NUMERICAL QUENCHING FOR A SLOW DIFFUSION SYSTEM COUPLED AT THE BOUNDARY

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Abstract. This paper concerns the study of a numerical approximation for the following problem, \( u_t = u_{xx}, \ v_t = v_{xx}, \ 0 < x < 1, \ 0 < t < T; \ u_x(0, t) = (u^{-p_1}v^{-q_1})(0, t), \ v_x(0, t) = (u^{-p_2}v^{-q_2})(0, t) \) and \( u_x(1, t) = v_x(1, t) = 0, \ 0 < t < T, \) with \( p_1, q_1, p_2 \) and \( q_2 \) real parameters. We show that the solution of the semidiscrete scheme, obtained by the finite differences method quenches in a finite time only at first node of the mesh. We also prove that the time derivative of the solution blows up at quenching node and establish some conditions under which occurs the non-simultaneous or simultaneous quenching for the solution of the semidiscrete problem. After show the convergence of the quenching time, we finally present some numerical results to illustrate certain point of our work.

1. Introduction

Consider the following Newton filtration equations

\[
\begin{align*}
(1.1) \quad u_t(x, t) &= u_{xx}(x, t), \quad v_t(x, t) = v_{xx}(x, t), \quad (x, t) \in (0, 1) \times (0, T),
\end{align*}
\]

coupled with the boundary singularities at the left border

\[
\begin{align*}
(1.2) \quad u_x(0, t) &= (u^{-p_1}v^{-q_1})(0, t), \quad u_x(1, t) = 0, \quad t \in (0, T),
\end{align*}
\]
subject to smooth initial data

\begin{align}
(u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) > 0, \quad x \in [0,1],
\end{align}

where \( p_2, q_1 \geq 0, p_1, q_2 > 0, u'_0(x), v'_0(x) > 0, u''_0(x), v''_0(x) \leq 0, \) \( x \in (0,1), \) and \( u_0, v_0 \) compatible on the boundary.

The equations in (1.1) describe the heat-conduction of electron in the plasma body and the radiation heat-conduction, where the thermal conductivity increases while the temperature is decreasing. The nonlinear boundary conditions can be understood as that the heat convection occurs on the surfaces of bodies [24].

The problem (1.1)–(1.4) is said to be quench at time \( T \) if the two components \( u \) and \( v \) of the solution \((u, v)\) of (1.1)–(1.4) are nonnegative for all \((x, t) \in [0, 1] \times [0, T)\) and

\[
\liminf_{t \to T^-} \min \{u(\cdot, t), v(\cdot, t)\} = 0^+.
\]

The quenching phenomenon was first observed by Kawarada [16]. Since then, it has attracted a lot of attention, both for scalar and coupling cases. Many studies have concentrated on the quenching solutions, including quenching criteria, quenching locations, quenching rates, and quenching profiles, see [3]-[7], [10]-[18], [22], [25] and references therein. Ji and Zheng [15] studied the problem (1.1)-(1.4), they obtain that if \( p_2 \geq p_1 + 1, q_1 \geq q_2 + 1, \) then quenching is always simultaneous, while for \( p_2 < p_1 + 1 \) with \( q_1 \geq q_2 + 1, \) or \( q_1 < q_2 + 1 \) with \( p_2 \geq p_1 + 1, \) the quenching must be non-simultaneous. When \( p_2 < p_1 + 1 \) and \( q_1 < q_2 + 1, \) both simultaneous and non-simultaneous quenching is possible.

Unless we are mistaken, the studies carried out so far do not concern the numerical approximation of the problem (1.1)–(1.4). We will therefore deal in this paper with the numerical study using a semidiscrete form of (1.1)–(1.4), particularly in study of simultaneous and non-simultaneous quenching. We start by the construction of the semidiscrete scheme. Let \( I \geq 3 \) be a positive integer and let \( h = 1/I. \) Define the grid \( x_i = ih \) with \( i = 0, \ldots, I. \) Approximate the solution \((u, v)\) of (1.1)–(1.4) by the solution \((U_h(t) = (U_0(t), \ldots, U_I(t))^T, V_h(t) = (V_0(t), \ldots, V_I(t))^T)\) and approximate the initial data \((u_0, v_0)\) of the same problem by \((\varphi_{1,h} = (\varphi_{1,0}, \ldots, \varphi_{1,I})^T, \varphi_{2,h} = (\varphi_{2,0}, \ldots, \varphi_{2,I})^T)\) of the following system
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of ODEs whose is obtain using the finite difference method

\begin{align}
U_i'(t) &= \delta^2 U_i(t) - b_i(U^{p_1}V^{-q_1})_i(t), \quad i = 0, \ldots, I, \quad t \in (0, T_h), \\
V_i'(t) &= \delta^2 V_i(t) - b_i(U^{p_2}V^{-q_2})_i(t), \quad i = 0, \ldots, I, \quad t \in (0, T_h), \\
U_i(0) &= \varphi_{1,i}, \quad V_i(0) = \varphi_{2,i}, \quad i = 0, \ldots, I,
\end{align}

(1.5) \quad (1.6) \quad (1.7)

where

\begin{align*}
p_2, q_1 &\geq 0, \quad p_1, q_2 > 0, \quad 0 < \varphi_{1,i} \leq M, \quad 0 < \varphi_{2,i} \leq M, \quad i = 0, \ldots, I, \\
\delta^2 U_0(t) &= \frac{2U_1(t) - 2U_0(t)}{h^2}, \quad \delta^2 U_i(t) = \frac{2U_{i-1}(t) - 2U_i(t)}{h^2}, \quad t \in (0, T_h), \\
\delta^2 U_i(t) &= \frac{U_{i-1}(t) - 2U_i(t) + U_{i+1}(t)}{h^2}, \quad 1 \leq i \leq I - 1, \quad t \in (0, T_h), \\
b_0 &= \frac{2}{h}, \quad \text{and} \quad b_i = 0, \quad i = 1, \ldots, I.
\end{align*}

Here \([0, T_h)\) is the maximal time interval such that

\[ \forall t \in [0, T_h), \quad \inf_{0 \leq i \leq I} \min \{U_i(t), V_i(t)\} > 0. \]

For \(0 \leq i \leq I\), we have

\[ \lim_{t \to T_h^-} \inf_{0 \leq i \leq I} \{U_i(t), V_i(t)\} = 0^+. \]

The time \(T_h\) can be finite or infinite. On the one hand, if \(T_h\) is finite, we say that the solution \((U_h, V_h)\) of (1.5)–(1.7) quenches in a finite time and \(T_h\) is called the semidiscrete quenching time of \((U_h, V_h)\). We say on the other hand that the solution \((U_h, V_h)\) quenches globally when \(T_h\) is infinite.

Numerical approximations of heat equations with non-linear boundary conditions have been the focus of many authors in recent years. We refer to \([1], [2], [8], [19]– [21], [23]\) and the references cited therein for our work.

The paper is organized as follows. In the next section, we give some properties concerning our semidiscrete scheme. In Section 3, under some conditions, we prove that the solution of the semidiscrete scheme (1.5)–(1.7) quenches in a finite time, we give a result on numerical quenching set. We also show that the time derivative of the solution blows up at quenching node. A criterion to identify simultaneous and non-simultaneous quenching is proposed in section 4. In Section 5, we show the convergence of the solution of the semidiscrete scheme and the convergence of the quenching times to the theoretical one when
the mesh size goes to zero. Finally, in last section, we give some numerical experiments.

