r^* \) set with minimum and maximum cardinality. Therefore, \( l_{R_r^*}(G \odot \bar{K}_m) = u_{R_r^*}(G \odot \bar{K}_m) = 2 \).

Hence the result. \( \square \)

**Corollary 3.1.** For \( C_n \odot \bar{K}_m \) with \( n \geq 3, m > 1 \),

(i) \( l_{r_r}(C_n \odot \bar{K}_m) = u_{r_r}(C_n \odot \bar{K}_m) = n(m - 1) \).

(ii) \( l_{r_r^*}(C_n \odot \bar{K}_m) = u_{r_r^*}(C_n \odot \bar{K}_m) = 0 \).

(iii) \( l_{R_r}(C_n \odot \bar{K}_m) = u_{R_r}(C_n \odot \bar{K}_m) = n(m - 1) \).

(iv) \( l_{R_r^*}(C_n \odot \bar{K}_m) = u_{R_r^*}(C_n \odot \bar{K}_m) = 2 \).
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DEPARTMENT OF MATHEMATICS
DR. AMBEDKAR INSTITUTE OF TECHNOLOGY, BENGALURU
AFFILIATED TO VISVESVARAYA TECHNOCLOGICAL UNIVERSITY, BELAGAVI
KARNATAKA STATE, INDIA
Email address: jayachatra@yahoo.co.in

DEPARTMENT OF MATHEMATICS
DR. AMBEDKAR INSTITUTE OF TECHNOLOGY, BENGALURU
AFFILIATED TO VISVESVARAYA TECHNOCLOGICAL UNIVERSITY, BELAGAVI
KARNATAKA STATE, INDIA
Email address: padmaprakash.mm@gmail.com