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COMMON FIXED POINT THEOREMS IN 2-NORMED SPACES

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ABSTRACT. In this paper, we prove some common fixed point theorems in linear 2-normed spaces for weakly compatible mappings using E.A. property. Moreover, we provide an example in support of main result.

1. INTRODUCTION

In 1963, Gahler [9] introduced the notion of linear 2-normed spaces and studied its properties. In [7], Jungck gave the concept of compatibility in metric spaces and since then several authors established many fixed point results under this notion. Later Jungck and Rhoades introduced the concept of weakly compatibility in metric spaces and studied some common fixed point theorems. For more work on fixed point results, we refer to [2-6]. On the other hand, in 2002, Aamri and Moutawakil [8] introduced the notion of E.A. property which contains the class of noncompatible maps in metric spaces.

In this paper, we define compatibility , weakly compatibility and EA.property in linear 2-normed spaces with which we prove three fixed point theorems and give an example to illustrate the main result.

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2. Preliminaries

Here we begin with the following definition.

Definition 2.1. *[1] Let N be a real linear space with dim$N \geq 2$.Then a function $\|.,.\| : N \to [0,\infty)$ is said to be 2-norm on N iff it satisfies the following for all $\alpha, \beta, \xi \in N$ and s$\in \mathbb{R}$*

P1) $\|\alpha, \beta\|$=*0 iff* $\{\alpha, \beta\}$ *is a linearly dependent set;*
P2) $\|\alpha, \beta\|$=$\|\beta, \alpha\|$*;*
P3) $\|s\alpha, \beta\| = |s| \, \|\alpha, \beta\|$*;*
P4) $\|\alpha + \beta, \xi\| \leq \|\alpha, \xi\| + \|\beta, \xi\|$.

Here $(N, \|.,.\|)$ is called a linear 2-normed space.

Let N denote the linear 2-normed space throughout this paper.

Definition 2.2. *[1] We say that a sequence (α_n) in N converges to some $\alpha \in N$ iff $\|\alpha_n - \alpha, \xi\| \to 0$ as n$\to \infty$ for every $\xi \in N$.We write it as $\lim\limits_{n\to\infty} \alpha_n = \alpha$ or simply $\alpha_n \to \alpha$.*

Definition 2.3. *[1] We say that a sequence (α_n) in N is a Cauchy sequence in N iff $\|\alpha_n - \alpha_m, \xi\| \to 0$ as n,m$\to \infty$ for every $\xi \in N$.*

Definition 2.4. *[1] We say that N is a complete iff every Cauchy sequence in N converges in N.*

Definition 2.5. *We say that (P, K) of self mappings on N is compatible iff (α_n) is a sequence in N such that $\lim\limits_{n\to\infty} \|P\alpha_n - \theta, \xi\| = 0$ and $\lim\limits_{n\to\infty} \|K\alpha_n - \theta, \xi\| = 0$ for some $\theta \in N$ imply that $\lim\limits_{n\to\infty} \|PK\alpha_n - KP\alpha_n, \xi\| = 0$ for every $\xi \in N$.*

Definition 2.6. *We say that (P, K) of self mappings on N is weakly compatible iff Pθ=Kθ for $\theta \in N$ implies that PKθ=KPθ.*

Definition 2.7. *We say that (P, K) of self mappings on N holds property E.A. iff there is some sequence $(\gamma_n') \subset N$ such that $\lim\limits_{n\to\infty} \left\|P\gamma_n' - \theta, \xi\right\| = 0$ and $\lim\limits_{n\to\infty} \|K\gamma_n' - \theta, \xi\| = 0$ for some $\theta \in N$ and for every $\xi \in N$.*

Example 1. *Consider $(N, \|.,.\|)$, where $N = \mathbb{R}^2$ and $\|(\alpha, \theta)\| := |\alpha_1\theta_2 - \alpha_2\theta_1|$ for $\alpha := (\alpha_1, \alpha_2)$ and $\theta := (\theta_1, \theta_2)$ in $\mathbb{R}^2$.Define P,K:$\mathbb{R}^2 \to \mathbb{R}^2$ by P$(\alpha, \beta) := (\frac{\alpha}{25}, \frac{\beta}{25})$ and K$(\alpha, \beta) := (\frac{\alpha}{3}, \frac{\beta}{3})$ for $(\alpha, \beta) \in \mathbb{R}^2$. Then PK$(\alpha, \beta) = (\frac{\alpha}{75}, \frac{\beta}{75})$ and KP$(\alpha, \beta) = (\frac{\alpha}{75}, \frac{\beta}{75})$.Let P$(\alpha_n, \beta_n) \to \theta$ and K$(\alpha_n, \beta_n) \to \theta$ in $\mathbb{R}^2$.For $(\alpha, \beta) \in \mathbb{R}^2$, we consider*

$\|PK(\alpha_n,\beta_n) - KP(\alpha_n,\beta_n),(\alpha,\beta)\| = \|(0,0),(\alpha,\beta)\| = 0 \to 0$,*as n$\to \infty$. Therefore P and K are compatible.*

3. Main Results

Our main results are as follows.

