SOME RESULTS ON CYCLIC AND NEGACYCLIC CODES OVER FORMAL POWER SERIES RINGS AND FINITE CHAIN RINGS

MRIGANKA S. DUTTA¹ AND HELEN K. SAIKIA

ABSTRACT. In this article, relationship between cyclic codes of composite length $mn$ over formal power series ring and $u$–constacyclic code of length $m$ over $R_\infty[x]_{<x^n-1>}$ has been established by constructing an isomorphism. For two odd numbers $m$ and $n$, relationship between $u$–constacyclic code of length $m$ over $R_\infty[x]_{<x^n-1>}$ and $u$–constacyclic code of length $m$ over $R_\infty[x]_{<x^n+1>}$ has been obtained. The ideals of the rings $R_\infty[x]_{<x^n-1>}$ and $R_\infty[x]_{<x^n+1>}$ have also been determined.

1. INTRODUCTION

Due to the rich algebraic structure, cyclic codes play an important role in coding theory as seen in [1, 7]. Initially, the researchers studied the properties of Cyclic codes over the binary field $\mathbb{F}_2$, then they extended the study to $\mathbb{F}_q$ with $q = p^r$ for some prime $p$ and $r \geq 1$. The structure of cyclic codes was obtained by viewing a cyclic code $C$ of length $n$ over a finite field $\mathbb{F}_q$ as an ideal of the ring $R_\infty[x]_{<x^n-1>}$. Dinh and Lopez-Permouth [2] in the year 2004 published a paper on structure of cyclic and negacyclic codes over finite chain rings. Dougherty, Liu, and Park [5] in 2011 defined a series of finite chain rings and introduced the concept of $\gamma$–adic codes over formal power series rings. In 2011 Dougherty and Liu [4] have given the concept of $\lambda$–cyclic code of length $n$ over formal...
power series rings. They established a relation between cyclic codes and negacyclic codes over formal power series rings. They obtained a relation between cyclic codes over formal power series rings and cyclic codes over finite chain rings. Dougherty and Ling [3] in the year 2006 proved that a cyclic shift in \( \mathbb{Z}_2^{4n} \) corresponds to a \( u-\)constacyclic shift in \((\mathbb{Z}_4[u]_{u^2-1})^n\) by constructing a module isomorphism between \((\mathbb{Z}_4[u]_{u^2-1})^n\) and \(\mathbb{Z}_4^{4n}\). Dutta and Saikia [6] have introduced the concept of \( \Phi_\lambda \)-cyclic code of length \( n \) over a formal power series ring and derived some related results. Sobhani and Molakarimi [8] in the year 2013 constructed a one-to-one correspondence between cyclic codes of length \( 2n \) over the ring \( R_{k-1,m} \) and cyclic codes of length \( n \) over the ring \( R_{k,m} \) for odd \( n \) and determined the number of ideals of the ring \( R_{2,m} \) and \( R_{3,m} \). Hence in [8] they have obtained the number of cyclic codes of odd length over \( R_{2,m} \) and \( R_{3,m} \) as a corollary. In this article, we have constructed an isomorphism between \( \frac{R_{\infty}[u]}{<x^n-u>} \) and \( \frac{R_{\infty}[x]}{<x^{m-1}>} \) and proved that cyclic codes of composite length \( mn \) over the formal power series ring \( R_\infty \) corresponds to \( u-\)constacyclic code of length \( m \) over \( \frac{R_{\infty}[x]}{<x^{n-1}>} \). Here, considering both \( m \) and \( n \) as odd numbers we have proved that \( u-\)constacyclic codes of length \( m \) over \( \frac{R_{\infty}[x]}{<x^{n-1}>} \) corresponds to \( u-\)constacyclic code of length \( m \) over \( \frac{R_{\infty}[x]}{<x^{n+1}>} \). Thus corresponding to every cyclic code of odd length \( mn \) over \( R_\infty \) there exists a negacyclic code of same length over \( R_\infty \). Finally, we have also determined the types of ideals of the ring \( \frac{R_{\infty}[u]}{<x^n-u>} \) as well as the ring \( \frac{R_{\infty}[u]}{<x^n-1>} \) that will give us cyclic codes over \( R_\infty \) and \( R_i \) respectively.

2. **Finite Chain Ring and Formal Power Series Ring**

In this article, we assume that all rings are commutative with identity \( 1 \neq 0 \).

