BOUNDS ON AG TOPOLOGICAL INDICES OF SOME GRAPH OPERATIONS

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Abstract. The AG index of a connected graph $G$ is $AG(G) = \sum_{uv \in E(G)} \frac{du + dv}{2 \sqrt{du \cdot dv}}$ where $du$ and $dv$ represent the degrees of the vertices of the edge $uv$. In this paper some bounds of AG index are presented.

1. Introduction

The topological indices are numerical values associated with molecular graphs. These graph invariants are called molecular descriptors. They play a vital role in chemical documentation, isomer discrimination, relationship analysis like QSAR and QSPR. In 1947, [7] Weiner used his topological index named as Weiner index to calculate the boiling point of paraffins. Then in 1972, [5] Gutman and Trinajstic defined the Zagreb indices which are popular. Thereafter many indices are defined namely [1] [4] Randic index, topological index etc. In 2016 [6] V.S. Shigehalli and Rachanna Kanavur introduced arithmetic-geometric indices.

Throughout this paper we consider only connected graphs without loops or multiple edges called simple connected graphs. For a graph $G$, $V(G)$ and $E(G)$ denote the set of all vertices and edges respectively. For a graph $G$ the degree of a vertex $v$ is the number of edges incident to $v$ and is denoted by $d(v)$. The composition (also called Lexicographic product) of graphs $G_1$ and $G_2$ with disjoint vertex set $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph with

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vertex set $V(G_1) \times V(G_2)$ and $(u_i, v_j)$ is adjacent with $u_k$ or $u_i = u_k$ and $v_j$ is adjacent with $v_l$.

The Cartesian product \cite{2} of $G_1 \times G_2$ of graphs $G_1$ and $G_2$ has the vertex set $V(G_1) \times V(G_2)$ and $(u_i, v_j), (u_k, v_l)$ is an edge of $G_1 \times G_2$ if $u_i = u_k$ and $(v_j, v_l) \in E(G_2)$ or $(u_i, u_k) \in E(G_1)$ and $v_j = v_l$.

In this paper bounds for the AG indices of Corona product, Cartesian product and Composition of graphs are derived.

**Definition 1.1.** Arithmetico-Geometrico topological index for a non-empty graph $G$ is denoted by $AG(G)$ and is defined as $AG(G) = \sum_{uv \in E(G)} \frac{du + dv}{2\sqrt{du \cdot dv}}$, where $du$ and $dv$ represent the degrees of the vertices of the edge $uv$.

2. **AG Indices of Graph Operations**

**Definition 2.1.** The eccentricity $e_e G(v)$ of a vertex $v$ in a connected graph $G$ is the greatest geodesic distance between $v$ and any other vertex. The diameter $D(G)$ of $G$ is defined as $d(G) = \max \{e_e G(v)|v \in V(G)\}$. Also the radius $\text{rad}(G)$ is defined as the $d(G) = \min \{e_e G(v)|v \in V(G)\}$.

**Definition 2.2.** The Cartesian product $G_1 \times G_2$ of $G_1$ and $G_2$ is a graph with vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(u_i, v_j), (u_k, v_l)$ are adjacent in $G_1 \times G_2$ if $u_i = u_k$ and $v_j, v_l \in E(G_2)$ or $u_i, u_k \in E(G_1)$ and $v_j = v_l$.