Theorem 3.1. *Let (P,K) be weakly compatible mappings of N of which one of their ranges is complete and satisfying E.A property such that $P(N)\subset K(N)$, $\|Pu - Py,\xi\| < \max\{\|Ku - Ky,\xi\|, \frac{\|Pu-Ku,\xi\|+\|Py-Ky,\xi\|}{2}, \frac{\|Pu-Ky,\xi\|+\|Py-Ku,\xi\|}{2}\}$ for $u \neq y \in N$. Then P and K have a unique common fixed point.*

Proof. Let $K(N)$ be complete. Since (P,K) holds property E.A, we have $\lim_{n\to\infty} \|P\alpha_n - \theta,\xi\| = 0$ and $\lim_{n\to\infty} \|K\alpha_n - \theta,\xi\| = 0$ for some sequence (α_n) in N and for some $\theta \in N$. This will imply that $\theta \in K(N)$, since $K(N)$ is complete. It follows that $\theta = Ka$ for some $a \in N$. Therefore, $\lim_{n\to\infty} P\alpha_n = Ka$ and $\lim_{n\to\infty} K\alpha_n = Ka$. Now let us show that $Pa = Ka$. For this, let $Pa \neq Ka$. Now we consider

$$\|P\alpha_n - Pa,\xi\| < \max\Bigg\{ \|K\alpha_n - Ka,\xi\|, \frac{\|P\alpha_n - K\alpha_n,\xi\| + \|Pa - Ka,\xi\|}{2}, \frac{\|P\alpha_n - Ka,\xi\| + \|Pa - K\alpha_n,\xi\|}{2}\Bigg\},$$

for every $n \in \mathbb{N}$. Now letting $n \to \infty$, we have

$$\begin{aligned}
\|Ka - Pa,\xi\| &< \max\Bigg\{ \|Ka - Ka,\xi\|, \frac{\|Ka - Ka,\xi\| + \|Pa - Ka,\xi\|}{2}, \frac{\|Ka - Ka,\xi\| + \|Pa - Ka,\xi\|}{2}\}\\
&= \max\{\|Ka - Ka,\xi\|, \frac{\|Pa - Ka,\xi\|}{2}, \frac{\|Pa - Ka,\xi\|}{2}\Bigg\}\\
&\leq \frac{\|Pa - Ka,\xi\|}{2}\\
&= \frac{\|Ka - Pa,\xi\|}{2}.
\end{aligned}$$

Therefore $\|Ka - Pa,\xi\| \leq \frac{\|Ka-Pa,\xi\|}{2}$. It follows that $\|Ka - Pa,\xi\| = 0$ and hence $Ka = Pa$. This implies that $KPa = PKa$, since (P,K) is weakly compatible. It follows $K\theta = P\theta$, since $Ka = \theta$. Now let us show that $Pa = PPa$. For this, let $Pa \neq PPa$. Then we have,

$$\begin{aligned}
\|Pa - PPa, \xi\| &< \max\Bigg\{ \|Ka - KPa, \xi\|, \\
&\frac{\|Pa - Ka, \xi\| + \|PPa - KPa, \xi\|}{2}, \frac{\|PPa - Ka, \xi\| + \|Pa - KPa, \xi\|}{2} \Bigg\} \\
&= \max\left\{ \|\theta - K\theta, \xi\|, \frac{\|Pa - Ka, \xi\| + \|P\theta - K\theta, \xi\|}{2}, \frac{\|P\theta - \theta, \xi\| + \|\theta - K\theta, \xi\|}{2} \right\} \\
&= \max\{\|\theta - P\theta, \xi\|, \|\theta - P\theta, \xi\|\} \\
&= \|\theta - P\theta, \xi\|.
\end{aligned}$$