**Definition 2.1.** [4] Let \( R \) be a ring and \( I \) be an ideal of \( R \). \( I \) is called a principal ideal if it is generated by a singleton set.

**Definition 2.2.** [4] A finite ring is called a chain ring if all its ideals are linearly ordered by inclusion.

**Theorem 2.1.** [4] All the ideals of a finite chain ring are principal.

**Remark 2.1.** Let \( R \) be a finite chain ring. As \( R \) is finite, it must have finitely many ideals. Again \( R \) is a chain ring. Thus all the ideals of \( R \) must be linearly ordered.
by inclusion. Hence every finite chain ring \( R \) has a unique maximal ideal. Let \( I \) be the unique maximal ideal of \( R \). As all the ideals of \( R \) are principal, \( I \) must have some generator. Let \( \gamma \) be a generator of \( I \).

**Definition 2.3.** [4] Let \( i \) be an arbitrary positive integer and \( \mathbb{F} \) be a finite field. The ring \( R_i \) is a finite chain ring and is defined as

\[
R_i = \{ a_0 + a_1 \gamma + ... + a_{i-1} \gamma^{i-1} \mid a_i \in \mathbb{F} \},
\]

where \( \gamma^{i-1} \neq 0 \), but \( \gamma^i = 0 \) in \( R_i \). The operations over \( R_i \) are defined as follows:

\[
\sum_{l=0}^{i-1} a_l \gamma^l + \sum_{l=0}^{i-1} b_l \gamma^l = \sum_{l=0}^{i-1} (a_l + b_l) \gamma^l; \quad (\sum_{l=0}^{i-1} a_l \gamma^l)(\sum_{l=0}^{i-1} b_l \gamma^l) = \sum_{s=0}^{i-1} \left( \sum_{l+l'=s} a_l b_{l'} \right) \gamma^s.
\]

**Definition 2.4.** [4] The ring \( R_\infty \) is called a formal power series ring which is defined as

\[
R_\infty = \mathbb{F}[[\gamma]] = \{ \sum_{l=0}^\infty a_l \gamma^l \mid a_l \in \mathbb{F} \}.
\]

Addition and multiplication over \( R_\infty \) are defined by extending the addition and multiplication of polynomials, namely, term-by-term addition

\[
\sum_{l=0}^\infty a_l \gamma^l + \sum_{l=0}^\infty b_l \gamma^l = \sum_{l=0}^\infty (a_l + b_l) \gamma^l,
\]

and the Cauchy product

\[
(\sum_{l=0}^\infty a_l \gamma^l)(\sum_{l=0}^\infty b_l \gamma^l) = \sum_{s=0}^\infty \left( \sum_{l+l'=s} a_l b_{l'} \right) \gamma^s.
\]

**Lemma 2.1.** [4] If \( a \) and \( b \) are any two elements of \( R_\infty \) such that both not zero, then the greatest common divisor \( \gcd(a, b) \) exists.

**Corollary 2.1.** [4] If \( a_1, a_2, \ldots, a_n \in R_\infty \) such that \( a_j \neq 0 \) for some \( 0 \leq J \leq n \), then the greatest common divisor \( \gcd(a_1, a_2, \ldots, a_n) \) exists. If \( a_j \) is a unit for some \( j \), then, \( \gcd(a_1, a_2, \ldots, a_n) = 1 \).

**Definition 2.5.** [4] Let \( i, j \) be two integers with \( i \leq j \). In [4], the mapping \( \Psi_i^j \) is defined by

\[
\Psi_i^j : R_j \rightarrow R_i, \quad \sum_{l=0}^{j-1} a_l \gamma^l \mapsto \sum_{l=0}^{i-1} a_l \gamma^l.
\]
Definition 2.6. [4] Let $i$ be any positive integer. In [4], the mapping $\Psi_i$ is defined by
\[ \Psi_i : R_\infty \to R_i, \sum_{l=0}^{\infty} a_l \gamma^l \mapsto \sum_{l=0}^{i-1} a_l \gamma^l. \]
It can be proved that $\Psi_j^i$ and $\Psi_i$ are homomorphisms. We can extend $\Psi_j^i$ naturally from $R_j^n$ to $R_i^n$. Similarly $\Psi_i$ can be extended naturally from $R_\infty^n$ to $R_i^n$.