2. Properties of the semidiscrete scheme

In this section, we give some auxiliary results for the problem (1.5)–(1.7).

\textbf{Definition 2.1.} We say that $(U_h, V_h) \in \left( C^1([0, T_h), \mathbb{R}^{I+1}) \right)^2$ is a lower solution of (1.5)–(1.7) if

\begin{align*}
U_i'(t) &\leq \delta^2 U_i(t) - b_i(U_i^{-p_1}(t)V_i^{-q_1}(t)), \quad i = 0, \ldots, I, \quad t \in (0, T_h), \\
V_i'(t) &\leq \delta^2 V_i(t) - b_i(U_i^{-p_2}(t)V_i^{-q_2}(t)), \quad i = 0, \ldots, I, \quad t \in (0, T_h), \\
0 &< U_i(0) \leq \varphi_{1,i}, \quad 0 < V_i(0) \leq \varphi_{2,i}, \quad i = 0, \ldots, I,
\end{align*}

where $(U_h, V_h)$ is the solution of (1.5)–(1.7). On the other hand, we say that $(\overline{U_h}, \overline{V_h}) \in \left( C^1([0, T_h), \mathbb{R}^{I+1}) \right)^2$ is an upper solution of (1.5)–(1.7) if these inequalities are reversed.

The following lemma is a discrete form of the maximum principle.

\textbf{Lemma 2.1.} Let $e_h, c_h, \alpha_h, \beta_h \in C^0([0, T_h), \mathbb{R}^{I+1})$ and $U_h, V_h \in C^1([0, T_h), \mathbb{R}^{I+1})$ such that

\begin{align*}
U_i'(t) - \delta^2 U_i(t) + e_i(t)U_i(t) + c_i(t)V_i(t) &\geq 0, \quad i = 0, \ldots, I, \quad t \in (0, T_h), \\
V_i'(t) - \delta^2 V_i(t) + \alpha_i(t)U_i(t) + \beta_i(t)V_i(t) &\geq 0, \quad i = 0, \ldots, I, \quad t \in (0, T_h), \\
U_i(0) &\geq 0, \quad V_i(0) \geq 0, \quad i = 0, \ldots, I.
\end{align*}

Then we have

\begin{align*}
U_i(t) &\geq 0, \quad V_i(t) \geq 0, \quad i = 0, \ldots, I, \quad t \in (0, T_h).
\end{align*}

\textbf{Proof.} Let $T_0 < T_h$ and let $(Z_h(t), W_h(t)) = (e^\lambda U_h(t), e^\lambda V_h(t))$ where $\lambda$ is a real. We find that $(Z_h(t), W_h(t))$ satisfies the following inequalities:

\begin{align*}
Z_i'(t) - \delta^2 Z_i(t) + (e_i(t) - \lambda)Z_i(t) + c_i(t)W_i(t) &\geq 0, \\
&\quad i = 0, \ldots, I, \quad t \in (0, T_h), \\
W_i'(t) - \delta^2 W_i(t) + \alpha_i(t)Z_i(t) + (\beta_i(t) - \lambda)W_i(t) &\geq 0, \\
&\quad i = 0, \ldots, I, \quad t \in (0, T_h),
\end{align*}

(2.1)
(2.2) \[ Z_i(0) \geq 0, \quad W_i(0) \geq 0, \quad i = 0, \ldots, I. \]

Set \( m = \min \{ \min_{0 \leq i \leq I, t \in [0,T]} Z_i(t), \min_{0 \leq i \leq I, t \in [0,T]} W_i(t) \} \). Since for \( i \in \{0, \ldots, I\} \), \( Z_i \) and \( W_i \) are continuous functions on a compact, we can assume that \( m = Z_{i_0}(t_{i_0}) \) for a certain \( i_0 \in \{0, \ldots, I\} \).

Assume \( m < 0 \). Taking \( \lambda \) negative such that
\[
(e_{i_0}(t_{i_0}) - \lambda)Z_{i_0}(t_{i_0}) + c_{i_0}(t_{i_0})W_{i_0}(t_{i_0}) < 0.
\]

If \( t_{i_0} = 0 \), then \( Z_{i_0}(0) < 0 \), which contradicts (2.2), hence \( t_{i_0} \neq 0 \); if \( 0 \leq i_0 \leq I \), we have
\[
Z_{i_0}'(t_{i_0}) = \lim_{k \to 0} \frac{Z_{i_0}(t_{i_0}) - Z_{i_0}(t_{i_0} - k)}{k} \leq 0.
\]

Moreover by a straightforward computation we get
\[
Z_{i_0}'(t_{i_0}) - \delta^2 Z_{i_0}(t_{i_0}) + (e_{i_0}(t_{i_0}) - \lambda)Z_{i_0}(t_{i_0}) + c_{i_0}(t_{i_0})W_{i_0}(t_{i_0}) < 0,
\]
but this inequality contradicts (2.1) and the proof is completed. \( \square \)

**Lemma 2.2.** Let \((U_h, V_h), (\overline{U}_h, \overline{V}_h) \in (C^1([0,T_h], \mathbb{R}^{I+1}))^2\) be lower and upper solutions of (1.5)–(1.7) respectively such that, \((U_h(0), V_h(0)) \leq (\overline{U}_h(0), \overline{V}_h(0))\) then
\[
(U_h(t), V_h(t)) \leq (\overline{U}_h(t), \overline{V}_h(t)).
\]

**Proof.** Let us define \((Z_h(t), W_h(t)) = (\overline{U}_h(t), \overline{V}_h(t)) - (U_h(t), V_h(t))\). By a straightforward computation and using the Mean value theorem, we obtain
(2.3) \[ Z_i'(t) - \delta^2 Z_i(t) - p_1 b_i(\mu_i(t))^{-p_1-1} Z_i(t) - q_1 b_i(\nu_i(t))^{-q_1-1} W_i(t) \geq 0, \]
\[ i = 0, \ldots, I \]
(2.4) \[ W_i'(t) - \delta^2 W_i(t) - p_2 b_i(\mu_i(t))^{-p_2-1} Z_i(t) - q_2 b_i(\nu_i(t))^{-q_2-1} W_i(t) \geq 0, \]
\[ i = 0, \ldots, I \]
where \( \mu_i(t), \nu_i(t) \) lie, respectively, between \( U_i(t) \) and \( \overline{U}_i(t) \), and between \( V_i(t) \)
and \( \overline{V}_i(t) \), for \( i \in \{0, \ldots, I\} \).

We can rewrite (2.3)–(2.4) as
\[
Z_i'(t) - \delta^2 Z_i(t) + e_i(t) Z_i(t) + c_i(t) W_i(t) \geq 0, \quad i = 0, \ldots, I, \ t \in (0,T_h),
\]
\[
W_i'(t) - \delta^2 W_i(t) + \alpha_i(t) Z_i(t) + \beta_i(t) W_i(t) \geq 0, \quad i = 0, \ldots, I, \ t \in (0,T_h),
\]
where \(c_i(t) = -p_1b_i(\mu_i(t))^{-p_1-1}, \alpha_i(t) = -p_2b_i(\mu_i(t))^{-p_2-1}\) and \(\beta_i(t) = -q_2b_i(\nu_i(t))^{-q_2-1}, \ i = 0, \ldots, I \ \forall t \in (0, T_h).\) According to Lemma 2.1, \(Z_i(t) \geq 0, W_i(t) \geq 0, \) for \(i = 0, \ldots, I, \forall t \in (0, T_h)\) and the proof is completed. \(\square\)

The next lemma gives the properties of the semidiscrete solution.

