DIMENSIONAL FORMULÆ FOR THE RELATIVE INVARIANTS OF PRE-MATHIEU AND MATHIEU GROUPS

P. VANCHINATHAN, S. RADHA, AND SANJIT DAS

ABSTRACT. In this paper, we have computed the dimensions of the relative symmetric polynomials associated with all the irreducible characters of Pre Mathieu groups \(M_9, M_{10}\) and Small Mathieu groups \(M_{11}, M_{12}\).

1. INTRODUCTION

The theory of symmetric polynomials is one of the most classical part of algebra and it plays an important role in the branches of algebra like Representation theory, Galois theory and Algebraic Combinatorics. A symmetric polynomial on \(n\) variables is a function that is unchanged by any permutation of its variables (ie) a polynomial \(f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})\) for any \(\sigma \in S_n\). Analogous to symmetric polynomials, anti-symmetric polynomials are not invariant under the action of \(S_n\) but they change sign according to the sign of the permutation (ie) \(f(x_1, x_2, \ldots, x_n) = \xi(\sigma) f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})\) where \(\xi(\sigma)\) denotes the sign of the permutation. M. Shahryari [3] introduced the notion of Relative Symmetric polynomials as a generalization of symmetric and anti-symmetric polynomials.

Let \(G\) be a subgroup of the symmetric group \(S_n\) and \(\chi\) be an irreducible character of \(G\). Let \(H_d(x_1, x_2, \ldots, x_n)\) denote the complex vector space of homogeneous polynomial of degree \(d\) in \(n\) variables. The image of the Reynold’s operator on

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$H_d(x_1, x_2, \ldots, x_n)$ will be the vector space of all symmetric polynomials of degree $d$. Let $\chi_0$ denote the trivial irreducible character of $G$. Then the Reynolds operator can alternatively be expressed as

$$\frac{\chi_0(1)}{|G|} \sum_{g \in G} \chi_0(g)g$$

This reformulation allows straightforward generalization to other characters.

**Definition 1.1.** For an irreducible character $\chi$ of $G(\subset S_n)$, the following operator $T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g)g$ is a projection on $H_d(x_1, x_2, \ldots, x_n)$ for every $d$. The image of this projection operator denoted by $H_d(G, \chi)$, is defined to be the vector space of relative symmetric polynomials, or relative invariants, of $G$ with respect to its irreducible character $\chi$.

Dimensions of the vector space of relative symmetric polynomials for various subgroups of the symmetric group $S_n$ like $S_n, A_n$, Young Subgroups [4] were found by Babaei, Zamani and Shahryari. Later in a series of papers Babaei and Zamani gave the formulæ for the cyclic group [5], dicyclic group [6] and dihedral group [7]. Radha S and Vanchinathan P [2] gave an alternative dimensional formulæ for the relative invariants of the dihedral group by using theorems from number theory and combinatorial arguments. They have also constructed generating functions for the dimensional formulæ. As an application, a specific Supercharacter theory for the dihedral group was also constructed.

In this paper we focus on Mathieu groups $M_{11}, M_{12}$. They are the first two of five sporadic simple groups discovered by Mathieu in 1884. They are multiply transitive permutation groups of degree 11 and 12 respectively. The groups $M_9, M_{10}$ which act transitively on sets 9 and 10 elements respectively are subgroups of the symmetric groups $S_9, S_{10}$. They are not sporadic simple groups but they are used to construct the Mathieu groups, by a process known as one-point extension. In this paper, we have determined the dimensions of the vector spaces of relative invariants for Pre-Mathieu groups $M_9, M_{10}$ and also for the Small Mathieu groups $M_{11}, M_{12}$. A notable feature is that all the dimensional formulæ appear as summations of binomial coefficients.
2. Dimensional Formulae for the Pre-Mathieu Groups $M_9$ and $M_{10}$

2.1. **Pre-Mathieu group** $M_9$. There are 6 conjugacy classes and hence 6 irreducible characters for $M_9$. The character table for $M_9$ [1] is given below:

<table>
<thead>
<tr>
<th>$M_9$</th>
<th>$1^9$</th>
<th>$2^4$</th>
<th>$3^3$</th>
<th>$4_{A}^2$</th>
<th>$4_{B}^2$</th>
<th>$4_{C}^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>2</td>
<td>-2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>8</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Here we state the first result of our paper. The dimensional formula for the relative invariants of the Pre-Mathieu group $M_9$ is given in the theorem 2.1 below.