3. Polynomial Rings over $R_\infty$ and $R_i$

The polynomial ring over $R_\infty$ is given by
\[ R_\infty[x] = \{ a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid a_i \in R_\infty, \ n \geq 0 \}. \]
Since $R_\infty$ is a domain, $R_\infty[x]$ is also a domain [4]. We shall consider a polynomial
\[ f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in R_\infty[x]. \]
We can define the following mapping:
\[ \psi_j : R_\infty[x] \to R_j[x], \ f(x) \mapsto \psi_j(f(x)), \]
where
\[ \psi_j(f(x)) = \psi_j(a_0) + \psi_j(a_1) x + \cdots + \psi_j(a_n) x^n \in R_j[x]. \]
Thus by projecting the coefficients of the elements in $R_\infty[x]$ onto the coefficients of the elements in $R_j[x]$, we got the ring of polynomials over $R_j$ from the ring of polynomials over $R_\infty$ [4].

Again we shall consider
\[ f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in R_j[x]. \]
Now we can define a mapping as follows:
\[ \psi_i^j : R_j[x] \to R_i[x], \ f(x) \mapsto \psi_i^j(f(x)), \]
where
\[ \psi_i^j(f(x)) = \psi_i^j(a_0) + \psi_i^j(a_1) x + \cdots + \psi_i^j(a_n) x^n \in R_i[x]. \]

Definition 3.1. [4] If $f(x) \in R_\infty[x]$ such that $\deg(f(x)) > 0$ and $\gcd(a_1, a_2, \ldots, a_n) = 1$, then $f(x)$ is called a primitive element.
Lemma 3.1. [4] If \( f(x) \in R_\infty[x] \) such that \( \deg(f(x)) > 0 \), then \( f(x) \) is a primitive polynomial if \( f(\psi_i(f(x)) \neq 0 \forall i < \infty \).

Theorem 3.1. [4] If \( f(x) \in R_\infty[x] \) such that \( \deg(f(x)) > 0 \), then there exist a unique \( s \) and a primitive polynomial \( g(x) \), such that \( f(x) = \gamma^s g(x) \).

Definition 3.2. [4] If \( \langle f(x), g(x) \rangle + \langle x^n - \lambda \rangle = R_i[x] \), then the polynomials \( f(x), g(x) \in R_i[x] \) are called coprime, where \( i < \infty \) or equivalently, if there exists \( u(x), v(x) \in R_i[x] \) such that \( f(x)u(x) + g(x)v(x) = 1 \), then the polynomials \( f(x), g(x) \in R_i[x] \) are called coprime.

4. Linear, Cyclic and Negacyclic Codes

Definition 4.1. [4] Let \( R \) be a ring and \( R^n \) be the \( R \)-module. A \( R \)-submodule \( C \) of \( R^n \) is called a linear code of length \( n \) over \( R \).

Note that in this study all codes are linear.

Definition 4.2. [4] Let \( x, y \) be vectors in \( R^n \). The inner product of \( x \) and \( y \) is defined by
\[
[x, y] = x_1y_1 + x_2y_2 + \ldots + x_ny_n.
\]

Definition 4.3. [4] For a code \( C \) of length \( n \) over \( R \), the dual code of \( C \) is defined by
\[
C^\perp = \{ x \in R^n | [x, c] = 0, \forall c \in C \}.
\]

Remark 4.1. \( C^\perp \) is linear whether or not \( C \) is linear.

In our study \( p \) is the characteristic of the finite field \( \mathbb{F} \). Thus \( p \) is prime. We assume that \( n \) is relatively prime to \( p \).

Let \( \lambda \) be an arbitrary unit of \( R_\infty \) and let
\[
\frac{R_\infty[x]}{< x^n - \lambda >} = \{ f(x) + < x^n - \lambda > | f(x) \in R_\infty[x] \}
\]

Let
\[
f(x) + < x^n - \lambda >, g(x) + < x^n - \lambda > \in \frac{R_\infty[x]}{< x^n - \lambda >},
\]
such that \( 0 \leq \deg(f(x)), \deg(g(x)) < n \), and \( f(x) + < x^n - \lambda > = g(x) + < x^n - \lambda > \). Then, we have \( f(x) - g(x) \in < x^n - \lambda > \). Which implies that \( f(x) = g(x) \) as \( R_\infty \) is a domain. Hence, for each \( f(x) + < x^n - \lambda > \in \frac{R_\infty[x]}{< x^n - \lambda >} \), there is a unique
Let us define a mapping

\[ P_\lambda : R^n_\infty \rightarrow \frac{R_\infty[x]}{<x^n-\lambda>} \]

given by

\[ (a_0, a_1, \ldots, a_{n-1}) \mapsto a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + <x^n-\lambda>. \]