**Lemma 2.3.** Let \((U_h, V_h) \in \left(C^1([0, T_h], \mathbb{R}^{I+1})\right)^2\) be the solution of (1.5)–(1.7) with an initial data \((\varphi_{1,h}, \varphi_{2,h})\) upper solution such that \(0 < \varphi_{1,i} < \varphi_{1,i+1} \leq M\) and \(0 < \varphi_{2,i} < \varphi_{2,i+1} \leq M\) for \(i = 0, \ldots, I - 1.\) Then we have

(i) \(0 < U_i(t) \leq \varphi_{1,i} \leq M\) and \(0 < V_i(t) \leq \varphi_{2,i} \leq M\) for \(i = 0, \ldots, I, t \in [0, T_h];\)

(ii) \((U_{i+1}(t), V_{i+1}(t)) > (U_i(t), V_i(t)), i = 0, \ldots, I - 1, t \in (0, T_h);\)

(iii) \((U'_i(t), V'_i(t)) \leq 0, i = 0, \ldots, I, t \in (0, T_h).\)

**Proof.**

(i) Since \((\varphi_{1,h}, \varphi_{2,h})\) is an an upper solution of (1.5)–(1.7), by the Lemma 2.1 and 2.2 we have \(0 < U_i(t) \leq \varphi_{1,i} \leq M\) and \(0 < V_i(t) \leq \varphi_{2,i} \leq M\) for \(i = 0, \ldots, I, t \in [0, T_h).\)

(ii) We argue by contradiction. Assume that \(t_0\) is the first \(t > 0,\) such that \((K_i, L_i)(t) = (U_{i+1} - U_i, V_{i+1} - V_i)(t) > 0,\) for \(0 \leq i \leq I - 1,\) but \(\min\{K_{i_0}(t_0), L_{i_0}(t_0)\} = 0\) for a certain \(i_0 \in \{0, \ldots, I - 1\}.\) Assume that \(K_{i_0}(t_0) = U_{i_0+1}(t_0) - U_{i_0}(t_0) = 0.\)

Without lost of generality, we can suppose that \(t_0\) is the smallest integer which satisfies the above equality. We have

\[
K'_0(t) = \frac{U_2(t) - 2U_1(t) + U_0(t)}{h^2} - \left(\frac{2U_1(t) - 2U_0(t)}{h^2} - \frac{2}{h} (U_0^{-p_1}(t)V_0^{-q_1}(t))\right)
\]

\[
K'_i(t) = \frac{U_{i+2}(t) - 2U_{i+1}(t) + U_i(t)}{h^2} - \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \ 1 \leq i \leq I - 2
\]

\[
K'_{I-1}(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2} - \frac{U_I(t) - 2U_{I-1}(t) + U_{I-2}(t)}{h^2}
\]

\[
\begin{cases}
K'_0(t) = \frac{K_1(t) - 3K_0(t)}{h^2} + \frac{2}{h} (U_0^{-p_1}(t)V_0^{-q_1}(t)) \\
K'_i(t) = \frac{K_{i+1}(t) - 2K_i(t) + K_{i-1}(t)}{h^2}, \ 1 \leq i \leq I - 2 \\
K'_{I-1}(t) = \frac{K_{I-2}(t) - 3K_{I-1}(t)}{h^2}
\end{cases}
\]

(2.5)
According to the hypotheses on \( t_0 \), we have the following inequalities:

\[
K_{i_0}'(t_0) = \lim_{\epsilon \to 0} \frac{K_{i_0}(t_0) - K_{i_0}(t_0 - \epsilon)}{\epsilon} \leq 0,
\]

\[
\frac{K_{i_0+1}(t_0) - 2K_{i_0}(t_0) + K_{i_0-1}(t_0)}{h^2} > 0 \text{ if } 1 \leq i_0 \leq I - 2,
\]

\[
\frac{K_{i_0+1}(t_0) - 3K_{i_0}(t_0)}{h^2} > 0 \text{ if } i_0 = 0,
\]

\[
\frac{-3K_{i_0}(t_0) + K_{i_0-1}(t_0)}{h^2} > 0 \text{ if } i_0 = I - 1,
\]

which implies,

\[
K_{i_0}'(t_0) - \frac{K_{i_0+1}(t_0) - 2K_{i_0}(t_0) + K_{i_0-1}(t_0)}{h^2} < 0 \text{ if } 1 \leq i_0 \leq I - 2,
\]

\[
K_{i_0}'(t_0) - \frac{K_{i_0+1}(t_0) - 3K_{i_0}(t_0)}{h^2} - \frac{2}{h^2} \left( U_0^{-p_1}(t)V_0^{-q_1}(t) \right) < 0 \text{ if } i_0 = 0,
\]

\[
K_{i_0}'(t_0) - \frac{-3K_{i_0}(t_0) + K_{i_0-1}(t_0)}{h^2} < 0 \text{ if } i_0 = I - 1.
\]

Thus, we have a contradiction with (2.5), which leads to the desired result.

(iii) Denote \( F_i(t) = U_i(t) - U_i(t + \varepsilon) \) and \( G_i(t) = V_i(t) - V_i(t + \varepsilon) \), for \( i = 0, \ldots, I \), using (i) and (1.7) we obtain \( F_i(0) \geq 0, G_i(0) \geq 0 \) for \( i = 0, \ldots, I \). It is not hard to see that

\[
F'_i(t) - \delta^2 F_i(t) + p_1 b_i(\xi(t))^{-p_1 - 1} F_i(t) + q_1 b_i(\eta(t))^{-q_1 - 1} G_i(t) \geq 0,
\]

\[
G'_i(t) - \delta^2 G_i(t) + p_2 b_i(\xi(t))^{-p_2 - 1} F_i(t) + q_2 b_i(\eta(t))^{-q_2 - 1} G_i(t) \geq 0,
\]

where \( \xi_i(t), \eta_i(t) \) lie, respectively, between \( U_i(t + \varepsilon) \) and \( U_i(t) \) and between \( V_i(t + \varepsilon) \) and \( V_i(t) \). From Lemma 2.1 we get

\[
F_i(t) \geq 0 \text{ and } G_i(t) \geq 0 \text{ for } i = 0, \ldots, I, \ t \in (0, T_h).
\]

This fact implies the desired result. \( \square \)
3. SEMIDISCRETE QUENCHING SOLUTION

Let \((U_h, V_h)\) be the solution of (1.5)–(1.7) with \(0 < \varphi_{1,i} \leq M, 0 < \varphi_{2,i} \leq M\) for \(i = 0, \ldots, I\). Using [4] and [9], we show that \((U_h, V_h)\) quenches in a finite time and \((U_h', V_h')\) blows up at quenching node.

**Theorem 3.1.** For every initial data, the solution \((U_h, V_h)\) of the system (1.5)–(1.7) quenches in finite time with the only quenching node \(\{i = 0\}\).