Therefore $\|Pa - PPa, \xi\| < \|Pa - PPa, \xi\|$-contradiction and hence Pa is a common fixed point of K and P. Similarly,one shows the result, when the range of P is considered as complete, since $P(N) \subset K(N)$. Now let $y \in N$ be also common fixed point of K and P with $a \neq y$. Then we have

$$\begin{aligned}
\|a - y, \xi\| = \|Pa - Py, \xi\| &< \max\Bigg\{ \|Ka - Ky, \xi\|, \\
&\frac{\|Pa - Ka, \xi\| + \|Py - Ky, \xi\|}{2}, \frac{\|Pa - Ky, \xi\| + \|Py - Ka, \xi\|}{2} \Bigg\} \\
&= \max\left\{ \|a - y, \xi\|, \frac{\|a - a, \xi\| + \|y - y, \xi\|}{2}, \frac{\|a - y, \xi\| + \|y - a, \xi\|}{2} \right\} \\
&= \|a - y, \xi\|.
\end{aligned}$$

Therefore,$\|a - y, \xi\| < \|a - y, \xi\|$-contradiction and hence the proof completed. □

Theorem 3.2. *Let (f, R) and (g, T) be weakly compatible mappings of N of which one of their ranges is complete and satisfying property E.A, $f(N) \subset T(N)$, $g(N) \subset R(N)$ and $\|fu - gy, \xi\| \leq \phi(\max\{\|Ru - Ty, \xi\|, \|Ru - gy, \xi\|, \|Ty - gy, \xi\|\})$ for $u, y \in N$,where $\phi : N \to [0, \infty)$ is with $\phi(0) = 0$ and $0 < \phi(k) < k$ for all $k > 0$. Then f,g, R and T have a unique common fixed point.*

Proof. Since (f, R) holds property E.A, we have $\lim_{n\to\infty} \|f\alpha_n - \theta, \xi\| = 0$ and $\lim_{n\to\infty} \|R\alpha_n - \theta, \xi\| = 0$ for some sequence (α_n) in N and for some $\theta \in N$. Since (α_n)

in N and $f(N)\subset T(N)$, one can find a sequence $(\beta_n^{'})$ in N such that $f\alpha_n = T\beta_n^{'}$ for all $n \in \mathbb{N}$ and hence $\lim\limits_{n\to\infty} \left\|T\beta_n^{'} - \theta, \xi\right\| = 0$. Let $T(N)$ be complete. This implies that $\theta \in T(N)$ and hence $\theta = Ta$ for some $a \in N$. For each $n \in \mathbb{N}$, we consider

$$\|f\alpha_n - ga, \xi\| \leq \phi(\max\{\|R\alpha_n - Ta, \xi\|, \|R\alpha_n - ga, \xi\|, \|Ta - ga, \xi\|\}).$$

Now letting $n \to \infty$, we have

$$\begin{aligned}\|Ta - ga, \xi\| &\leq \phi(\max\{\|Ta - Ta, \xi\|, \|Ta - ga, \xi\|, \|Ta - ga, \xi\|\}) \\ &= \phi(\|Ta - ga, \xi\|).\end{aligned}$$

Therefore $\|Ta - ga, \xi\| \leq \|Ta - ga, \xi\|$. If $\|Ta - ga, \xi\| \neq 0$, by definition of ϕ, $\phi(\|Ta - ga, \xi\|) < \|Ta - ga, \xi\|$ and hence $\|Ta - ga, \xi\| < \|Ta - ga, \xi\|$-contradiction. Therefore $Ta = ga = \theta$. Since $ga \in gN$ and $g(N) \subset R(N)$, $ga = R\chi$ for some $\chi \in N$.Let us show that $R\chi = f\chi$. For this, we consider

$$\begin{aligned}&\|f\chi - R\chi, \xi\| = \|f\chi - ga, \xi\| \\ &\leq \phi(\max\{\|R\chi - Ta, \xi\|, \|R\chi - ga, \xi\|, \|Ta - ga, \xi\|\}) \\ &= \phi(0) = 0.\end{aligned}$$

Thus $\|f\chi - R\chi, \xi\| \leq 0$ and hence $f\chi = R\chi = ga = Ta = \theta$.Since (f, R) and (g, T) are weakly compatible, we must have $f\theta = R\theta$ and $g\theta = T\theta$. Now we show that $g\theta = \theta$. For, this, we have

$$\begin{aligned}\|f\chi - g\theta, \xi\| &\leq \phi(\max\{\|R\chi - T\theta, \xi\|, \|R\chi - g\theta, \xi\|, \|Ta - g\theta, \xi\|\}) \\ &= \phi(\|f\chi - g\theta, \xi\|).\end{aligned}$$

Therefore $\|f\chi - g\theta, \xi\| \leq\phi(\|f\chi - g\theta, \xi\|)$. If $\|f\chi - g\theta, \xi\| \neq 0$, then we must have $\|f\chi - g\theta, \xi\| < \|f\chi - g\theta, \xi\|$-contradiction. Thus $\|f\chi - g\theta, \xi\| = 0$ and hence $f\chi = g\theta = \theta$. Similarly, let us show that $f\theta = \theta$. For this, we have

$$\begin{aligned}&\|f\theta - ga, \xi\| \leq \phi(\max\{\|R\theta - Ta, \xi\|, \\ &\|R\theta - ga, \xi\|, \|Ta - ga, \xi\|\}) = \phi(\|f\theta - ga, \xi\|).\end{aligned}$$