It can be seen that $|E(G_1 \times G_2)| = |E(G_1)||V(G_2)| + |E(G_2)||V(G_1)|$ and

$$d_{G_1 \times G_2}(u, v) = d_{G_1}(u) + d_{G_2}(v).$$

**Theorem 2.1.** Let $G_1$ and $G_2$ be two graphs with orders $n_1$ and $n_2$ and size $m_1$ and $m_2$ respectively. Then

$$AG(G_1 \times G_2) \leq \frac{\Delta_1 + \Delta_2}{\delta_1 + \delta_2} (m_2 n_1 + m_1 n_2),$$

where $\delta_1$ and $\delta_2$ are the minimum degrees of the vertices of $G_1$ and $G_2$ and $\Delta_1$ and $\Delta_2$ are their maximum degrees.
Proof.

\[ AG(G_1 \times G_2) = \]

\[ \sum_{(u,v_j) \in E(G_1 \times G_2), (u,v_i) \not\sim (u_k,v_i)} \frac{d_{G_1 \times G_2}(u_i,v_j) + d_{G_1 \times G_2}(u_k,v_i)}{2 \sqrt{d_{G_1 \times G_2}(u_i,v_j) \cdot d_{G_1 \times G_2}(u_k,v_i)}} \]

\[ + \sum_{(u,v_j),(u_k,v_i) \in E(G_1 \times G_2), (v_j,v_i) \not\in E(G_2)} \frac{d_{G_1 \times G_2}(u_i,v_j) + d_{G_1 \times G_2}(u_i,v_i)}{2 \sqrt{d_{G_1 \times G_2}(u_i,v_j) \cdot d_{G_1 \times G_2}(u_i,v_i)}} \]

\[ + \sum_{(u,v_j),(u_k,v_i) \in E(G_1 \times G_2), (u_i,u_k) \not\in E(G_1)} \frac{d_{G_1 \times G_2}(u_i,v_j) + d_{G_1 \times G_2}(u_k,v_j)}{2 \sqrt{d_{G_1 \times G_2}(u_i,v_j) \cdot d_{G_1 \times G_2}(u_k,v_j)}} \]

\[ = \sum_{(u,v_j),(u_i,v_i) \in E(G_1 \times G_2), (v_j,v_i) \in E(G_2)} \frac{d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_i) + d_{G_2}(v_j)}{2 \sqrt{(d_{G_1}(u_i) + d_{G_2}(v_j))(d_{G_1}(u_i) + d_{G_2}(v_j))}} \]

\[ + \sum_{(u,v_j),(u_i,v_i) \in E(G_1 \times G_2), (v_j,v_i) \in E(G_2)} \frac{2d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_2}(v_j)}{2 \sqrt{(d_{G_1}(u_i) + d_{G_2}(v_j))(d_{G_1}(u_i) + d_{G_2}(v_j))}} \]

\[ + \sum_{(u,v_j),(u_i,v_i) \in E(G_1 \times G_2)} \frac{d_{G_1}(u_i) + d_{G_1}(u_k) + 2d_{G_2}(v_j)}{2 \sqrt{(d_{G_1}(u_i) + d_{G_2}(v_j))(d_{G_1}(u_k) + d_{G_2}(v_j))}} \]

(2.1)

Suppose \( \delta_1 \) and \( \delta_2 \) be minimum degrees of the vertices of \( G_1 \) and \( G_2 \) and \( \Delta_1 \) and \( \Delta_2 \) be their maximum degrees. Then \( \delta_1 \leq d_{G_1}(u_i) \leq \Delta_1 \) and \( \delta_2 \leq d_{G_1}(u_i) \leq \Delta_2 \). So,

\[ AG(G_1 \times G_2) \leq \]

\[ \sum_{(u,v_j),(u_i,v_i) \in E(G_1 \times G_2), (v_j,v_i) \in E(G_2)} \frac{2 \Delta_1 + 2 \Delta_2}{2 \sqrt{(\delta_1 + \delta_2)^2}} + \sum_{(u_i,v_j),(u_k,v_i) \in E(G_1 \times G_2), (u_i,u_k) \in E(G_1)} \frac{2 \Delta_1 + 2 \Delta_2}{2 \sqrt{(\delta_1 + \delta_2)^2}} \]