**Proof.** Integrating (1.5) in time we have
\[
U_i(t) - U_i(0) = \int_0^t (\delta^2 U_i(\tau) - b_i (U_i^{-p_1}(\tau)V_i^{-q_1}(\tau))) \, d\tau
\]
summing up the above equality we arrive at
\[
\sum_{i=0}^I hU_i(t) = \sum_{i=0}^I hU_i(0) + \int_0^t \left( \frac{U_{i-1}(\tau) - U_i(\tau)}{h} + \frac{U_1(\tau) - U_0(\tau)}{h} - 2(U_0^{-p_1}(\tau)V_0^{-q_1}(\tau)) \right) \, d\tau.
\]
(1.5) implies that
\[
\frac{h}{2} U_I(t) - \frac{h}{2} U_I(0) = \int_0^t \frac{U_{I-1}(\tau) - U_I(\tau)}{h} \, d\tau, \text{ and}
\]
\[
\frac{h}{2} U_0(t) - \frac{h}{2} U_0(0) = \int_0^t \left( \frac{U_1(\tau) - U_0(\tau)}{h} - (U_0^{-p_1}(\tau)V_0^{-q_1}(\tau)) \right) \, d\tau.
\]
Thus
\[
\frac{h}{2} U_I(t) + \sum_{i=1}^{I-1} hU_i(t) + \frac{h}{2} U_0(t) = \frac{h}{2} U_I(0) + \sum_{i=1}^{I-1} hU_i(0) + \frac{h}{2} U_0(0) - \int_0^t (U_0^{-p_1}(\tau)V_0^{-q_1}(\tau)) \, d\tau,
\]
therefore
\[
\frac{h}{2} U_I(t) + \sum_{i=1}^{I-1} hU_i(t) + \frac{h}{2} U_0(t) \leq M - (M^{-p_1}M^{-q_1}) t.
\]
By the same way, we also prove that
\[
\frac{h}{2} V_I(t) + \sum_{i=1}^{I-1} hV_i(t) + \frac{h}{2} V_0(t) \leq M - (M^{-p_2}M^{-q_2}) t,
\]
which yield a contradiction because $U_h$ and $V_h$ are positive for all times in $[0, T_h)$. Then there exists $0 < T_h < \infty$ such that
\[
\lim_{t \to T_h^-} \min\{U_0(t), V_0(t)\} = 0^+.
\]

Now we will show that $\{i = 0\}$ is the unique quenching node. In everything that follows $i \in \{0, \ldots, I - 1\}$ and $t \in (0, T_h)$. Set $g(U_i(t)) = U_i^{-\rho_1}(t)$, $f(V_i(t)) = V_i^{-\rho_1}(t)$, $d(U_i(t)) = U_i^{-\rho_2}(t)$, and
\[
Z_i(t) = \frac{U_{i+1}(t) - U_i(t)}{h} - \phi_i(g(U_i(t)) + f(V_i(t)))
\]
\[
W_i(t) = \frac{V_{i+1}(t) - V_i(t)}{h} - \phi_i(d(U_i(t)) + j(V_i(t)))
\]
where $\phi_i, \delta^2 \phi_i \geq 0$, $\delta^+ \phi_i \leq 0$, $\phi_I = 0$, $\phi_0 = 1$, $\phi_i(g(U_i(0)) + f(V_i(0))) \leq \delta^+ U_i(0)$, and $\phi_i(d(U_i(0)) + j(V_i(0))) \leq \delta^+ V_i(0)$.

By means of Taylor expansions inspired by [9] we have
\[
\delta^2(\phi_i k(J_i(t))) = \phi_i k'(J_i(t))\delta^2 J_i(t) + k(J_i(t))\delta^2 \phi_i + k'(J_i(t))\delta^+ \phi_i \delta^+ J_i(t) + \frac{k''(\lambda_i(t))}{2} \delta^2 J_i(t)
\]
\[
\delta^2(\phi_i k(J_i(t))) = \phi_i k'(J_i(t))\delta^2 J_i(t) + k(J_i(t))\delta^2 \phi_i + k'(J_i(t))\delta^+ \phi_i \delta^+ J_i(t) + \frac{k''(\lambda_i(t))}{2} \delta^2 J_i(t), \quad i = 1, \ldots, I - 1,
\]
\[
\delta^2(\phi_0 k(J_0(t))) = \phi_0 k'(J_0(t))\delta^2 J_0(t) + k(J_0(t))\delta^2 \phi_0 + 2k'(J_0(t))\delta^+ \phi_0 \delta^+ J_0(t) + \phi_0(\delta^+ J_0(t))^2 k''(\rho_0(t)).
\]

If we use the fact that $J_i$, $\delta^+ J_i(t)$ and $\delta^2 J_i(t)$ are nonnegative and the hypothesis on $\phi_h$, we arrive at
\[
\delta^2(\phi_i k(J_i(t))) \geq \phi_i k'(J_i(t))\delta^2 J_i(t), \quad i = 0, \ldots, I - 1.
\]

By using (3.1) we can get
\[
Z_i'(t) - \delta^2 Z_i(t) \geq \frac{b_i}{h} (g(U_i) + f(V_i)) + b_i \phi_i' g(U_i) (g(U_i) + f(V_i)) + b_i \phi_i' f(U_i) (d(U_i) + j(V_i)).
\]

The above inequalities implies that
\[
Z_i'(t) - \delta^2 Z_i(t) + b_i g(U_i(t)) Z_i(t) + b_i f'(V_i(t)) W_i(t) \geq \frac{1}{h} (g(U_i(t)) + f(V_i(t))) + f'(U_i(t))(d(U_i(t)) + j(V_i(t))) + g'(U_i(t))(g(U_i(t)) + f(V_i(t))).
\]
We obtain
\[ Z'_i(t) - \delta^2 Z_i(t) + b_i g'(U_i(t)) Z_i(t) + b_i f'(V_i(t)) W_i(t) \geq 0, \]
for the parameter \( h \) small enough. Thus we have
\[ Z'_i(t) - \delta^2 Z_i(t) + b_i g'(U_i(t)) Z_i(t) + b_i f'(V_i(t)) W_i(t) \geq 0, \]
\[ W'_i(t) - \delta^2 W_i(t) + b_i d'(U_i(t)) Z_i(t) + b_i j'(V_i(t)) W_i(t) \geq 0, \]
\[ Z_i(0) \geq 0, \ W_i(0) \geq 0. \]
Using the Lemma 2.1 we have \( Z_i(t) \geq 0 \) and \( W_i(t) \geq 0 \), for \( i = 0, \ldots, I - 1 \) and \( t \in (0, T_h) \). This implies that \( \frac{U_{i+1}(t) - U_i(t)}{h} \geq \phi_i(g(U_i(t)) + f(V_i(t))) \geq \frac{1}{2} \left( \frac{1}{M^{p_i}} + \frac{1}{M^{q_i}} \right) \) for \( i = 0, \ldots, J \), with \( \phi_J = \frac{1}{2} \), where \( J \in \{1, \ldots, I - 1\} \). Thus by summing we get
\[ U_i(t) \geq U_0 + \frac{ih}{2} \left( \frac{1}{M^{p_i}} + \frac{1}{M^{q_i}} \right) \geq \frac{ih}{2} \left( \frac{1}{M^{p_i}} + \frac{1}{M^{q_i}} \right) \text{ whenever } i > 0. \]
We deal with \( V_h \) by the same way. \( \square \)

**Theorem 3.2.** If \( \lim_{t \to T_h} U_0(t) = 0 \left( \lim_{t \to T_h} V_0(t) = 0 \right) \), then \( U_h'(t) \) blows up (\( V_h'(t) \) blows up).