Putting \( \lambda = 1 \) and \( \lambda = -1 \) we get \( P_1 \) and \( P_{-1} \) as follows:

\[ P_1 : R^n_\infty \rightarrow \frac{R_\infty[x]}{<x^n-1>} \]

given by

\[ (a_0, a_1, \ldots, a_{n-1}) \mapsto a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + <x^n-1>, \]

and

\[ P_{-1} : R^n_\infty \rightarrow \frac{R_\infty[x]}{<x^n+1>} \]

given by

\[ (a_0, a_1, \ldots, a_{n-1}) \mapsto a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + <x^n-1>. \]

Let \( C \) be an arbitrary subset of \( R^n_\infty \). We denote the image of \( C \) under the map \( P_\lambda \) by \( P_\lambda(C) \). We use \( a(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} \) to denote the image of \( (a_0, a_1, \ldots, a_{n-1}) \) under \( P_\lambda, P_1 \) and \( P_{-1} \) respectively ([4]).

**Definition 4.4.** [4] Let \( C \) be a linear code of length \( n \) over \( R_\infty \). The code \( C \) is called a \( \lambda \)-cyclic code over \( R_\infty \), if

\[ c = (c_0, c_1, \ldots, c_{n-1}) \in C \implies (\lambda c_{n-1}, c_0, \ldots, c_{n-2}) \in C. \]

If \( \lambda = 1 \) then \( C \) is called a cyclic code and if \( \lambda = -1 \), then \( C \) is called a negacyclic code, otherwise, it is called a constacyclic code. Thus

\[ P_\lambda(C) = \{c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} + <x^n-\lambda> | (c_0, c_1, \ldots, c_{n-1}) \in C\}. \]

Now the following lemma can be easily proved.

**Lemma 4.1.** [4] A linear code \( C \) of length \( n \) over \( R_\infty \) is a \( \lambda \)-cyclic code if \( f \) \( P_\lambda(C) \) is an ideal of \( \frac{R_\infty[x]}{<x^n-\lambda>} \).
From Lemma 4.1 we get the following corollary:

**Corollary 4.1.** [4] Assuming the notations given above

(i) A linear code $C$ of length $n$ over $R_\infty$ is a cyclic code if $f \in P_1(C)$ is an ideal of $\frac{R_\infty[x]}{<x^n-1>}$, 

(ii) A linear code $C$ of length $n$ over $R_\infty$ is a negacyclic code if $f \in P_{-1}(C)$ is an ideal of $\frac{R_\infty[x]}{<x^n+1>}$. 

Let us consider the following ring homomorphism:

$$
\psi_i : \frac{R_\infty[x]}{<x^n-1>} \rightarrow \frac{R_i[x]}{<x^n-1>}
$$
given by

$$
f(x) \mapsto \psi_i(f(x)).
$$

Since $\psi_i$ is a ring homomorphism, therefore if $I$ is an ideal of $\frac{R_\infty[x]}{<x^n-1>}$, then $\psi_i(I)$ is an ideal of $\frac{R_i[x]}{<x^n-1>}$.

**Theorem 4.1.** [4] If $C$ is a cyclic code over $R_\infty$, then, $\psi_i(C)$ is a cyclic code over $R_i$ for all $i < \infty$.

Now we are going to establish an important result which is the central result of our present work. Let $F$ be a finite field and $p$ be the characteristic of $F$. Thus $p$ is a prime. Let $R_\infty = F[[\gamma]] = \{ \sum_{i=0}^{\infty} a_i \gamma^i | a_i \in F \}$ be the formal power series ring over $F$, where $\gamma$ is the indeterminate. Let $\lambda$ be an arbitrary unit of $R_\infty$. If we consider $m$ and $n$ to be two positive integers relatively prime to $p$, then we have the following result:

**Theorem 4.2.** Assuming the notations given above we have

$$
\frac{R_\infty[x]}{<u^n-\lambda>} \cong \frac{R_\infty[x]}{<x^{mn}-\lambda>}.
$$

