NOTE ON COEFFICIENTS FOR THE CLASS \( \mathcal{U}(\lambda) \)

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ABSTRACT. Let \( \mathcal{A} \) be the class of analytic function in the unit disc \( \mathbb{D} = \{ z \in \mathbb{C}, |z| < 1 \} \), normalized by \( f'(0) - 1 = f(0) = 0 \). For \( 0 < \lambda \leq 1 \), let \( \mathcal{U}(\lambda) = \{ f \in \mathcal{A}, \left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda, z \in \mathbb{D} \} \). The aim of the present article is to present some coefficients results for the class \( \mathcal{U}(\lambda) \). We give a new proof for a known results on the second coefficient \( a_2 \) (Corollary 2.1) and for the Koebe domain (Corollary 2.3) for the class \( \mathcal{U}(\lambda) \). At the end of the note a problem is proposed.

1. INTRODUCTION

Let \( \mathcal{A} \) be the class of analytic function in the unit disc \( \mathbb{D} = \{ z \in \mathbb{C}, |z| < 1 \} \), normalized by \( f'(0) - 1 = f(0) = 0 \). Let \( \mathcal{S} \) denote the set of functions in \( \mathcal{A} \) which are univalent. For general theory of univalent function see [2] and [5].

For \( 0 < \lambda \leq 1 \), let

\[
\mathcal{U}(\lambda) = \{ f \in \mathcal{A}, |U_f(z)| < \lambda, z \in \mathbb{D} \},
\]

where \( U_f(z) = \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \), for \( f \in \mathcal{A} \) with \( \frac{f(z)}{z} \neq 0 \) for \( z \in \mathbb{D} \). In [6], Nunokawa and Ozaki proved the inclusion \( \mathcal{U} := \mathcal{U}(1) \subset \mathcal{S} \). The classe \( \mathcal{U}(\lambda) \) has been extensively studied in the recent years. For more details concerning this classe, see [4, 7, 8] and references therein. In [8], the authors proved, among other results, that the class \( \mathcal{U}(\lambda) \) is preserved under some transformations such

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as rotations, dilation, conjugation and omitted-value transformations. They also proposed the following conjecture:

**Conjecture 1.** Suppose that \( f = z + \sum_{n=1}^{\infty} a_n z^n \in U(\lambda) \) for some \( 0 < \lambda < 1 \), then

\[
|a_n| \leq \sum_{k=0}^{n-1} \lambda^k, \quad n \geq 2.
\]

The conjecture 1 has been proved in the case \( n = 2 \) by Vasudevarao and Yanagihara ([10], Theorem 2.6). In [7], the authors proved (Theorem 3) the Conjecture 1, by using subordination techniques, for \( n = 2, 3 \) and 4.

The following theorem is due to Rogosinski ([2], p.192):

**Theorem 1.1** (Rogosinski’s Theorem). Let \( f(z) = \sum_{n=1}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=1}^{\infty} b_n z^n \) be analytic in \( D \), and suppose \( g \prec f \). Then

\[
\sum_{n=2}^{n} |b_k|^2 \leq \sum_{n=1}^{n} |a_k|^2, \quad n \geq 1.
\]

Here, "\( \prec \)" is the symbol of subordination: \( g \prec f \) means that

\[
g(z) = f(w(z)), \quad z \in D,
\]

for some analytic function \( w \) with \( |w(z)| \leq |z| \) for \( z \in D \). For \( f \in U(\lambda) \), we have the following relation of subordination ([8], Theorem 4):

\[
(1.1) \quad \frac{f(z)}{z} = 1 + \sum_{n=2}^{\infty} a_n z^{n-1} = \frac{1}{(1-z)(1-\lambda z)} = 1 + \sum_{n=1}^{\infty} \frac{1 - \lambda^{n+1}}{1 - \lambda} z^n, \quad 0 < \lambda < 1.
\]

2. **RESULTS**

**Theorem 2.1.** Let \( f = z + \sum_{n=1}^{\infty} a_n z^n \in U(\lambda) \) for some \( 0 < \lambda \leq 1 \). Then

\[
(2.1) \quad \sqrt{\sum_{k=2}^{n} |a_k|^2} \leq \sqrt{n-1} \sum_{k=0}^{n-1} \lambda^k.
\]

**Proof.** For \( \lambda = 1 \), the theorem is a consequence, according to the fact that \( U \subset S \), of the de Brange’s theorem.

For \( 0 < \lambda < 1 \), we have from (1.1), by applying the Rogosinskis’s theorem,

\[
(2.2) \quad \sum_{k=2}^{n} |a_k|^2 \leq \sum_{k=1}^{n-1} \left( \frac{1 - \lambda^{k+1}}{1 - \lambda} \right)^2.
\]
Since $0 < \lambda < 1$, we have

$$(2.3) \quad 1 - \lambda^{k+1} \leq 1 - \lambda^n, \text{ for } 1 \leq k \leq n - 1.$$ 

Taking (2.3) in (2.2), we obtain

$$\sum_{k=2}^{n} |a_k|^2 \leq (n - 1) \left( \frac{1 - \lambda^n}{1 - \lambda} \right)^2$$

which yields

$$\sqrt{\sum_{k=2}^{n} |a_k|^2} \leq \sqrt{n - 1} \frac{1 - \lambda^n}{1 - \lambda} = \sqrt{n - 1} \sum_{k=0}^{n-1} \lambda^k.$$ 