**Proof.** Suppose that \( U_h'(t) \) is bounded. Then, there exists a nonnegative constant \( C \) such that \( U_h'(t) > C \) and we have
\[ \sum_{i=0}^{I-1} \sum_{j=0}^{i} h^2 U_j'(t) > \sum_{i=0}^{I-1} \sum_{j=0}^{i} h^2 C. \]
\[ \sum_{i=0}^{I-1} \sum_{j=0}^{i} h^2 C = \sum_{i=0}^{I-1} (i + 1) h^2 C = \frac{I}{2} (h^2 C + I h^2 C) = \frac{hC}{2} + C. \]
\[ \sum_{i=0}^{I-1} \sum_{j=0}^{i} h^2 U_j'(t) = \sum_{i=1}^{I-1} \left( \sum_{j=1}^{i} h^2 U_j'(t) + h^2 U_0'(t) \right) + h^2 U_0'(t). \]
From (1.5) we arrive at
\[ \sum_{i=0}^{I-1} \sum_{j=0}^{i} h^2 U_j'(t) = U_1(t) - U_0(t) - (U_0^{-p_1}(t) V_0^{-q_1}(t)) + \frac{h}{2} U_0'(t) \]
and using the Lemma 2.3 we obtain

\[ U_I(t) - U_0(t) - (U_0^{-p_1}(t)V_0^{-q_1}(t)) > hC + \frac{C}{2}. \]

As \( t \to T_h^- \), the left-hand side tends to infinity while the right-side is finite. This contradiction proves that \( U_h' \) blows up. \( \square \)

4. SIMULTANEOUS VERSUS NON-SIMULTANEOUS QUENCHING

We identify simultaneous and non-simultaneous quenching in this section. We consider \( (U_h, V_h) \) the solution of (1.5)–(1.7) with \( h \) fixed.

**Theorem 4.1.** If \( U_h \) quenches and \( V_h \) does not quench in (1.5)–(1.7) then \( p_2 < p_1 + 1 \).

**Proof.** We suppose that \( V_h \) does not quench. By (1.5) there exists \( c > 0 \) such that

\[ U_0'(t) \geq -c U_0^{-p_1}(t), \]

integrating this inequality from \( t \) to \( T_h \), we obtain

\[ U_0(t) \leq C(T_h - t)^{\frac{1}{p_1+1}}, \] where \( C = ((p_1 + 1)c)^{\frac{1}{p_1+1}}. \]

Now we use (4.1) and (1.6) and we arrive at

\[ V_0'(t) \leq \delta^2 V_0(t) - b_0 \left( V_0^{-q_2}(t)C(T_h - t)^{\frac{-q_2}{p_1+1}} \right). \]

Thus \( V_0(T_h) \leq C_1 - C_2 \int_0^{T_h} (T_h - t)^{\frac{-q_2}{p_1+1}} dt \). We remark that this integral diverges if \( p_2 \geq p_1 + 1 \), which is a contradiction and the proof is completed. \( \square \)

Theorem 4.1 implies the following corollary:

**Corollary 4.1.** If \( p_2 \geq p_1 + 1 \) and \( q_1 \geq q_2 + 1 \), then any quenching in (1.5)–(1.7) must be simultaneous.

**Lemma 4.1.** Let \( (U_h, V_h) \) be the solution of (1.5)–(1.7). Assume that \( U_h \) quenches at time \( T_h \) (\( V_h \) quenches at time \( T_h \)), \( 0 < \varphi_{1,i} \leq M, 0 < \varphi_{2,i} \leq M \) for \( i = 0, \ldots, I \) and

\[ \delta^2 \varphi_{1,i} - b_i (\varphi_{1,i}^{-p_1} \varphi_{2,i}^{-q_1}) + c (\varphi_{1,i}^{-p_1} \varphi_{2,i}^{-q_1}) \leq 0, \]

\[ \delta^2 \varphi_{2,i} - b_i (\varphi_{1,i}^{-p_2} \varphi_{2,i}^{-q_2}) + c (\varphi_{1,i}^{-p_2} \varphi_{2,i}^{-q_2}) \leq 0. \]
Then there exists a positive constant $C$ such that for $t \in (0, T_h)$
\[
\frac{U^{p_1+1}_0(t)}{C(p_1 + 1)} \geq T_h - t \quad \left( \frac{V^{q_2+1}_0(t)}{C(q_2 + 1)} \geq T_h - t \right),
\]
\[
U_0(t) \geq C(T_h - t)^{\frac{1}{p_1+1}} \quad \left( V_0(t) \geq C(T_h - t)^{\frac{1}{q_2+1}} \right).
\]

Proof. Set for $i = 0, \ldots, I$, $t \in [0, T_h)$,
\[
Z_i(t) = U_i'(t) + c \left( U_i^{-p_1}(t)V_i^{-q_1}(t) \right) \quad \text{and} \quad W_i(t) = V_i'(t) + c \left( U_i^{-p_2}(t)V_i^{-q_2}(t) \right).
\]

A straightforward calculation and also the Mean value theorem give
\[
Z_i'(t) - \delta^2 Z_i(t) + \alpha_i(t)Z_i(t) + \beta_i(t)W_i(t) \leq 0, \quad i = 0, \ldots, I, \quad t \in (0, T_h),
\]
\[
W_i'(t) - \delta^2 W_i(t) + \alpha_i(t)Z_i(t) + b_i(t)W_i(t) \leq 0, \quad i = 0, \ldots, I, \quad t \in (0, T_h),
\]
\[
Z_i(0) \leq 0, \quad W_i(0) \leq 0, \quad i = 0, \ldots, I.
\]

Using the Lemma 2.1, we have
\[
Z_i(t) \leq 0, \quad W_i(t) \leq 0, \quad i = 0, \ldots, I, \quad t \in (0, T_h).
\]

Thus we get
\[
(4.4) U_i'(t) \leq -cU_i^{-p_1}(t) \quad \text{and} \quad V_i'(t) \leq -cV_i^{-q_2}(t), \quad i = 0, \ldots, I, \quad t \in (0, T_h).
\]

Using the fact that $U_i$ quenches ($V_i$ quenches) and integrating (4.4) from $t$ to $T_h$, we arrive at
\[
\frac{U^{p_1+1}_0(t)}{C(p_1 + 1)} \geq T_h - t \quad \left( \frac{V^{q_2+1}_0(t)}{C(q_2 + 1)} \geq T_h - t \right),
\]
and we have so
\[
U_0(t) \geq C(T_h - t)^{\frac{1}{p_1+1}} \quad \left( V_0(t) \geq C(T_h - t)^{\frac{1}{q_2+1}} \right).
\]

Theorem 4.2. If $q_1 < q_2 + 1$ ($p_2 < p_1 + 1$), then there exist initial data such that $V_h$ ($U_h$) quenches but $U_h$ ($V_h$) doesn’t.
Proof. Here we argue by contradiction. Assuming that $U_h$ and $V_h$ quench simultaneously at time $T_h$ for any initial data. We can write

$$
\int_0^t U'(s) ds \geq \int_0^{T_h} U'(s) ds - \frac{2}{h} \int_0^{T_h} (U_1(s) - U_0(s)) ds - \frac{2}{h} \int_0^{T_h} (U_1^{-p_1}(s)V_1^{-q_1}(s)) ds.
$$

Using the Lemma 4.1, we have

$$
U_0(t) \geq U_0(0) + \frac{2}{h^2} \int_0^{T_h} (U_1(s) - U_0(s)) ds - \frac{2}{h} \int_0^{T_h} (T_h - s)^{-\frac{p_1}{p_1+1}} (T_h - s)^{-\frac{q_1}{q_2+1}} ds.
$$