**Proof.** Let us define a mapping $\Phi : \frac{R_\infty[x]}{<u^n-u>} \rightarrow \frac{R_\infty[x]}{<x^{mn}-\lambda>}$ given by

$$
\Phi \left( \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j} u^i \right) x^j \right) = \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j} (x^m)^i \right) x^j.
$$

Now for

$$
a_{0,0} + a_{0,1} x + \cdots + a_{0,m-1} x^{m-1} + a_{1,0} x^m + a_{1,1} x^{m+1} + \cdots + a_{1,m-1} x^{2m-1}
$$
there exists
\[
\sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j} u^i \right) x^j \in \frac{R_\infty[x]}{<x^{mn} >},
\]
such that
\[
\Phi \left( \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j} u^i \right) x^j \right) = \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j} x^m \right) x^j
\]
\[
= \sum_{j=0}^{m-1} \left( a_{0,j} x^0 + a_{1,j} x^m + \cdots + a_{n-1,j} x^{m(n-1)} \right) x^j
\]
\[
= a_{0,0} x + a_{0,1} x + \cdots + a_{0,m-1} x^{m-1} + a_{1,0} x^m + a_{1,1} x^{m+1} + \cdots + a_{1,m-1} x^{2m-1} + \cdots + a_{n-1,0} x^{m(n-1)} + \cdots + a_{n-1,m-1} x^{mn-1}
\]

Therefore the mapping \( \Phi \) is onto.

To prove \( \Phi \) is one-one, we take

\[
\Phi \left( \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j} u^i \right) x^j \right) = \Phi \left( \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} b_{i,j} u^i \right) x^j \right)
\]

\[
\Rightarrow \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} a_{i,j} x^m \right) x^j = \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} b_{i,j} x^m \right) x^j
\]

\[
\Rightarrow \sum_{j=0}^{m-1} \left( a_{0,j} x^0 + a_{1,j} x^m + \cdots + a_{n-1,j} x^{m(n-1)} \right) x^j
\]

\[
= \sum_{j=0}^{m-1} \left( b_{0,j} x^0 + b_{1,j} x^m + \cdots + b_{n-1,j} x^{m(n-1)} \right) x^j
\]

\[
\Rightarrow a_{0,0} + a_{0,1} x + \cdots + a_{0,m-1} x^{m-1} + a_{1,0} x^m + a_{1,1} x^{m+1} + \cdots + a_{1,m-1} x^{2m-1} + \cdots + a_{n-1,0} x^{m(n-1)} + \cdots + a_{n-1,m-1} x^{mn-1}
\]

\[
= b_{0,0} + b_{0,1} x + \cdots + b_{0,m-1} x^{m-1} + b_{1,0} x^m + b_{1,1} x^{m+1} + \cdots + b_{1,m-1} x^{2m-1} + \cdots + b_{n-1,0} x^{m(n-1)} + \cdots + b_{n-1,m-1} x^{mn-1}
\]
\[ a_{0,0} = b_{0,0}, a_{0,1} = b_{0,1}, \ldots, a_{n-1,m-1} = b_{n-1,m-1} \]

\[ \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} a_{i,j}u^j = \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} b_{i,j}u^j. \]

Thus \( \Phi \) is one-one and hence it is a bijection.

Now for

\[ \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} a_{i,j}u^j, \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} b_{i,j}u^j \in \frac{R_{\infty}[u]}{<u^m-u>} [x] \]

\[ \Phi(\sum_{j=0}^{m-1} \sum_{i=0}^{n-1} a_{i,j}u^j + \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} b_{i,j}u^j) = \Phi(\sum_{j=0}^{m-1} \sum_{i=0}^{n-1} (a_{i,j} + b_{i,j})u^j) \]

\[ \Rightarrow \Phi(\sum_{j=0}^{m-1} \sum_{i=0}^{n-1} a_{i,j}u^j + \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} b_{i,j}u^j) = \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} (a_{i,j} + b_{i,j})(x^m)^j u^j \]

\[ \Rightarrow \Phi(\sum_{j=0}^{m-1} \sum_{i=0}^{n-1} a_{i,j}u^j + \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} b_{i,j}u^j) = \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} a_{i,j}(x^m)^j u^j \]

\[ \Rightarrow \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} b_{i,j}(x^m)^j u^j \]

Hence \( \Phi \) preserves addition.