\[ \leq \frac{\Delta_1 + \Delta_2}{\delta_1 + \delta_2} \sum_{(u,v_j),(u_i,v_i) \in E(G_1 \times G_2), (v_j,v_i) \in E(G_2)} 1 + \frac{\Delta_1 + \Delta_2}{\delta_1 + \delta_2} \sum_{(u_i,v_j),(u_k,v_i) \in E(G_1 \times G_2), (u_i,u_k) \in E(G_1)} 1 \]
From (2.1), we have

\[ AG(G_1 \times G_2) = \frac{\Delta_1 + \Delta_2}{\delta_1 + \delta_2} |E(G_2)||V(G_1)| + \frac{\Delta_1 + \Delta_2}{\delta_1 + \delta_2} |E(G_1)||V(G_2)|. \]

Hence, \( AG(G_1 \times G_2) \leq \frac{\Delta_1 + \Delta_2}{\delta_1 + \delta_2} (m_2 n_1 + m_1 n_2). \)

\( \square \)

Theorem 2.2. Let \( G_1 \) and \( G_2 \) be two graphs with orders \( n_1 \) and \( n_2 \) and size \( m_1 \) and \( m_2 \) respectively. Then

\[ AG(G_1 \times G_2) \leq (n_1 m_2 + n_2 m_1) \left( \frac{n_1 + n_2 - rad(G_1) - rad(G_2)}{\delta_1 + \delta_2} \right). \]

Proof. From (2.1), we have

\[ AG(G_1 \times G_2) = \sum_{(u, v) \in E(G_1) \times E(G_2)} \frac{2d_{G_1}(u) + d_{G_2}(v)}{2\sqrt{(d_{G_1}(u) + d_{G_2}(v))(d_{G_1}(u) + d_{G_2}(v))}} + \sum_{(u, v) \in E(G_1) \times E(G_2)} \frac{d_{G_1}(u) + d_{G_2}(v)}{2\sqrt{(d_{G_1}(u) + d_{G_2}(v))(d_{G_1}(u) + d_{G_2}(v))}} = A_1 + A_2. \]

Now,

\[ A_1 = \sum_{(u, v) \in E(G_1) \times E(G_2)} \frac{2d_{G_1}(u) + d_{G_2}(v)}{2\sqrt{(d_{G_1}(u) + d_{G_2}(v))(d_{G_1}(u) + d_{G_2}(v))}} \leq \sum_{(u, v) \in E(G_1) \times E(G_2)} \frac{2(n_1 - ecc_{G_1}(u)) + (n_2 - ecc_{G_2}(v))}{2\sqrt{(\delta_1 + \delta_2)^2}} \leq \sum_{(u, v) \in E(G_1) \times E(G_2)} \frac{2(n_1 - rad(G_1)) + (n_2 - rad(G_2))}{2(\delta_1 + \delta_2)}. \]
Similarly,

\[
A_2 = \sum_{(u, v_j), (u, v_k) \in E(G_1 \times G_2), (u, v_k) \in E(G_1)} \frac{d_{G_1}(u_i) + d_{G_1}(u_k) + 2d_{G_2}(v_j)}{2\sqrt{(d_{G_1}(u_i) + d_{G_2}(v_j))(d_{G_1}(u_k) + d_{G_2}(v_j))}}
\leq \sum_{(u, v_j), (u, v_k) \in E(G_1 \times G_2), (u, v_k) \in E(G_1)} \frac{(n_1 - \text{ecc}_{G_1}(u_i)) + (n_1 - \text{ecc}_{G_1}(u_k)) + 2(n_2 - \text{ecc}_{G_2}(v_j))}{2(\delta_1 + \delta_2)}
\leq \sum_{(u, v_j), (u, v_k) \in E(G_1 \times G_2), (u, v_k) \in E(G_1)} \frac{(n_1 - \text{rad}(G_1)) + (n_1 - \text{rad}(G_1)) + 2(n_2 - \text{rad}(G_2))}{2(\delta_1 + \delta_2)}
= n_2m_1 \left(\frac{n_1 + n_2 - \text{rad}(G_1) - \text{rad}(G_2)}{\delta_1 + \delta_2}\right).
\]