$q_1 < q_2 + 1$ implies that this integral is converged and

$$
U_0(t) \geq C_1 - C_2 T_h^{\frac{1}{p_1+1} - \frac{q_1}{q_2+1}}, \text{ with } C_1, C_2 > 0.
$$

By summation of (1.6) we observe that

$$
-\frac{h}{2} V'(t) - \frac{h}{2} V'(t) - \sum_{i=1}^{I-1} hV'_i(t) = U_0^{-p_2}(t)V_0^{-q_2}(t),
$$

(4.5)

integrate (4.5) from 0 to $T_h$, we can obtain

$$
V_1(0) (U_0^{-p_2}(0)V_0^{-q_2}(0))^{-1} \geq T_h,
$$

then if $T_h$ is sufficiently small (depending on $U_h(0)$ and $V_h(0)$), $U_0(T_h) \geq c_0 > 0$. We have so a contradiction with the hypothesis that $U_h$ quenches and leads us to the desired result. □

Theorem 4.3. If $p_2 \leq \frac{q_2(p_1 + 1)}{q_2 + 1}$ and $q_1 \geq q_2 + 1 \left( q_1 \leq \frac{p_1(q_2 + 1)}{p_1 + 1} \text{ and } p_2 \geq p_1 + 1 \right)$ then $U_h$ (or $V_h$) quenches alone under any positive initial data.
Proof. Assume that there exists initial data such that $U_h$ and $V_h$ quench simultaneously at time $T_h$. We can suppose without lost of generality that this initial data satisfies (4.2)–(4.3). According to (1.6)

$$V_0'(t) = \delta^2 V_0(t) - b_0(U_0^{-p_2}(t)V_0^{-q_2}(t)),$$

$$V_0'(t) \geq -b_0(U_0^{-p_2}(t)V_0^{-q_2}(t)),$$

$$V_0(t) \leq b_0 \int_0^{T_h} U_0^{-p_2}(s)V_0^{-q_2}(s)ds.$$  

We know from Lemma 4.1 that $U_0(t) \geq C_1(T_h - t)^{-\frac{1}{p_1+1}}, V_0(t) \geq C_2(T_h - t)^{-\frac{1}{q_2+1}}$. As $p_2 \leq \frac{q_2(p_1 + 1)}{q_2 + 1}$, there exists $\alpha > 0$ such that $V_0(t) \leq \alpha(T_h - t)^{-\frac{1}{q_2+1}}$. (1.5) implies

$$U_0'(t) = \delta^2 U_0(t) - b_0(U_0^{-p_1}(t)V_0^{-q_1}(t)),$$

$$U_0'(t) \leq \delta^2 U_0(t) - b_0 V_0^{-q_1}(t),$$

$$U_0'(t) \leq \delta^2 U_0(t) - b_0 \alpha^{-q_1}(T_h - t)^{-\frac{q_1}{q_2+1}}.$$  

Integrating both sides from 0 to $T_h$, we obtain $U_0(0) \geq -c_1 + c_2 \int_0^{T_h} (T_h - t)^{-\frac{q_1}{q_2+1}} dt$. It is clear that the integral diverges if $q_1 \geq q_2 + 1$, which is a contradiction. We have so the desired result.  

Remark 4.1. We can see of the Lemma 4.1 and the proof of Theorem 4.3 that if $U_h$ ($V_h$) quenches at time $T_h$, then $U_0(t) \sim (T_h - t)^{-\frac{1}{p_1+1}}$ ($V_0(t) \sim (T_h - t)^{-\frac{1}{q_2+1}}$) for $t$ close enough to $T_h$ where $(U_h, V_h)$ is the solution of (1.5)–(1.7) such that the initial data satisfies (4.2)–(4.3).

5. CONVERGENCE OF THE SEMIDISCRETE QUENCHING TIME

Under some assumptions, we show in this section that the semidiscrete quenching time converges to the real one when the mesh size goes to zero. To obtain the convergence of semidiscrete quenching time, we firstly prove the following theorem about the convergence of the semidiscrete scheme. Before, we denote

$$u_h(t) = (u(x_0, t), \ldots, u(x_I, t))^T, \quad v_h(t) = (v(x_0, t), \ldots, v(x_I, t))^T,$$

$$\|U_h(t)\|_{\infty} = \max_{0 \leq i \leq I} |U_i(t)|, \quad \|U_h(t)\|_{\inf} = \min_{0 \leq i \leq I} |U_i(t)|.$$
Theorem 5.1. Assume that the problem (1.1)–(1.4) has solution \((u, v) \in (C^{4,1}([0, 1] \times [0, T^*]))^2\) and the initial data \((\varphi_{1,h}, \varphi_{2,h})\) of (1.5)–(1.7) satisfies

\[
\|\varphi_{1,h} - u_h(0)\|_{\infty} = o(1), \quad \|\varphi_{2,h} - v_h(0)\|_{\infty} = o(1) \quad h \to 0.
\]

Then, for \(h\) sufficiently small, the problem (1.5)–(1.7) has a unique solution \((U_h, V_h) \in (C^1([0, T^*], \mathbb{R}^{l+1}))^2\) such that

\[
\max_{t \in [0, T^*]} \|U_h(t) - u_h(t)\|_{\infty} = O(\|\varphi_{1,h} - u_h(0)\|_{\infty} + \|\varphi_{2,h} - v_h(0)\|_{\infty} + h), \quad as \ h \to 0,
\]

\[
\max_{t \in [0, T^*]} \|V_h(t) - v_h(t)\|_{\infty} = O(\|\varphi_{1,h} - u_h(0)\|_{\infty} + \|\varphi_{2,h} - v_h(0)\|_{\infty} + h), \quad as \ h \to 0.
\]

Proof. Let \(\rho > 0\) be such that

\[
(\|u\|_{\infty}, \|v\|_{\infty}) < \rho, \quad t \in [0, T^*].
\]

Then the problem (1.5)–(1.7) has for each \(h\), a unique solution \((U_h, V_h) \in (C^1([0, T^*], \mathbb{R}^{l+1}))^2\). Let \(t(h) \leq T^*\) be the greatest value of \(t > 0\) such that

\[
\max \{\|U_h(t) - u_h(t)\|_{\infty}, \|V_h(t) - v_h(t)\|_{\infty}\} < 1.
\]

The relation (5.1) implies \(t(h) > 0\) for \(h\) small enough. Using the triangle inequality, we obtain

\[
\|U_h(t)\|_{\infty} \leq 1 + \rho \quad and \quad \|V_h(t)\|_{\infty} \leq 1 + \rho \quad for \ t \in (0, t(h)).
\]