Let us consider

\[ a_{i,j}u^j, b_{r,s}u^s x^s \in \frac{R_{\infty}[u]}{<u^m-u>} [x]. \]

Now we have

\[ a_{i,j}u^j, b_{r,s}u^s x^s = a_{i,j}, b_{r,s}u^{i+r} x^{j+s} \in \frac{R_{\infty}[u]}{<u^m-u>} [x], \]

(4.1) \[ \Phi(a_{i,j}u^j), \Phi(b_{r,s}u^s x^s) = a_{i,j}, b_{r,s}x^{m(i+r) + j + s} \]

(4.2) \[ \Phi(a_{i,j}u^j, b_{r,s}u^s x^s) = \Phi(a_{i,j}, b_{r,s}u^{i+r} x^{j+s}) = a_{i,j}, b_{r,s}x^{m(i+r) + j + s}. \]
Hence from (4.1) and (4.2)
\[ \Phi(a_{i,j}u^i x^j \cdot b_{r,s} u^r x^s) = \Phi(a_{i,j}u^i x^j) \cdot \Phi(b_{r,s} u^r x^s). \]
This implies that \( \Phi \) preserves multiplication. Thus it is proved that \( \Phi \) is an isomorphism. Therefore
\[ \frac{R_\infty[u]}{<u^n - \lambda>} [x] \cong \frac{R_\infty[x]}{<x^m - \lambda>}. \]

Putting \( \lambda = 1 \) and \( \lambda = -1 \), we get the following two corollaries:

**Corollary 4.2.** Assuming the notations given above we have
\[ \frac{R_\infty[u]}{<u^n - 1>} [x] \cong \frac{R_\infty[x]}{<x^m - 1>}. \]

**Corollary 4.3.** Assuming the notations given above we have
\[ \frac{R_\infty[u]}{<u^n + 1>} [x] \cong \frac{R_\infty[x]}{<x^m + 1>}. \]

Thus we have established that cyclic codes of composite length \( mn \) over the formal power series ring \( R_\infty \) corresponds to \( u \)-constacyclic code of length \( m \) over \( \frac{R_\infty[u]}{<u^n - 1>} \). Similarly negacyclic codes of composite length \( mn \) over the formal power series ring \( R_\infty \) corresponds to \( u \)-constacyclic code of length \( m \) over \( \frac{R_\infty[u]}{<u^n + 1>} \).

**Theorem 4.3.** Assuming the notations given above we have
\[ \frac{R_i[u]}{<u^n - \lambda>} [x] \cong \frac{R_i[x]}{<x^m - \lambda>}. \]

**Proof.** The proof of this theorem is similar to the proof of the Theorem 4.1. \( \square \)

Putting \( \lambda = 1 \) and \( \lambda = -1 \), we get the following two corollaries:

**Corollary 4.4.** Assuming the notations given above we have
\[ \frac{R_i[u]}{<u^n - 1>} [x] \cong \frac{R_i[x]}{<x^m - 1>}. \]

**Corollary 4.5.** Assuming the notations given above we have
\[ \frac{R_i[u]}{<u^n + 1>} [x] \cong \frac{R_i[x]}{<x^m + 1>}. \]
Theorem 4.4. Let $m$ and $n$ are two odd numbers and $\gcd(m, p) = 1, \gcd(n, p) = 1$. Then
\[
\frac{R_{\infty}[u]}{[x^{m-u}]^{<u^n-1>}} \cong \frac{R_{\infty}[u]}{[x^{m-u}]^{<u^n+1>}}.
\]