Hence the conclusion,

\[
A(G_1 \times G_2) \leq n_1m_2 \left(\frac{n_1 + n_2 - \text{rad}(G_1) - \text{rad}(G_2)}{\delta_1 + \delta_2}\right) + n_2m_1 \left(\frac{n_1 + n_2 - \text{rad}(G_1) - \text{rad}(G_2)}{\delta_1 + \delta_2}\right).
\]

\[\square\]

**Definition 2.3.** The Corona product \([3] G_1 \circ G_2\) and \(G_1\) and \(G_2\) is a graph obtained by taking \(|V(G_1)|\) copies of \(G_2\) and joining each vertex of the \(i^{th}\) copy with vertex \(v_i \in V(G_1)\). Then

\[|V(G_1 \circ G_2)| = |V(G_1)|(|1 + |V(G_2)|)|\]
\[ |E(G_1 \circ G_2)| = |E(G_1)| + |V(G_1)||V(G_2)| + |E(G_2)|. \]

Also, for a vertex in \( V(G_1 \circ G_2) \),
\[
d_{G_1 \circ G_2}(u) = \begin{cases} 
  d_{G_1}(u) + |V(G_2)| & ; u \in V(G_1) \\
  d_{G_2}(u) + 1 & ; u \in V(G_2) 
\end{cases}
\]

**Theorem 2.3.** Let \( G_2(i = 1, 2, \ldots, |V(G_1)|) \) represent the \( i^{th} \) copy of \( G_2 \) attached to the \( i^{th} \) vertex of \( G_1 \) and \( \delta_i \) and \( \Delta_i \) are minimum and maximum degrees of the vertices of \( G_i \), \( i = 1, 2 \). Then for the corona product \( G_1 \circ G_2 \) of \( G_1 \) and \( G_2 \),
\[
AG(G_1 \circ G_2) \leq \frac{m_1(\Delta_1 + n_2)}{\delta_1 + n_2} + \frac{(\Delta_2 + 1)n_1m_2}{\delta_2 + 1} + \frac{\Delta_2 + \Delta_1 + n_1 + 1}{2\sqrt{(\delta_2 + 1)(\delta_1 + n_1)}}n_1m_2
\]

**Proof.** The edge sets of \( G_1 \circ G_2 \) can be partitioned into three sets,
\[
E_1 = \{ e = uv \in E(G_1 \circ G_2), e \in E(G_1) \},
\]
\[
E_2 = \{ e = uv \in E(G_1 \circ G_2), e \in E(G_2), i = 1, 2, \ldots, |V(G_1)| \},
\]
\[
E_3 = \{ e = uv \in E(G_1 \circ G_2), u \in V(G_2), i = 1, 2, \ldots, |V(G_1)| \text{and, } v \in V(G_1) \}.
\]