Let \((e_{1,h}, e_{2,h})(t) = (U_h - u_h, V_h - v_h)(t) \forall t \in [0, T^*]\) be the discretization error. These error functions verify

\[
e'_{1,i}(t) = \delta^2 e_{1,i}(t) + p_1 b_i(\theta_i(t))^{-p_1} e_{1,i}(t) + q_1 b_i(\Theta_i(t))^{-q_1} e_{2,i}(t) + O(h),
\]

\[
e'_{2,i}(t) = \delta^2 e_{2,i}(t) + p_2 b_i(\theta_i(t))^{-p_2} e_{1,i}(t) + q_2 b_i(\Theta_i(t))^{-q_2} e_{2,i}(t) + O(h),
\]

where \(\theta_i(t)\) and \(\Theta_i(t)\) lie, respectively, between \(U_i(t)\) and \(u(x_i, t)\), and between \(V_i(t)\) and \(v(x_i, t)\), for \(i \in \{0, \ldots, l\}\). Using (5.2) and (5.4), there exist \(K\) and \(L\) positive constants such that

\[
e'_{1,i}(t) \leq \delta^2 e_{1,i}(t) + b_i L|e_{1,i}(t)| + b_i L|e_{2,i}(t)| + Kh,
\]

\[
e'_{2,i}(t) \leq \delta^2 e_{2,i}(t) + b_i L|e_{1,i}(t)| + b_i L|e_{2,i}(t)| + Kh.
\]

Let \((z, w) \in (C^{4,1}([0, 1] \times [0, T^*]))^2\) be such that

\[
z(x, t) = (\|\varphi_{1,h} - u_h(0)\|_{\infty} + \|\varphi_{2,h} - v_h(0)\|_{\infty} + Qh) e^{(M+2)t-(1-x)^2}
\]
and \( w = z \forall (x, t) \in [0, 1] \times [0, T^*] \), with \( M, Q \) positive constants. By the Lemma 2.2, we can prove that
\[
|e_{1,i}(t)| < z(x_i, t) \quad \text{and} \quad |e_{2,i}(t)| < w(x_i, t) \quad \text{with} \quad 0 \leq i \leq I \quad \text{for} \quad t \in (0, t(h)).
\]
Thus we get
\[
\|U_h(t) - u_h(t)\|_\infty \leq (\|\varphi_{1,h} - u_h(0)\|_\infty + \|\varphi_{2,h} - v_h(0)\|_\infty + Qh) e^{(M+2)t},
\]
\[
\|V_h(t) - v_h(t)\|_\infty \leq (\|\varphi_{1,h} - u_h(0)\|_\infty + \|\varphi_{2,h} - v_h(0)\|_\infty + Qh) e^{(M+2)t},
\]
where \( t \in (0, t(h)) \). Suppose that \( T^* > t(h) \). From (5.3), we obtain
\[
1 = \|U_h(t(h)) - u_h(t(h))\|_\infty \leq (\|\varphi_{1,h} - u_h(0)\|_\infty + \|\varphi_{2,h} - v_h(0)\|_\infty + Qh) e^{(M+2)t}.
\]
Since the term on the right hand side of the above inequality goes to zero as \( h \) tends to zero, we deduce that \( 1 \leq 0 \), which is impossible. Consequently \( t(h) = T^* \) and we conclude the proof. \( \square \)

**Theorem 5.2.** Let \( (u, v) \in (C^{4,1}([0, 1] \times [0, T]))^2 \) be solution of (1.1)–(1.4) with quenches time \( T \) and the initial data at (1.5)–(1.7) satisfies (4.2)–(4.3) and (5.1). Then the solution \( (U_h, V_h) \) of (1.5)–(1.7) quenches in a finite time \( T_h \) and we have \( \lim_{h \to 0^+} T_h = T \).

**Proof.** Set \( \varepsilon > 0 \), there exists \( \eta > 0 \) such that
\[
y \leq \eta \quad \text{for} \quad t \in [0, T],
\]
\[
y \leq \eta \quad \text{for} \quad t \in [0, T].
\]
Applying the triangle inequality, we get
\[
\|U_h(T_1)\|_{\infty} \leq \|U_h(T_1) - u_h(T_1)\|_{\infty} + \|u_h(T_1)\|_{\infty} \leq \eta.
\]
We know from Theorem 3.1 that \( (U_h, V_h) \) quenches in a finite time \( T_h \). Assuming that \( U_h \) quenches, we can deduce from Lemma 4.1 and (5.5) that
\[
|T_h - T| \leq |T_h - T_1| + |T_1 - T| \leq \frac{\|U_h(T_1)\|_{\infty}^{1+p_1}}{C(p_1 + 1)} + \frac{\varepsilon}{2} \leq \varepsilon.
\]
The case where \( V_h \) quenches is analogous. \( \square \)
6. Numerical experiments

In this section, we present some numerical approximations to the quenching time of (1.5)–(1.7) for the initial data \( \varphi_{1,i} = \varphi_{2,i} = 1 + \frac{2}{\pi} \sin \left( \frac{\pi}{2} \frac{i}{h} \right) \) for \( i = 0, \ldots, I \), with different values of \( p_1, p_2, q_1 \) and \( q_2 \).

By setting \( W_i(t) = (U_i(t))^{-1} \) and \( W_{I+1+i}(t) = (V_i(t))^{-1} \), \( i = 0, \ldots, I \), we obtain the following differential system

\[
\begin{align*}
W_0'(t) &= \frac{2}{h^2} \left( W_0(t) - \frac{(W_0(t))^2}{W_1(t)} \right) + \frac{2}{h}(W_0(t))^{p_1+2}(W_{I+1}(t))^{q_1} \\
W_i'(t) &= \frac{1}{h^2} \left( 2W_i(t) - \frac{(W_i(t))^2}{W_{i-1}(t)} - \frac{(W_i(t))}{W_{i+1}(t)} \right), \ i = 1, \ldots, I - 1 \\
W_I'(t) &= \frac{2}{h^2} \left( W_I(t) - \frac{(W_I(t))^2}{W_{I-1}(t)} \right) \\
W_{I+1}'(t) &= \frac{2}{h^2} \left( W_{I+1}(t) - \frac{(W_{I+1}(t))^2}{W_{I+2}(t)} \right) + \frac{2}{h}(W_0(t))^{p_2}(W_{I+1}(t))^{q_2+2} \\
W_{I+i+1}'(t) &= \frac{1}{h^2} \left( 2W_{I+i+1}(t) - \frac{(W_{I+i+1}(t))^2}{W_{I+i}(t)} - \frac{(W_{I+i+1}(t))^2}{W_{I+i+2}(t)} \right), \ i = 1, \ldots, I - 1 \\
W_{2I+1}'(t) &= \frac{2}{h^2} \left( W_{2I+1}(t) - \frac{(W_{2I+1}(t))^2}{W_{2I}(t)} \right)
\end{align*}
\]

where \( W_i(0) = (\varphi_{1,i})^{-1} \) and \( W_{I+i+1}(0) = (\varphi_{2,i})^{-1} \) for \( i = 0, \ldots, I \). We can see that \( W_h \) blows up when \( (U_h, V_h) \) quenches. Let \( \eta \) be the arc length of \( W_h \). Considering the variables \( t \) and \( W_h \) as functions of \( \eta \), we obtain the following system of differential equations

\[
\begin{align*}
\frac{dt}{d\eta} &= \frac{1}{\sqrt{1 + \sum_{i=0}^{2I+1} f_i^2}}, \\
\frac{dW_i}{d\eta} &= \frac{f_i}{\sqrt{1 + \sum_{i=0}^{2I+1} f_i^2}}, \ i = 0, \ldots, 2I + 1, \\
t(0) &= 0, \ W_i(0) = (\varphi_{1,i})^{-1}, \ W_{I+i+1}(0) = (\varphi_{2,i})^{-1}, \ i = 0, \ldots, I,
\end{align*}
\]

where \( 0 < \eta < \infty \) and \( f_i(t) = W_i'(t) \) since \( dt^2 = dt^2 + dW_0^2 + \cdots + dW_{2I+1}^2 \). It is well known (Hirota & Ozawa, 2006) that

\[
\lim_{\eta \to \infty} t(\eta) = T_h \quad \text{and} \quad \lim_{\eta \to \infty} \|W_h(\eta)\|_{\infty} = \infty.
\]
For the numerical computation, let us define $\eta = \eta_l$ by

$$\eta_l = 2^{16} \cdot 2^l$$

for each value of $l$, we apply DOP54 (see Hairer, Nørsett & Wanner, 1993) to system (6.1) and we get a linearly convergent sequence to the blow-up time $\{t_l^{(k)}\}_{k=1}^{l+1}$. We also accelerate the sequence recursively by Aitken method's:

$$t_{l+2}^{(k)} = t_{l+1}^{(k)} - \frac{\left( t_{l+2}^{(k)} - t_{l+1}^{(k)} \right)^2}{t_{l+2}^{(k)} - 2t_{l+1}^{(k)} + t_l^{(k)}},$$