Proof. Since $m$ and $n$ both are odds, $mn$ is also odd. Again $\gcd(m, p) = 1$ and $\gcd(n, p) = 1$. Therefore $\gcd(mn, p) = 1$. We define the map
\[
\eta : \frac{R_{\infty}[x]}{[x^{mn+1}]} \rightarrow \frac{R_{\infty}[x]}{[x^{mn-1}]}
\]
given by
\[
f(x) + [x^{mn+1}] \mapsto -f(-x) + [x^{mn-1}].
\]
Now if
\[
f(x) + [x^{mn+1}] = g(x) + [x^{mn+1}],
\]
then we have
\[
f(x) - g(x) \in [x^{mn+1}].
\]
Therefore
\[
f(x) - g(x) = (x^{mn} + 1)q(x) \text{ for some } q(x)
\]
and
\[
f(-x) - g(-x) = ((-x)^{mn} + 1)q(-x) = (-x^{mn} + 1)q(-x)
\]
\[
= (x^{mn} - 1)(-q(-x)) \in [x^{mn} - 1].
\]
This implies that
\[
\eta(f(x) + [x^{mn+1}]) = f(-x) + [x^{mn} - 1] = g(-x) + [x^{mn} - 1]
\]
\[
= \eta(g(x) + [x^{mn} + 1]).
\]
Thus, the correspondence $\eta$ is a well-defined map. Now
\[
\eta((f(x) + [x^{mn} + 1]) + (g(x) + [x^{mn} + 1]))
\]
\[
= \eta((f(x) + g(x)) + [x^{mn} + 1]) = (f(-x) + g(-x)) + [x^{mn} - 1]
\]
\[
=f(-x) + [x^{mn} - 1] + g(-x) + [x^{mn} - 1]
\]
\[
= \eta(f(x) + [x^{mn} + 1]) + \eta(g(x) + [x^{mn} + 1]).
\]
Thus, $\eta$ preserves addition.
Again
\[ \eta((f(x) + <x^{mn} + 1>). (g(x) + <x^{mn} + 1>)) \]
\[ = \eta((f(x). g(x) + <x^{mn} + 1>) = (f(-x). g(-x)) + <x^{mn} - 1>) \]
\[ = \eta(f(x) + <x^{mn} + 1>). \eta(g(x) + <x^{mn} + 1>) \]

Thus \( \eta \) preserves multiplication.

For \( f(-x) + <x^{mn} - 1> \in \frac{R_\infty[x]}{<x^{mn} - 1>} \) there exists \( f(x) + <x^{mn} + 1> \in \frac{R_\infty[x]}{<x^{mn} + 1>} \) such that
\[ \eta(f(x) + <x^{mn} + 1>) = f(-x) + <x^{mn} - 1> \]

Hence \( \eta \) is onto.

Let
\[ \eta(f(x) + <x^{mn} + 1>) = \eta(g(x) + <x^{mn} + 1>) \]
\[ \implies f(-x) + <x^{mn} - 1> = g(-x) + <x^{mn} - 1> \]
\[ \implies f(x) + <x^{mn} - 1> = g(x) + <x^{mn} - 1> \]

(Replacing \( x \) by \(-x\) and since \( mn \) is odd)
\[ \implies f(x) + <x^{mn} + 1> = g(x) + <x^{mn} + 1> . \]

Hence \( \eta \) is bijective. Thus it is an isomorphism. Therefore
\[ \frac{R_\infty[x]}{<x^{mn} + 1>} \cong \frac{R_\infty[x]}{<x^{mn} - 1>} . \]

Because
\[ \frac{R_\infty[x]}{<x^{m-u}+u-1>} \cong \frac{R_\infty[x]}{<x^{m-u}>} \text{ and } \frac{R_\infty[u]}{<u^{n+1}>} \cong \frac{R_\infty[x]}{<x^{mn}+1>} . \]

Therefore
\[ \frac{R_\infty[u]}{<u^{n-1}>} \cong \frac{R_\infty[u]}{<x^{m-u}>} . \]

\[ \square \]

**Theorem 4.5.** A linear code \( C \) of length \( mn \) over \( R_\infty \) is a \( \lambda \)-cyclic code if and only if \( \Phi^{-1}(P_{\lambda}(C)) \) is an ideal of \( \frac{R_\infty[x]}{<x^{mn}-\lambda>} \).

**Proof.** From Lemma 4.1 we know that, a linear code \( C \) of length \( mn \) over \( R_\infty \) is a \( \lambda \)-cyclic code, if, and only if, \( P_{\lambda}(C) \) is an ideal of \( \frac{R_\infty[x]}{<x^{mn}-\lambda>} \). Again \( \Phi \) is an isomorphism between \( \frac{R_\infty[u]}{<u^{n-1}>} \) and \( \frac{R_\infty[x]}{<x^{mn}-\lambda>} \). Thus \( \Phi^{-1} \) is an isomorphism.
So \( \Phi^{-1}(P_{\lambda}(C)) \) is an ideal of \( \frac{R_{\infty}[x]}{<x^{m}-u>} \), if, and only if, \( (P_{\lambda}(C)) \) is an ideal of \( \frac{R_{\infty}[x]}{<x^{m}-\lambda>} \). Thus a linear code \( C \) of length \( mn \) over \( R_{\infty} \) is a \( \lambda \)-cyclic code if and only if \( \Phi^{-1}(P_{\lambda}(C)) \) is an ideal of \( \frac{R_{\infty}[x]}{<x^{m}-u>} \).