Now,
\[
AG(G_1 \circ G_2) = \sum_{uv \in E(G_1 \circ G_2)} \frac{d_{G_1 \circ G_2}(u) + d_{G_1 \circ G_2}(v)}{2\sqrt{d_{G_1 \circ G_2}(u)d_{G_1 \circ G_2}(v)}}
\]
\[
= \sum_{uv \in E_1} \frac{d_{G_1 \circ G_2}(u) + d_{G_1 \circ G_2}(v)}{2\sqrt{d_{G_1 \circ G_2}(u)d_{G_1 \circ G_2}(v)}} + \sum_{uv \in E_2} \frac{d_{G_1 \circ G_2}(u) + d_{G_1 \circ G_2}(v)}{2\sqrt{d_{G_1 \circ G_2}(u)d_{G_1 \circ G_2}(v)}}
\]
\[
+ \sum_{uv \in E_3} \frac{d_{G_1 \circ G_2}(u) + d_{G_1 \circ G_2}(v)}{2\sqrt{d_{G_1 \circ G_2}(u)d_{G_1 \circ G_2}(v)}}
\]
\[
= A_1 + A_2 + A_3.
\]
\[ A_1 = \sum_{uv \in E_1} \frac{d_{G_1 \circ G_2}(u) + d_{G_1 \circ G_2}(v)}{2 \sqrt{d_{G_1 \circ G_2}(u)d_{G_1 \circ G_2}(v)}} \]
\[ = \sum_{uv \in E_1} \frac{d_{G_1}(u) + |V(G_2)| + d_{G_1}(v) + |V(G_2)|}{2 \sqrt{(d_{G_1}(u) + |V(G_2)|)(d_{G_1}(v) + |V(G_2)|)}} \]
\[ = \sum_{uv \in E_1} \frac{d_{G_1}(u) + d_{G_1}(v) + 2n_2}{2 \sqrt{(d_{G_1}(u) + n_2)(d_{G_1}(v) + n_2)}} \]
\[ \leq \sum_{uv \in E_1} \frac{\Delta_1 + \Delta_1 + 2n_2}{2 \sqrt{(\delta_1 + n_2)(\delta_1 + n_2)}} \]
\[ \leq \sum_{uv \in E_1} \frac{\Delta_1 + n_2}{\delta_1 + n_2} \]
\[ = \frac{\Delta_1 + n_2}{\delta_1 + n_2} \sum_{uv \in E_1} 1 = \frac{m_1(\Delta_1 + n_2)}{\delta_1 + n_2}. \]  

Hence, \( A_1 \leq \frac{m_1(\Delta_1 + n_2)}{\delta_1 + n_2}. \)

\[ A_2 = \sum_{uv \in E_2} \frac{d_{G_1 \circ G_2}(u) + d_{G_1 \circ G_2}(v)}{2 \sqrt{d_{G_1 \circ G_2}(u)d_{G_1 \circ G_2}(v)}} \]
\[ = \sum_{uv \in E_2} \frac{d_{G_2}(u) + 1 + d_{G_2}(v) + 1}{2 \sqrt{(d_{G_2}(u) + 1)(d_{G_2}(v) + 1)}} \]
\[ \leq \sum_{uv \in E_2} \frac{\Delta_2 + 1 + \Delta_2 + 1}{2 \sqrt{(\delta_2 + 1)(\delta_2 + 1)}} \]
\[ = \sum_{uv \in E_2} \frac{\Delta_2 + 1}{\delta_2 + 1} \]
\[ = \frac{\Delta_2 + 1}{\delta_2 + 1} \sum_{uv \in E_2} 1 \]
\[ = \frac{\Delta_2 + 1}{\delta_2 + 1} (|V(G_1)| |E(G_2)|) \]

Hence, \( A_2 \leq \frac{n_1 m_2 (\Delta_2 + 1)}{\delta_2 + 1}. \)
\[ A_3 = \sum_{uv \in E_3} \frac{d_{G_1 \circ G_2}(u) + d_{G_1 \circ G_2}(v)}{2\sqrt{d_{G_1 \circ G_2}(u)d_{G_1 \circ G_2}(v)}} \]
\[ = \sum_{uv \in E_3} \frac{d_{G_2}(u) + 1 + d_{G_1}(v) + |V(G_2)|}{2\sqrt{(d_{G_2}(u) + 1)(d_{G_1}(v) + |V(G_2)|)}} \]
\[ \leq \sum_{uv \in E_3} \frac{\Delta_2 + 1 + \Delta_1 + n_2}{2\sqrt{(\delta_2 + 1)(\delta_1 + n_2)}} \sum_{uv \in E_3} 1 \]
\[ = \frac{\Delta_2 + 1 + \Delta_1 + n_2}{2\sqrt{(\delta_2 + 1)(\delta_1 + n_2)}} |V(G_1)||V(G_2)| \]
i.e.,
\[ A_3 \leq \frac{\Delta_2 + \Delta_1 + n_2 + 1}{2\sqrt{(\delta_2 + 1)(\delta_1 + n_2)}} n_1 n_2. \]