**Theorem 2.1.** Let $\chi_0, \chi_1, \chi_2, \chi_3, \chi_4, \chi_5$ be the irreducible characters of $M_9$ with degrees $1, 1, 1, 1, 2, 8$ respectively. Then the dimensions of $H_d(M_9, \chi_i)$ for $i = 0, 1, 2, 3, 4, 5$, the vector spaces of relative symmetric polynomials with respect to all irreducible characters of the Pre-Mathieu group $M_9$ is given below:

1. $\dim(H_d(M_9, \chi_0)) = f_{11}(d) + 9f_{12}(d) + 8f_{13}(d) + 18f_{14}(d) + 18f_{14}(d) + 18f_{14}(d) = f_{11}(d) + 9f_{12}(d) + 8f_{13}(d) + 54f_{14}(d)$
2. $\dim(H_d(M_9, \chi_1)) = f_{11}(d) + 9f_{12}(d) + 8f_{13}(d) + 18f_{14}(d) + 18f_{14}(d) + 18f_{14}(d) = f_{11}(d) + 9f_{12}(d) + 8f_{13}(d) + 18f_{14}(d)$
3. $\dim(H_d(M_9, \chi_2)) = f_{11}(d) + 9f_{12}(d) + 8f_{13}(d) + 18f_{14}(d)$
4. $\dim(H_d(M_9, \chi_3)) = f_{11}(d) + 9f_{12}(d) + 8f_{13}(d) + 18f_{14}(d)$
5. $\dim(H_d(M_9, \chi_4)) = 2f_{11}(d) + 18f_{12}(d) + 16f_{13}(d)$
6. $\dim(H_d(M_9, \chi_5)) = 8f_{11}(d) - 8f_{13}(d)$

where $f_{11}(d), f_{12}(d), f_{13}(d)$ and $f_{14}(d)$ respectively denote the number of monomials fixed by the identity permutation, permutations of the type $2^4, 3^3$ and $4^2$. 

- $f_{11}(d) = \binom{d+6}{8}$
- $f_{12}(d) = \sum_{t=0}^{[d/2]} \binom{t+3}{3} = \frac{1}{4}([d/2]^4 + 10[d/2]^3 + 35[d/2]^2 + 50[d/2] + 24)$
- $f_{13}(d) = \binom{2^4}{d} = \frac{1}{18}(d + 3)(d + 6)$
- $f_{14}(d) = \sum_{t=0}^{[d/4]} (t + 1) = \frac{1}{2}([d/4] + 1)([d/4] + 2)$
Now derivation of \( f_{12}(d) \) is given here. For a monomial to be invariant under the action of the permutation of type \( 2^4 + 1 \) say \((12)(34)(56)(78)(9)\), the variables \( x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \) should assume degrees \( d_1, d_1, d_2, d_2, d_3, d_3, d_4, d_4, d_5 \) respectively. But sum of all the degrees is \( d \). Hence we have \( d_1 + d_1 + d_2 + d_2 + d_3 + d_3 + d_4 + d_4 + d_5 = d \). (ie) \( 2(d_1 + d_2 + d_3 + d_4) + d_5 = d \). Hence, \( d_1 + d_2 + d_3 + d_4 = \frac{d - d_5}{2} \).

Put \( \frac{d - d_5}{2} = l \). Now \( d_1 + d_2 + d_3 + d_4 = l \) denotes the number of partitions of \( l \) into 4 parts with \( l \) ranging from 0 to \( \lfloor d/2 \rfloor \). Hence the number of monomials to be invariant under the action of \( 2^4 + 1 \) is given by \( f_{12}(d) = \sum_{l=0}^{\lfloor d/2 \rfloor} (l+3)^3 \).

The derivations of other formulæ are easy and hence omitted.