□

**Remark 2.1.** The theorem 2.1 is a necessary condition for the Conjecture 1 to be true.

**Corollary 2.1.** Let $f$ as in the Theorem 2.1. Then

$$(2.4) \quad |a_n| \leq \sqrt{n - 1} \sum_{k=0}^{n-1} \lambda^k, \text{ for } n \geq 2.$$ 

In particular, we have

$$(2.5) \quad |a_2| \leq 1 + \lambda.$$ 

**Proof.** (2.4) is an immediate consequence of (2.1). (2.5) follows from (2.4) applying to $n = 2$. □

As a second consequence of the Theorem 2.1, we have the following asymptotic behavior of the coefficient $a_n$:

**Corollary 2.2.** Let $f = z + \sum_{n=1}^{\infty} a_n z^n \in \mathcal{U}(\lambda)$ for some $0 < \lambda < 1$. Then

$$\lim_{n \to \infty} \frac{|a_n|}{n} = 0.$$ 

**Proof.** From (2.4), we have

$$(2.6) \quad \frac{|a_n|}{n} \leq \sqrt{\frac{n - 1}{n^2} \sum_{k=0}^{n-1} \lambda^k}, \text{ for } n \geq 2.$$ 

Since $0 < \lambda < 1$, we have $\lim_{n \to \infty} \sum_{k=0}^{n-1} \lambda^k = \frac{1}{1-\lambda}$. Thus the limit of the right side of (2.6) is 0. This gives the desired result. □
Remark 2.2. The Corollary 2.2 shows that all functions in $U(\lambda)$ have the same Hayman number which is zero.

In the following corollary we give, based on the upper bound 2.5 for the second coefficient, a new proof of a result due to Vasudevarao and Yanagihara ([10], Theorem 3.4) concerning the Koebe domain for the class $U(\lambda)$.

Corollary 2.3. Let $f$ be a function in $U(\lambda)$. Then the range of $f$ contains the open disc of center the origin and of radius $\frac{1}{2(1+\lambda)}$. The result is sharp.

Proof. Let $w$ be an omitted-value for $f$. Since $U(\lambda)$ is preserved by omitted-value transformation, then the function

$$F(z) = \frac{w f(z)}{w - f(z)}, \quad z \in \mathbb{D}$$

belongs to $U(\lambda)$. we have

$$F(z) = w f(z), \quad z \in \mathbb{D}$$  \hspace{1cm} (2.7)

Deriving twice the both sides of (2.7), we get, by a little calculation,

$$F''(z) (w - f(z)) - 2 f'(z) F'(z) - F(z) f''(z) = w f''(z), \quad z \in \mathbb{D},$$

which gives

$$w F''(0) - 2 = w f''(0).$$

This yields

$$1 = w(a_2(f) - a_2(F)).$$

Hence we obtain

$$1 \leq |w|(|a_2(f)| + |a_2(F)|).$$  \hspace{1cm} (2.8)

Now, since $f, F \in U(\lambda)$, we obtain from (2.8) and (2.5) that

$$1 \leq 2|w|(1 + \lambda).$$

Hence, $|w| \geq \frac{1}{2(1+\lambda)}$. This gives the desired result.

For the sharpness of the result, let $f_\lambda = \frac{1}{(1-z)(1-\lambda z)}$. We have $f_\lambda \in U(\lambda)$ and a little calculation shows that the solutions of the equation

$$f_\lambda(z) = -\frac{1}{2(1 + \lambda)}$$
are -1 and $-\frac{1}{\lambda}$ which are out the disc $\mathbb{D}$. Thus $-\frac{1}{2(1+\lambda)}$ is an omitted value of $f_{\lambda}$ and hence the range of $f_{\lambda}$ does not contain a radius greater than $\frac{1}{2(1+\lambda)}$. This shows the sharpness. □

**Remark 2.3.** If $w$ is an omitted-value for a function $f \in \mathcal{U}(\lambda)$ then, by the Corollary 2.3, we have $2(1 + \lambda)|w| \geq 1$. Hence a necessary condition for the Conjecture 1 to be true for the coefficient $a_n$ of $f$ is that

$$|a_n| \leq 2(1 + \lambda)|w| \sum_{k=0}^{n-1} \lambda^k.$$ 

If $\lambda = 1$, the estimation above becomes

$$|a_n| \leq 4|w| n,$$

which is a particular case of the well known Littlewood’s Conjecture (see [3], p. 897) which is now a consequence of the de Branges’s Theorem [1].

The Remark 2.3 leads to propose the following problem which can be considered as the analogue, for the class $\mathcal{U}(\lambda)$, of the Littlewood’s Conjecture for the class $\mathcal{S}$ of univalent functions.

**Problem 1.** Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{U}(\lambda), \ 0 < \lambda \leq 1$. If $w$ is an omitted-value of $f$, then

$$|a_n| \leq 2(1 + \lambda)|w| \sum_{k=0}^{n-1} \lambda^k, \quad n \geq 2.$$ 

**References**


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