Hence we conclude:
\[ AG(G_1 \circ G_2) \leq \frac{m_1(\Delta_1 + n_2)}{\delta_1 + n_2} + \frac{(\Delta_2 + 1)n_1 m_2}{\delta_2 + 1} + \frac{\Delta_2 + \Delta_1 + n_2 + 1}{2\sqrt{(\delta_2 + 1)(\delta_1 + n_2)}} n_1 n_2. \]

\[ \square \]

**Theorem 2.4.** Let \( G_2, (i = 1, 2, \ldots, |V(G_1)|) \) represent the \( i \)th copy of \( G_2 \) attached to the \( i \)th vertex of \( G_1 \) and \( \delta_i \) and \( \Delta_i \) are minimum and maximum degrees of the vertices of \( G_i, i = 1, 2 \). Then for the corona product \( G_1 \circ G_2 \) of \( G_1 \) and \( G_2 \),
\[ AG(G_1 \circ G_2) \geq \frac{(\delta_1 + n_2)m_1}{\Delta_1 + n_2} + \frac{(\delta_2 + 1)n_1 m_2}{\Delta_2 + 1} + \frac{\delta_2 + \delta_1 + n_1 + 1)n_1 n_2}{2\sqrt{(\Delta_2 + 1)(\Delta_1 + n_1)}} n_1 n_2. \]

**Proof.** We have from (2.2)
\[ A_1 = \sum_{uv \in E_1} \frac{d_{G_1}(u) + |V(G_2)| + d_{G_1}(u) + |V(G_2)|}{2\sqrt{(d_{G_1}(u) + |V(G_2)|)(d_{G_1}(u) + |V(G_2)|)}} \]
\[ \geq \sum_{uv \in E_1} \frac{\delta_1 + n_2 + \delta_1 + n_2}{2\sqrt{(\Delta_1 + n_2)(\Delta_1 + n_2)}} \]
\[ = \frac{\delta_1 + n_2}{\Delta_1 + n_2} \sum_{uv \in E_1} 1 \geq \frac{m_1(\delta_1 + n_2)}{\Delta_1 + n_2}. \]
Again, from (2.3)

\[ A_2 = \sum_{uv \in E_2} \frac{d_{G_2}(u) + 1 + d_{G_2}(v) + 1}{2\sqrt{(d_{G_2}(u) + 1)(d_{G_2}(v) + 1)}} \geq \sum_{uv \in E_2} \frac{\delta_2 + 1 + \delta_2 + 1}{2\sqrt{(\Delta_2 + 1)(\Delta_2 + 1)}} \]

\[ = \frac{\delta_2 + 1}{\Delta_2 + 1} \sum_{uv \in E_2} \geq \frac{(\delta_2 + 1)n_1m_2}{\Delta_2 + 1}. \]

From (2.4)

\[ A_3 = \sum_{uv \in E_3} \frac{d_{G_3}(u) + 1 + d_{G_3}(v) + |V(G_2)|}{2\sqrt{(d_{G_3}(u) + 1)(d_{G_3}(v) + |V(G_2)|)}} \geq \sum_{uv \in E_3} \frac{\delta_2 + 1 + \delta_1 + n_2}{2\sqrt{(\Delta_2 + 1)(\Delta_1 + n_2)}} \]

\[ = \frac{\delta_2 + 1 + \delta_1 + n_1}{2\sqrt{(\Delta_2 + 1)(\Delta_1 + n_1)}} \sum_{uv \in E_3} 1 = \frac{n_1n_2(\delta_2 + 1 + \delta_1 + n_2)}{2\sqrt{(\Delta_2 + 1)(\Delta_1 + n_2)}}. \]