2.2. Pre Mathieu group \( M_{10} \). There are 8 conjugacy classes for \( M_{10} \) and hence 8 irreducible characters. The character table for \( M_{10} \) [1] is given below:

<table>
<thead>
<tr>
<th>( M_{10} )</th>
<th>1^{10}</th>
<th>2^4</th>
<th>3^3</th>
<th>4_A^2</th>
<th>4_B^2</th>
<th>5^2</th>
<th>2^18_A^1</th>
<th>2^18_B^1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi_0 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( \chi_2 )</td>
<td>9</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( \chi_3 )</td>
<td>9</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_4 )</td>
<td>10</td>
<td>2</td>
<td>1</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_5 )</td>
<td>10</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \omega )</td>
<td>( \bar{\omega} )</td>
</tr>
<tr>
<td>( \chi_6 )</td>
<td>10</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \bar{\omega} )</td>
<td>( \omega )</td>
</tr>
<tr>
<td>( \chi_7 )</td>
<td>16</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Here we state the second result of our paper. The dimensional formulæ for the relative invariants of the Pre-Mathieu group \( M_{10} \) is given in the theorem 2.2 below.

**Theorem 2.2.** Let \( \chi_0, \chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7 \) denote the irreducible characters of \( M_{10} \) with degrees 1, 1, 9, 9, 10, 10, 10, 16 respectively. Then the dimensions of \( H_d(M_{10}, \chi_i) \) for \( i = 0, 1, 2, 3, 4, 5, 6, 7 \), the vector spaces of relative symmetric polynomials with respect to all irreducible characters of the Pre-Mathieu group \( M_{10} \) are given below:

\[
(1) \quad \dim(H_d(M_{10}, \chi_0)) = f_{21}(d) + 45f_{22}(d) + 80f_{23}(d) + 90f_{24}(d) + 180f_{25}(d) + 144f_{25}(d) + 90f_{26}(d) + 90f_{26}(d)
\]

\[
= f_{21}(d) + 45f_{22}(d) + 80f_{23}(d) + 270f_{24}(d) + 144f_{25}(d) + 180f_{26}(d)
\]
\[(2) \quad \dim(H_d(M_{10}, \chi_1)) = f_{21}(d) + 45f_{22}(d) + 80f_{23}(d) + 90f_{24}(d) - 180f_{25}(d) + 144f_{26}(d) - 90f_{26}(d) - 90f_{26}(d) = f_{21}(d) + 45f_{22}(d) + 80f_{23}(d) - 90f_{24}(d) + 144f_{25}(d) - 180f_{26}(d)\]

\[(3) \quad \dim(H_d(M_{10}, \chi_2)) = 9f_{21}(d) + 45f_{22}(d) + 90f_{24}(d) + 180f_{24}(d) - 144f_{25}(d) - 90f_{26}(d) = 9f_{21}(d) + 45f_{22}(d) + 270f_{24}(d) - 144f_{25}(d) - 180f_{26}(d)\]

\[(4) \quad \dim(H_d(M_{10}, \chi_3)) = 9f_{21}(d) + 45f_{22}(d) + 90f_{24}(d) - 180f_{24}(d) - 144f_{25}(d) + 90f_{26}(d) + 90f_{26}(d) = 9f_{21}(d) + 45f_{22}(d) - 90f_{24}(d) - 144f_{25}(d) + 180f_{26}(d)\]

\[(5) \quad \dim(H_d(M_{10}, \chi_4)) = 10f_{21}(d) + 90f_{22}(d) + 80f_{23}(d) - 180f_{24}(d)\]

\[(6) \quad \dim(H_d(M_{10}, \chi_5)) = 10f_{21}(d) - 90f_{22}(d) + 80f_{23}(d) + 90\omega f_{26}(d) + 90\overline{\omega} f_{26}(d) = 10f_{21}(d) - 90f_{22}(d) + 80f_{23}(d)\]

\[(7) \quad \dim(H_d(M_{10}, \chi_6)) = 10f_{21}(d) - 90f_{22}(d) + 80f_{23}(d) + 90\overline{\omega} f_{26}(d) + 90\omega f_{26}(d) = 10f_{21}(d) - 90f_{22}(d) + 80f_{23}(d)\]

\[(8) \quad \dim(H_d(M_{10}, \chi_7)) = 16f_{21}(d) - 160f_{23}(d) + 144f_{25}(d)\]