$l \geq 2k, \ k = 0, 1, 2, \ldots$

As in (Hirota & Ozawa, 2006), for our experiments we set $\text{RTOL} = \text{ATOL} = 1.d-15$ and $\text{ITOL} = 0$. Where the parameters RTOL and ATOL are the tolerances of the relative and absolute errors, respectively, and ITOL is used to choose the manner in which the errors are controlled.

**Tables and graphics:** $\varphi_{1,i} = \varphi_{2,i} = 1 + \frac{2}{\pi} \sin \left( \frac{\pi}{2} ih \right), \ i = 0, \ldots, I.$

In the following tables, in rows, we present the numerical quenching times $T_h$ and the numbers of iterations $n$ corresponding to meshes of 16, 32, 64, 128, 256, 512, 1024.

**Table 1.** Numerical quenching times and numbers of iterations obtained for $p_1 = 2, \ p_2 = 1, \ q_1 = 1.6$ and $q_2 = 0.5$

<table>
<thead>
<tr>
<th>$I$</th>
<th>$T_h$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.12669452</td>
<td>2063</td>
</tr>
<tr>
<td>32</td>
<td>0.12406180</td>
<td>3791</td>
</tr>
<tr>
<td>64</td>
<td>0.12326487</td>
<td>7168</td>
</tr>
<tr>
<td>128</td>
<td>0.12302885</td>
<td>13779</td>
</tr>
<tr>
<td>256</td>
<td>0.12296021</td>
<td>27181</td>
</tr>
<tr>
<td>512</td>
<td>0.12294057</td>
<td>59900</td>
</tr>
<tr>
<td>1024</td>
<td>0.12293502</td>
<td>176017</td>
</tr>
</tbody>
</table>

**Table 2.** Numerical quenching times and numbers of iterations obtained for $p_1 = 1, \ p_2 = 2, \ q_1 = 2$ and $q_2 = 1$

<table>
<thead>
<tr>
<th>$I$</th>
<th>$T_h$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.12097809</td>
<td>1871</td>
</tr>
<tr>
<td>32</td>
<td>0.11817407</td>
<td>3546</td>
</tr>
<tr>
<td>64</td>
<td>0.11732426</td>
<td>6782</td>
</tr>
<tr>
<td>128</td>
<td>0.11707255</td>
<td>13090</td>
</tr>
<tr>
<td>256</td>
<td>0.11699937</td>
<td>25861</td>
</tr>
<tr>
<td>512</td>
<td>0.11697843</td>
<td>56990</td>
</tr>
<tr>
<td>1024</td>
<td>0.11697252</td>
<td>167374</td>
</tr>
</tbody>
</table>
NUMERICAL QUENCHING FOR A SLOW DIFFUSION SYSTEM...

Table 3. Numerical quenching times and numbers of iterations obtained for $p_1 = 0.5$, $p_2 = 2.5$, $q_1 = 0.5$ and $q_2 = 1.5$

<table>
<thead>
<tr>
<th>$I$</th>
<th>$T_h$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.14183488</td>
<td>2282</td>
</tr>
<tr>
<td>32</td>
<td>0.13941895</td>
<td>4309</td>
</tr>
<tr>
<td>64</td>
<td>0.13868637</td>
<td>8238</td>
</tr>
<tr>
<td>128</td>
<td>0.13846865</td>
<td>15899</td>
</tr>
<tr>
<td>256</td>
<td>0.13840510</td>
<td>31361</td>
</tr>
<tr>
<td>512</td>
<td>0.13838685</td>
<td>68551</td>
</tr>
<tr>
<td>1024</td>
<td>0.13838169</td>
<td>199191</td>
</tr>
</tbody>
</table>

Table 4. Numerical quenching times and numbers of iterations obtained for $p_1 = 2$, $p_2 = 1$, $q_1 = 4$ and $q_2 = 2$

<table>
<thead>
<tr>
<th>$I$</th>
<th>$T_h$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.08282272</td>
<td>1649</td>
</tr>
<tr>
<td>32</td>
<td>0.08018808</td>
<td>2940</td>
</tr>
<tr>
<td>64</td>
<td>0.07938374</td>
<td>5479</td>
</tr>
<tr>
<td>128</td>
<td>0.07914533</td>
<td>10446</td>
</tr>
<tr>
<td>256</td>
<td>0.07907608</td>
<td>20407</td>
</tr>
<tr>
<td>512</td>
<td>0.07905629</td>
<td>43379</td>
</tr>
<tr>
<td>1024</td>
<td>0.07905071</td>
<td>119081</td>
</tr>
</tbody>
</table>

Figure 1. On the left, quenching of $U_h$ and on the right, no quenching of $V_h$ for $p_1 = 2$, $p_2 = 1$, $q_1 = 1.6$ and $q_2 = 0.5$.

Figure 2. On the left, quenching of $U_h$ and on the right, quenching of $V_h$ for $p_1 = 1$, $p_2 = 2$, $q_1 = 2$ and $q_2 = 1$. 
Figure 3. On the left, no quenching of $U_h$ and on the right, quenching of $V_h$ for $p_1 = 0.5$, $p_2 = 2.5$, $q_1 = 0.5$ and $q_2 = 1.5$.

Figure 4. On the left, quenching of $U_h$ and on the right, no quenching of $V_h$ for $p_1 = 2$, $p_2 = 1$, $q_1 = 4$ and $q_2 = 2$.

Remark 6.1. We observe that, the solution of our problem quenches in a finite time and the convergence of quenching time $T_h$ is given in different tables. Moreover, we can see that of the figure 1, $U_h$ quenches while $V_h$ doesn’t when $p_2 < p_1 + 1$, of the figure 2, $U_h$ and $V_h$ quench simultaneously when $p_2 \geq p_1 + 1$ and $q_1 \geq q_2 + 1$, of the figure 3, $V_h$ quenches while $U_h$ doesn’t when $q_1 < q_2 + 1$ and of the figure 4, $U_h$ quenches alone under any positive initial data when $p_2 \leq \frac{q_2(p_1+1)}{q_2+1}$ and $q_1 \geq q_2 + 1$.

These numerical results are in fact consistent with the Theorem 4.1, Corollary 4.1, Theorem 4.2 and Theorem 4.3.

References

[16] H. Kawarada: On the solutions of initial boundary value problem for $u_t = u_{xx} + \frac{1}{1-u}$, RIMS Kyoto University, 10 (1975), 729–736.


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