Hence,

\[ AG(G_1 \circ G_2) \geq \frac{(\delta_1 + n_2)m_1}{\Delta_1 + n_2} + \frac{(\delta_2 + 1)n_1m_2}{\Delta_2 + 1} + \frac{(\delta_2 + \delta_1 + n_2 + 1)n_1n_2}{2\sqrt{(\Delta_2 + 1)(\Delta_1 + n_2)}}n_1n_2. \]

\[ \square \]

**Definition 2.4.** The composition or lexicographic product \( G = G_1[G_2] \) of graphs \( G_1 \) and \( G_2 \) with disjoint vertex sets \( V(G_1) \) and \( V(G_2) \) and edge sets \( E(G_1) \) and \( E(G_2) \) is a graph with vertex set \( V(G_1) \times V(G_2) \) and \((u_i, v_j)\) is adjacent with \((u_k, v_i)\) whenever \( u_i \) is adjacent with \( u_k \) or \( u_i = u_k \) and \( v_j \) adjacent with \( v_i \).

By this definition, one can see that

\[ |E(G_1[G_2])| = |E(G_1)||V(G_2)|^2 + |E(G_2)||V(G_1)| \]

\[ d_{G_1[G_2]}(u, v) = |V(G_2)|d_{G_1}(u) + d_{G_2}(v). \]

**Theorem 2.5.** Let \( G_1 \) and \( G_2 \) be two connected graphs with order \( n_1 \) and \( n_2 \), size \( m_1 \) and \( m_2 \), \( \delta_1 \) and \( \Delta_1 \) are minimum and maximum degrees of the vertices \( G_i, i = 1, 2 \) respectively. Then \( AG(G_1[G_2]) \leq \frac{(n_2\Delta_1 + \Delta_2)(n_1m_2 + m_1n_2^2)}{n_2\delta_1 + \delta_2}. \)
Proof.

\[ AG(G_1[G_2]) = \sum_{(u_i, v_j), (u_k, v_l) \in E(G_1[G_2]), \{u, v\} \neq \{k, l\}} \frac{d_{G_1[G_2]}(u_i, v_j) + d_{G_1[G_2]}(u_k, v_l)}{2 \sqrt{d_{G_1[G_2]}(u_i, v_j) \cdot d_{G_1[G_2]}(u_k, v_l)}} \]

\[ = \sum_{(u_i, v_j), (u_k, v_l) \in E(G_1[G_2]), j \neq l} \frac{d_{G_1[G_2]}(u_i, v_j) + d_{G_1[G_2]}(u_k, v_l)}{2 \sqrt{d_{G_1[G_2]}(u_i, v_j) \cdot d_{G_1[G_2]}(u_k, v_l)}} \]

\[ + \sum_{(u_i, v_j), (u_k, v_l) \in E(G_1[G_2]), i \neq k} \frac{d_{G_1[G_2]}(u_i, v_j) + d_{G_1[G_2]}(u_k, v_l)}{2 \sqrt{d_{G_1[G_2]}(u_i, v_j) \cdot d_{G_1[G_2]}(u_k, v_l)}} \]

\[ = A_1 + A_2. \]

Consider

\[ A_1 = \sum_{(u_i, v_j), (u_k, v_l) \in E(G_1[G_2]), j \neq l} \frac{d_{G_1[G_2]}(u_i, v_j) + d_{G_1[G_2]}(u_k, v_l)}{2 \sqrt{d_{G_1[G_2]}(u_i, v_j) \cdot d_{G_1[G_2]}(u_k, v_l)}} \]

\[ = \sum_{(u_i, v_j), (u_k, v_l) \in E(G_1[G_2]), j \neq l} \frac{|V(G_2)| d_{G_1}(u_i) + d_{G_2}(v_j) + |V(G_2)| d_{G_1}(u_i) + d_{G_2}(v_l)}{2 \sqrt{(|V(G_2)| d_{G_1}(u_i) + d_{G_2}(v_j)) \cdot (|V(G_2)| d_{G_1}(u_i) + d_{G_2}(v_l))}} \]