where

- \(f_{21}(d) = \left(\frac{d+9}{9}\right)\)
- \(f_{22}(d) = \sum_{l=0}^{\lfloor d/2\rfloor} \left(\frac{d+3}{3}\right)(d-2l+1)\)
- \(f_{23}(d) = \sum_{l=0}^{\lfloor d/3\rfloor} \left(\frac{d+2}{2}\right) = \frac{1}{6}[[d/3] + 1)(d/3 + 1)]\)
- \(f_{24}(d) = \sum_{l=0}^{\lfloor d/4\rfloor} (l+1)(d-4l+1) = \frac{1}{6}[[d/4] + 1)(3d(d/4) - 8(d/4)]^2 - 13[d/4] + 6d + 6]\)
- \(f_{25}(d) = \left(\frac{d+1}{1}\right) = \begin{cases} \frac{d}{5} + 1 & \text{when } d \text{ is a multiple of } 5 \\ \frac{d}{5} & \text{otherwise} \end{cases}\)
- \(f_{26}(d) = \begin{cases} \frac{d}{5} + 1 & \text{when } d \text{ is even} \\ \frac{d}{5} & \text{otherwise} \end{cases}\)

The derivation of the formula \(f_{22}(d)\) is given here. For a monomial to be invariant under the action of the permutation of type \(2^4 \times 1^2\) say \((12)(34)(56)(78)(9)(10)\), the variables \(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\) should assume degrees \(d_1, d_2, d_3, d_3, d_4, d_4, d_5, d_6\) respectively. But sum of all the degrees is \(d\).

Hence we have \(d_1 + d_2 + d_2 + d_2 + d_3 + d_3 + d_4 + d_4 + d_5 + d_6 = d\). (ie) \(2(d_1 + d_2 + d_3 + d_4) + d_5 + d_6 = d\). Hence, \(d_1 + d_2 + d_3 + d_4 = \frac{d - d_5 - d_6}{2}\). Put \(d - d_5 - d_6 = l\). Now \(d_1 + d_2 + d_3 + d_4 = l\) denotes the number of partitions of \(l\) into 4 parts with \(l\) ranging from 0 to \(\lfloor d/2\rfloor\) and simultaneously dividing \(d_5 + d_6 = d - 2l\).
into two parts. Hence the number of monomials to be invariant under the action of \(2^4 + 1\) is given by
\[
f_{22}(d) = \sum_{l=0}^{\lfloor d/2 \rfloor} \binom{l+3}{3}(d - 2l + 1)
\]
The derivations of other formulæ are easy and hence omitted.

3. Dimensional Formulæ for the Small Mathieu Groups \(M_{11}\) and \(M_{12}\)

3.1. Small Mathieu group \(M_{11}\). There are 10 conjugacy classes and hence 10 irreducible characters for \(M_{11}\). The character table for \(M_{11}\) [1] is given below:

| \(M_{11}\) | \(1^{11}\) | \(2^4\) | \(3^3\) | \(4^2\) | \(5^2\) | \(2^13^16^1\) | \(2^18^1_A\) | \(2^18^1_B\) | \(11^1_A\) | \(11^1_B\) |
|---|---|---|---|---|---|---|---|---|---|
| \(\chi_0\) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \(\chi_1\) | 10 | 2 | 1 | 2 | 0 | -1 | 0 | 0 | -1 | -1 |
| \(\chi_2\) | 10 | -2 | 1 | 0 | 0 | 1 | \(\alpha\) | \(\bar{\alpha}\) | -1 | -1 |
| \(\chi_3\) | 10 | -2 | 1 | 0 | 0 | 1 | \(\bar{\alpha}\) | \(\alpha\) | -1 | -1 |
| \(\chi_4\) | 11 | 3 | 2 | -1 | 1 | 0 | -1 | -1 | 0 | 0 |
| \(\chi_5\) | 16 | 0 | -2 | 0 | 1 | 0 | 0 | 0 | \(\beta\) | \(\bar{\beta}\) |
| \(\chi_6\) | 16 | 0 | -2 | 0 | 1 | 0 | 0 | 0 | \(\bar{\beta}\) | \(\beta\) |
| \(\chi_7\) | 44 | 4 | -1 | 0 | -1 | 1 | 0 | 0 | 0 | 0 |
| \(\chi_8\) | 45 | -3 | 0 | 1 | 0 | 0 | -1 | -1 | 1 | 1 |
| \(\chi_9\) | 55 | -1 | 1 | -1 | 0 | -1 | 1 | 1 | 0 | 0 |

where \(\alpha = \sqrt{-2}\) and \(\beta = \frac{1}{2}(-1 + \sqrt{-11})\).