Therefore $\|f\theta - ga, \xi\| \leq \phi(\|f\theta - ga, \xi\|)$. If $\|f\theta - ga, \xi\| \neq 0$, then by definition ϕ, $\|f\theta - ga, \xi\| < \|f\theta - ga, \xi\|$-contradiction. Therefore, $\|f\theta - ga, \xi\| = 0$ and hence $f\theta = ga = \theta$. As $f\theta = R\theta$ and $g\theta = T\theta$, then $f\theta = R\theta = g\theta = T\theta = \theta$. This shows that θ is a common fixed point of f, g, R and T. Similarly, we can show the result when $R(N)$ is complete. The cases in which $f(N)$ or $g(N)$ is complete are similar to the cases $T(N)$ or $R(N)$ is complete respectively, since $f(N)\subset T(N)$ and $g(N)\subset$

$R(N)$. Now let $\mu \in N$ be also common fixed point of f, g, R and T. For this, we have

$$\|\theta - \mu, \xi\| = \|f\theta - g\mu, \xi\| \leq \phi(\max\{\|R\theta - T\mu, \xi\|,$$
$$\|R\theta - g\mu, \xi\|, \|T\mu - g\mu, \xi\|\}) = \phi(\|\theta - \mu, \xi\|).$$

Thus $\|\theta - \mu, \xi\| \leq \phi(\|\theta - \mu, \xi\|)$. If $\|\theta - \mu, \xi\| \neq 0$, then $\|\theta - \mu, \xi\| < \|\theta - \mu, \xi\|$-contradiction and hence $\mu = \theta$. $\square$

Example 2. *Consider* $(N, \|.,.\|)$ *, where* $N = \mathbb{R}^2$ *and* $\|(\alpha, \theta)\| := |\alpha_1\theta_2 - \alpha_2\theta_1|$ *for* $\alpha := (\alpha_1, \alpha_2)$ *and* $\theta := (\theta_1, \theta_2)$ *in* N*. Now we define four self maps* f, g, R *and* T *on* N *by* $f(\alpha, u) := (\frac{\alpha}{4}, \frac{\alpha}{4})$*,* $g(\alpha, u) := (\frac{u}{4}, \frac{u}{4})$ *,* $R(\alpha, u) := (\alpha, \alpha)$ *and* $T(\alpha, u) := (u, u)$ *for* $(\alpha, u) \in N$ *and* $\phi : [0, \infty) \to [0, \infty)$ *by* $\phi(k) := \frac{k}{2}$ *for* $k \in [0, \infty)$*.*

Case(i): *Let* α, β *and* $\gamma \in N$*,where* $\alpha := (\alpha_1, \alpha_2)$*,* $\beta := (\beta_1, \beta_2)$ *and* $\gamma := (\gamma_1, \gamma_2)$*. Consider* $\|f\alpha - g\beta, \gamma\| = \left\|(\frac{\alpha_1}{4}, \frac{\alpha_1}{4}) - (\frac{\beta_1}{4}, \frac{\beta_1}{4}), (\gamma_1, \gamma_2)\right\| = \frac{\|(\alpha_1 - \beta_1, \alpha_1 - \beta_1), (\gamma_1, \gamma_2)\|}{4}$ *and* $\|R\alpha - T\beta, \gamma\| = \|(\alpha_1 - \beta_1, \alpha_1 - \beta_1), (\gamma_1, \gamma_2)\|$*.*

Case(ii): *Let* $f(\alpha, \beta) = R(\alpha, \beta)$ *for* $(\alpha, \beta) \in N$*. Then we have* $\alpha = 0$ *and hence* $fR(\alpha, \beta) = Rf(\alpha, \beta) = (0, 0)$*.Clearly,* $f(N) \subset T(N)$ *and* $g(N) \subset R(N)$*. Since* $f(N)$ *is one dimensional, then it is complete.*

Case(iii): *By taking* $\alpha_n = (\frac{1}{n}, \frac{1}{n})$ *in* N *for* $n \in \mathbb{N}$*,*$f\alpha_n \to (0, 0)$ *and* $R\alpha_n \to (0, 0)$ *and* $g\alpha_n \to (0, 0)$ *and* $T\alpha_n \to (0, 0)$*,*$n \to \infty$*. Hence*f, g, R *and* T *have a common fixed point by Theorem 3.2, namely zero.*

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