\[ \leq \sum_{(u_i, v_j), (u_k, v_l) \in E(G_1[G_2]), j \neq l} \frac{2(n_2 \Delta_1 + \Delta_2)}{2 \sqrt{(n_2 \delta_1 + \delta_2)(n_2 \delta_1 + \delta_2)}} \]

\[ \leq \frac{(n_2 \Delta_1 + \Delta_2)}{(n_2 \delta_1 + \delta_2)} \sum_{(u_i, v_j), (u_k, v_l) \in E(G_1[G_2]), j \neq l} 1 \leq \frac{(n_2 \Delta_1 + \Delta_2)n_1 m_2}{(n_2 \delta_1 + \delta_2)}. \]

Now consider

\[ A_2 = \sum_{(u_i, v_j), (u_k, v_l) \in E(G_1[G_2]), i \neq k} \frac{d_{G_1[G_2]}(u_i, v_j) + d_{G_1[G_2]}(u_k, v_l)}{2 \sqrt{d_{G_1[G_2]}(u_i, v_j) \cdot d_{G_1[G_2]}(u_k, v_l)}} \]

\[ = \sum_{(u_i, v_j), (u_k, v_l) \in E(G_1[G_2]), i \neq k} \frac{|V(G_2)| d_{G_1}(u_i) + d_{G_2}(v_j) + |V(G_2)| d_{G_1}(u_k) + d_{G_2}(v_l)}{2 \sqrt{(|V(G_2)| d_{G_1}(u_i) + d_{G_2}(v_j)) \cdot (|V(G_2)| d_{G_1}(u_k) + d_{G_2}(v_l))}} \]

\[ \leq \sum_{(u_i, v_j), (u_k, v_l) \in E(G_1[G_2]), j \neq l} \frac{2(n_2 \Delta_1 + \Delta_2)}{2 \sqrt{(n_2 \delta_1 + \delta_2)(n_2 \delta_1 + \delta_2)}} \]
\[ \leq \frac{(n_2 \Delta_1 + \Delta_2)}{(n_2 \delta_1 + \delta_2)} \sum_{(u,v_j),(u_k,v_j) \in E(G_1[G_2])} 1 \]
\[ = \frac{(n_2 \Delta_1 + \Delta_2)}{(n_2 \delta_1 + \delta_2)} \sum_{(u,v_j) \in E(G_1)} \sum_{v_j \in V(G_2)} \sum_{(u_k, v_j) \in E(G_2)} 1 \]
\[ = \frac{(n_2 \Delta_1 + \Delta_2)m_1 n_2^2}{(n_2 \delta_1 + \delta_2)}. \]

Hence,
\[ AG(G_1[G_2]) \leq \frac{(n_2 \Delta_1 + \Delta_2)n_1 m_2}{(n_2 \delta_1 + \delta_2)} + \frac{(n_2 \Delta_1 + \Delta_2)m_1 n_2^2}{(n_2 \delta_1 + \delta_2)} \]
\[ = \frac{(n_2 \Delta_1 + \Delta_2)(n_1 m_2 + m_1 n_2^2)}{n_2 \delta_1 + \delta_2}. \]

\[ \square \]

**Theorem 2.6.** Let \( G_1 \) and \( G_2 \) be two connected graphs with order \( n_1 \) and \( n_2 \), size \( m_1 \) and \( m_2, \delta, \) and \( \Delta, \) are minimum and maximum degrees of the vertices \( G_i, i = 1, 2, \) respectively. Then
\[ AG(G_1[G_2]) \geq \frac{(n_2 \delta_1 + \delta_2)(n_1 m_2 + m_1 n_2^2)}{n_2 \Delta_1 + \Delta_2}. \]

**Proof.** Same as above. \( \square \)

**References**


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