Here we state the third result of our paper. The dimensional formulæ for the relative invariants of the Small Mathieu group \(M_{11}\) is given in the theorem 3.1 below.

**Theorem 3.1.** Let \(\chi_0, \chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7, \chi_8, \chi_9\) denote the irreducible characters of \(M_{11}\) with degrees 1, 10, 10, 10, 11, 16, 16, 44, 45, 55 respectively. Then the dimensions of \(H_d(M_{11}, \chi_i)\) for \(i = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\), the vector spaces of relative symmetric polynomials with respect to all irreducible characters of the Pre Mathieu group \(M_{11}\) are given below:

\[
\text{(1) } \dim(H_d(M_{11}, \chi_0)) = f_{31}(d) + 165f_{32}(d) + 440f_{33}(d) + 990f_{34}(d) + 1584f_{35}(d) + 1320f_{36}(d) + 1980f_{37}(d) + 1440f_{38}(d)
\]
(2) \( \text{dim}(H_d(M_{11}, \chi_1)) = 10f_{31}(d) + 330f_{32}(d) + 440f_{33}(d) + 1980f_{34}(d) - 1320f_{36}(d) - 1440f_{38}(d) \)

(3) \( \text{dim}(H_d(M_{11}, \chi_2)) = 10f_{31}(d) - 330f_{32}(d) + 440f_{33}(d) + 1320f_{36}(d) + 990\alpha f_{37}(d) + 990\alpha f_{37}(d) - 1440f_{38}(d) \)

(4) \( \text{dim}(H_d(M_{11}, \chi_3)) = 10f_{31}(d) - 330f_{32}(d) + 440f_{33}(d) + 1320f_{36}(d) + 990\alpha f_{37}(d) + 990\alpha f_{37}(d) - 1440f_{38}(d) \)

(5) \( \text{dim}(H_d(M_{11}, \chi_4)) = 11f_{31}(d) + 495f_{32}(d) + 880f_{33}(d) - 990f_{34}(d) + 1584f_{35}(d) - 1980f_{37}(d) \)

(6) \( \text{dim}(H_d(M_{11}, \chi_5)) = 16f_{31}(d) - 880f_{33}(d) + 1584f_{35}(d) + 720\beta f_{38}(d) + 720\beta f_{38}(d) = 16f_{31}(d) - 880f_{33}(d) + 1584f_{35}(d) - 720f_{38}(d) \)

(7) \( \text{dim}(H_d(M_{11}, \chi_6)) = 16f_{31}(d) - 880f_{33}(d) + 1584f_{35}(d) + 720\beta f_{38}(d) + 720\beta f_{38}(d) = 16f_{31}(d) - 880f_{33}(d) + 1584f_{35}(d) - 720f_{38}(d) \)

(8) \( \text{dim}(H_d(M_{11}, \chi_7)) = 44f_{31}(d) + 660f_{32}(d) - 440f_{33}(d) - 1584f_{35}(d) + 1320f_{36}(d) \)

(9) \( \text{dim}(H_d(M_{11}, \chi_8)) = 45f_{31}(d) - 495f_{32}(d) + 990f_{34}(d) - 1980f_{37}(d) + 1440f_{38}(d) \)

(10) \( \text{dim}(H_d(M_{11}, \chi_9)) = 55f_{31}(d) - 165f_{32}(d) + 440f_{33}(d) - 990f_{34}(d) - 1320f_{36}(d) + 1980f_{37}(d) \)