**Corollary 4.6.** Assuming the notations given above we have

(i) A linear code \( C \) of length \( mn \) over \( R_{\infty} \) is a cyclic code if and only if \( \Phi^{-1}(P_{1}(C)) \) is an ideal of \( \frac{R_{\infty}[x]}{<x^{m}-u>} \).

(ii) A linear code \( C \) of length \( mn \) over \( R_{\infty} \) is a negacyclic code if and only if \( \Phi^{-1}(P_{-1}(C)) \) is an ideal of \( \frac{R_{\infty}[x]}{<x^{m}-u>} \).

**Theorem 4.6.** If \( C \) is a cyclic code of length \( mn \) over \( R_{\infty} \), then \( \Phi^{-1}(\psi_{i}(P_{1}(C))) \) is an ideal of \( \frac{R_{i}[x]}{<x^{m}-u>} \).

**Proof.** From Theorem 4.1 we know that if \( C \) is a cyclic code over \( R_{\infty} \), then \( \psi_{i}(C) \) is a cyclic code over \( R_{i} \) for all \( i < \infty \). Thus if \( C \) is a cyclic code of length \( mn \) over \( R_{\infty} \) then \( \psi_{i}(P_{1}(C)) \) is an ideal of \( \frac{R_{i}[x]}{<x^{m}-1>} \). As \( \Phi \) is an isomorphism between \( \frac{R_{i}[x]}{<x^{m}-u>} \) and \( \frac{R_{i}[x]}{<x^{m}-1>} \), \( \Phi^{-1} \) is an isomorphism between \( \frac{R_{i}[x]}{<x^{m}-u>} \) and \( \frac{R_{i}[x]}{<x^{m}-1>} \). Hence \( \psi_{i}(P_{1}(C)) \) is an ideal of \( \frac{R_{i}[x]}{<x^{m}-1>} \) if and only if \( \Phi^{-1}(\psi_{i}(P_{1}(C))) \) is an ideal of \( \frac{R_{i}[x]}{<x^{m}-u>} \). Thus if \( C \) is a cyclic code of length \( mn \) over \( R_{\infty} \) then, \( \Phi^{-1}(\psi_{i}(P_{1}(C))) \) is an ideal of \( \frac{R_{i}[x]}{<x^{m}-u>} \).

5. **Conclusion**

In [4] Dougherty and Liu proved that corresponding to every cyclic code of odd length \( n \) over \( R_{\infty} \) there exists a negacyclic code of same length over \( R_{\infty} \). Here we have considered both \( m \) and \( n \) as odd numbers and proved that \( u \)-constacyclic codes of length \( m \) over \( \frac{R_{\infty}[x]}{<x^{n+1}>} \) corresponds to \( u \)-constacyclic code of length \( m \) over \( \frac{R_{\infty}[x]}{<x^{n+1}>} \). Neither a counter example have been found to disprove that \( u \)-constacyclic codes of length \( m \) over \( \frac{R_{\infty}[x]}{<x^{n+1}>} \) corresponds to \( u \)-constacyclic code of length \( m \) over \( \frac{R_{\infty}[x]}{<x^{n+1}>} \), nor any isomorphism has been constructed between \( \frac{R_{\infty}[x]}{<x^{m}-u>} \) and \( \frac{R_{\infty}[x]}{<x^{m}-u>} \) to prove that \( u \)-constacyclic codes of length \( m \) over \( \frac{R_{\infty}[x]}{<x^{n+1}>} \) corresponds to \( u \)-constacyclic codes of length \( m \) over \( \frac{R_{\infty}[x]}{<x^{n+1}>} \), when at least one of \( m \) or \( n \) is even. Hence still the problem whether
\[ \frac{\mathbb{R}_\infty[u]}{\langle x^m - u \rangle} \] is isomorphic to \[ \frac{\mathbb{R}_\infty[u]}{\langle x^n - u \rangle} \] or not is unsolved, when at least one of \( m \) or \( n \) is even.

REFERENCES


DEPARTMENT OF MATHEMATICS
NALBARI COLLEGE
NALBARI, PIN-781335, INDIA
Email address: dutta.mriganka82@gmail.com

DEPARTMENT OF MATHEMATICS
GAUHATI UNIVERSITY
GUWAHATI, PIN-781014, INDIA
Email address: hsaikia@yahoo.com