where

- \( f_{31}(d) = \binom{d+10}{10} \)
- \( f_{32}(d) = \sum_{l=0}^{\lfloor d/2 \rfloor} \binom{d+3}{3} \binom{d-2l+2}{2} \)
- \( f_{33}(d) = \sum_{l=0}^{\lfloor d/3 \rfloor} \binom{d+2}{2} (d - 3l + 1) \)
- \( f_{34}(d) = \sum_{l=0}^{\lfloor d/4 \rfloor} (l + 1) \binom{d-4l+2}{2} \)
- \( f_{35}(d) = \sum_{l=0}^{\lfloor d/5 \rfloor} (l + 1) = \frac{1}{2} \left[ \left\lfloor \frac{d}{5} \right\rfloor + 1 \right] \left( \left\lfloor \frac{d}{5} \right\rfloor + 2 \right) \)
- \( f_{36}(d) = \begin{cases} \sum_{l=0}^{\lfloor d/6 \rfloor} \left[ \left\lfloor \frac{d}{6} \right\rfloor + 1 \right] & \text{when } d \text{ is even} \\ \sum_{l=0}^{\lfloor d/6 \rfloor} \left[ \left\lfloor \frac{d}{6} \right\rfloor \div 3 \right] & \text{when } d \text{ is odd} \end{cases} \)
- \( f_{37}(d) = \begin{cases} 1 & \text{if } d \text{ is a multiple of } 11 \\ 0 & \text{elsewhere} \end{cases} \)
- \( f_{38}(d) = \begin{cases} 1 & \text{if } d \text{ is a multiple of } 11 \\ 0 & \text{elsewhere} \end{cases} \)

The derivation of the formula \( f_{33}(d) \) is given here. For a monomial to be invariant under the action of the permutation of type \( 3^3+1^2 \) say \( (123)(456)(789)(10)(11) \), the variables \( x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11} \) should assume degrees \( d_1, d_1, d_1, d_2, d_2, d_2, d_3, d_3, d_3, d_4, d_5 \) respectively. But sum of all the degrees is \( d \).
Hence we have $d_1 + d_1 + d_1 + d_2 + d_2 + d_2 + d_3 + d_3 + d_3 + d_4 + d_5 = d$. (ie) $3(d_1 + d_2 + d_3) + d_4 + d_5 = d$. Hence, $d_1 + d_2 + d_3 = \frac{d-d_4-d_5}{3}$. Put $\frac{d-d_4-d_5}{3} = l$. Now $d_1 + d_2 + d_3 = l$ denotes the number of partitions of $l$ into 3 parts with $l$ ranging from 0 to $\lfloor d/3 \rfloor$ and simultaneously dividing $d_4 + d_5 = d - 3l$ into two parts. Hence the number of monomials to be invariant under the action of $3^3 + 1^2$ is given by $f_{33}(d) = \sum_{l=0}^{\lfloor d/3 \rfloor} \binom{l+2}{2}(d - 3l + 1)$.

### 3.2. Small Mathieu group $M_{12}$. There are 15 conjugacy classes and hence 15 irreducible characters for $M_{12}$. The character table for $M_{12}$ [1] is given below:

<table>
<thead>
<tr>
<th>$M_{12}$</th>
<th>$1^{12}$</th>
<th>$2^4$</th>
<th>$2^6$</th>
<th>$3^4$</th>
<th>$3^4$</th>
<th>$2^2 2^4 2^2$</th>
<th>$2^2 3^4 6^4$</th>
<th>$2^2 8^4$</th>
<th>$4^8 1^1$</th>
<th>$2^2 10^1$</th>
<th>$11^1_1$</th>
<th>$11^1_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>11</td>
<td>3</td>
<td>-1</td>
<td>2</td>
<td>-1</td>
<td>3</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>11</td>
<td>3</td>
<td>-1</td>
<td>2</td>
<td>-1</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>16</td>
<td>4</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>$\omega$</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>16</td>
<td>4</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>$\omega$</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>45</td>
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<td>5</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_6$</td>
<td>54</td>
<td>6</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_7$</td>
<td>55</td>
<td>7</td>
<td>-5</td>
<td>1</td>
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<td>-1</td>
<td>-1</td>
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<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_8$</td>
<td>55</td>
<td>-5</td>
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<td>1</td>
<td>-1</td>
<td>3</td>
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<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_9$</td>
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<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_{10}$</td>
<td>66</td>
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<td>6</td>
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<td>0</td>
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<td>-2</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_{11}$</td>
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<td>3</td>
<td>-1</td>
<td>0</td>
<td>3</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_{12}$</td>
<td>120</td>
<td>-8</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_{13}$</td>
<td>144</td>
<td>4</td>
<td>0</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_{14}$</td>
<td>176</td>
<td>0</td>
<td>-4</td>
<td>-4</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

where $\omega = \frac{1}{2}(-1 + \sqrt{-11})$.

Here we state the fourth result of our paper. The dimensional formulæ for the relative invariants of the Small Mathieu group $M_{12}$ is given in the theorem 3.2 below.

**Theorem 3.2.** Let $\chi_0, \chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7, \chi_8, \chi_9, \chi_{10}, \chi_{11}, \chi_{12}, \chi_{13}, \chi_{14}$ denote the irreducible characters of $M_{12}$ with degrees 1, 11, 11, 16, 16, 45, 54, 55, 55, 55, 66, 99, 120, 144, 176 respectively. Then the dimensions of $H_d(M_{12}, \chi_i)$ for $i = 0, 1, 2, \ldots, 3, 14$, the vector spaces of relative symmetric polynomials with respect to all irreducible characters of the Pre Mathieu group $M_{12}$ are given below:

1. $\dim(H_d(M_{12}, \chi_0)) = g_1(d) + 495g_2(d) + 396g_3(d) + 1760g_4(d) + 2640g_5(d) + 2970[g_6(d) + g_7(d)] + 9504[g_6(d) + g_{13}(d)] + 15840g_9(d) + 7920g_{10}(d) + 11880[g_{11}(d) + g_{12}(d)] + 17280g_{14}(d)$
(2) \( \dim(H_d(M_{12}, \chi_1)) = 11g_1(d) + 1485g_2(d) - 396g_3(d) + 3520g_4(d) - 2640g_5(d) + 8910g_6(d) - 2970g_7(d) + 9504(g_8(d) - g_{13}(d)) - 7920g_{10}(d) + 11880[g_{11}(d) - g_{12}(d)] 

(3) \( \dim(H_d(M_{12}, \chi_2)) = 11g_1(d) + 1485g_2(d) - 396g_3(d) + 3520g_4(d) - 2640g_5(d) - 2970g_6(d) + 8910g_7(d) + 9504(g_8(d) - g_{13}(d)) - 7920g_{10}(d) + 11880[g_{12}(d) - g_{11}(d)] 

(4) \( \dim(H_d(M_{12}, \chi_3)) = 16g_1(d) + 1584g_3(d) - 3520g_4(d) + 2640g_5(d) + 9504[g_8(d) - g_{13}(d)] + 7920g_{10}(d) - 8640g_{14}(d) 

(5) \( \dim(H_d(M_{12}, \chi_4)) = 16g_1(d) + 1584g_3(d) - 3520g_4(d) + 2640g_5(d) + 9504[g_8(d) - g_{13}(d)] + 7920g_{10}(d) - 8640g_{14}(d) 

(6) \( \dim(H_d(M_{12}, \chi_5)) = 45g_1(d) - 1485g_2(d) + 1980g_3(d) + 7920g_5(d) + 2970[g_6(d) + g_7(d)] - 7920g_{10}(d) - 11880[g_{11}(d) + g_{12}(d)] + 17280g_{14}(d) 

(7) \( \dim(H_d(M_{12}, \chi_6)) = 54g_1(d) + 2970g_2(d) + 2376g_3(d) + 5940[g_6(d) + g_7(d)] + 9504[g_{13}(d) - g_8(d)] - 17280g_{14}(d) 

(8) \( \dim(H_d(M_{12}, \chi_7)) = 55g_1(d) + 3465g_2(d) - 1980g_3(d) + 1760g_4(d) + 2640g_5(d) - 2970[g_6(d) + g_7(d)] + 15840g_9(d) + 7920g_{10}(d) - 11880[g_{11}(d) + g_{12}(d)] 

(9) \( \dim(H_d(M_{12}, \chi_8)) = 55g_1(d) - 495g_2(d) - 1980g_3(d) + 1760g_4(d) + 2640g_5(d) - 2970g_6(d) + 8910g_7(d) - 15840g_9(d) + 7920g_{10}(d) + 11880[g_{12}(d) - g_{11}(d)] 

(10) \( \dim(H_d(M_{12}, \chi_9)) = 55g_1(d) - 495g_2(d) - 1980g_3(d) + 1760g_4(d) + 2640g_5(d) + 8910g_6(d) - 2970g_7(d) - 15840g_9(d) + 7920g_{10}(d) + 11880[g_{11}(d) - g_{12}(d)] 

(11) \( \dim(H_d(M_{12}, \chi_{10})) = 66g_1(d) + 990g_2(d) + 2376g_3(d) + 5280g_4(d) - 5940[g_6(d) + g_7(d)] + 9504[g_8(d) + g_{13}(d)] - 15840g_9(d) 

(12) \( \dim(H_d(M_{12}, \chi_{11})) = 99g_1(d) + 1485g_2(d) - 396g_3(d) + 7920g_5(d) - 2970[g_6(d) + g_7(d)] - 9504[g_8(d) + g_{13}(d)] - 7920g_{10}(d) + 11880[g_{11}(d) + g_{12}(d)] 

(13) \( \dim(H_d(M_{12}, \chi_{12})) = 120g_1(d) - 3960g_2(d) + 5280g_4(d) + 15840g_9(d) - 17280g_{14}(d) 

(14) \( \dim(H_d(M_{12}, \chi_{13})) = 144g_1(d) + 1584g_3(d) - 7920g_5(d) - 9504[g_8(d) + g_{13}(d)] + 7920g_{10}(d) + 17280g_{14}(d) 

(15) \( \dim(H_d(M_{12}, \chi_{14})) = 176g_1(d) - 1584g_3(d) - 7040g_4(d) - 2640g_5(d) + 9504[g_8(d) + g_{13}(d)] - 7920g_{10}(d) 

where 

\[ g_1(d) = \binom{d+11}{11} \] 
\[ g_2(d) = \sum_{l=0}^{[d/2]} \binom{l+3}{3} \binom{d-2l+3}{3} \]
- \( g_3(d) = \begin{cases} 
\binom{\frac{d+4}{2}}{5} & \text{when } d \text{ is even} \\
0 & \text{otherwise} 
\end{cases} \)

- \( g_4(d) = \sum_{l=0}^{\left\lfloor \frac{d}{3} \right\rfloor} \binom{l+2}{2} \binom{d-3l+2}{2} \)

- \( g_5(d) = \begin{cases} 
\binom{\frac{d+3}{3}}{3} & \text{when } d \text{ is a multiple of 3} \\
0 & \text{otherwise} 
\end{cases} \)

- \( g_6(d) = \sum_{l=0}^{\left\lfloor \frac{d}{4} \right\rfloor} (l+1) \binom{d-4l+3}{3} \)

- \( g_7(d) = \begin{cases} 
\sum_{l=0}^{\left\lfloor \frac{d}{4} \right\rfloor} \left\lceil \frac{d-4l}{2} \right\rceil + 1 \right\rfloor (l+1) & \text{when } d \text{ is even} \\
0 & \text{otherwise} 
\end{cases} \)

- \( g_8(d) = \sum_{l=0}^{\left\lfloor \frac{d}{5} \right\rfloor} (l+1)(d-5l+1) \)

- \( g_9(d) = \sum_{l=0}^{\left\lfloor \frac{d}{6} \right\rfloor} \sum_{k=0}^{\left\lfloor \frac{d-6l}{2} \right\rfloor} \left\lceil \frac{d-6l-3k}{2} \right\rceil + 1 \)

- \( g_{10}(d) = \begin{cases} 
\frac{d^6}{6} + 1 & \text{when } d \text{ is a multiple of 6} \\
0 & \text{otherwise} 
\end{cases} \)

- \( g_{11}(d) = \sum_{l=0}^{\left\lfloor \frac{d}{8} \right\rfloor} \sum_{k=0}^{\left\lfloor \frac{d-8l}{2} \right\rfloor} \left\lceil \frac{d-8l-k}{2} \right\rceil + 1 \)

- \( g_{12}(d) = \begin{cases} 
\left\lceil \frac{d}{8} \right\rceil + 1 & \text{when } d \text{ is a multiple of 4} \\
0 & \text{otherwise} 
\end{cases} \)

- \( g_{13}(d) = \begin{cases} 
\left\lceil \frac{d}{10} \right\rceil + 1 & d \text{ is even} \\
0 & \text{otherwise} 
\end{cases} \)

- \( g_{14}(d) = \left\lceil \frac{d}{11} \right\rceil + 1 \)

Proof for all the above formulae \( g_i(d), i = 0, 1, \ldots, 14 \) are calculated using the same procedure and derivations followed in \( M_9, M_{10} \) and \( M_{11} \) respectively.

**References**


