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ON INDEPENEDENCE NUMBER OF NIL GRAPH OF $\mathbb{Z}_n$, FOR SOME PARTICULAR VALUES OF n

SHAZIDA BEGUM[1] AND KUNTALA PATRA

ABSTRACT. Let R be a commutative ring. P.W. Chen [1] defined a kind of graph structure by considering the elements of R as the vertices of the graph. Any two elements $x, y \in R$ are adjacent if $xy \in N(R)$, where $N(R)$ denotes the set of nil elements of R. This definition was modified by A.H.Li and Q.S.Li [4] by considering the vertex set to be $R - \{0\}$. In our paper we adopt the modified definition given by A.H.Li and Q.S.Li. We call this graph as Nil Graph and determine the independence number of the nil graph $\Gamma_N(\mathbb{Z}_n)$, for some particular values of n.

1. INTRODUCTION

For the last few decades it has been an interesting and exciting topic of research to study algebraic structure by using properties of graph. In this field P.W.Chen [1] defined a new kind of graph structure for a commutative ring R, in which the set of vertices consists of all the elements of R and any two distinct vertices x and y are adjacent if and only if $xy \in N(R)$, $N(R)$ denotes the set of all nil-elements of R. This definition was later modified by A.H.Li and Q.S.Li [4] by considering the vertex set as $R^* = R - \{0\}$. Since the ring is commutative, the graph defined above is an undirected graph. Taking this concept M.J Nikmehr and S.Khojasteh [5] proved several results of $\Gamma_N(R)$ of matrix algebras.

[1]*corresponding author*

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A ring R is called non reduced if $N(R) \neq 0$. If R is a non reduced commutative ring, then for any $r \in R^*$ and $x \in N(R)^*$, r and x are adjacent in $\Gamma_N(R)$. Therefore $\mathcal{Z}_N(R)^* = |R| - 1$ by A.H.Li and Q.S.Li [4]. In our paper we consider a non reduced commutative ring R where $N(R) = \{x \in R \mid x^2 = 0\}$ and the graph $\Gamma_N(R)$ will be called nil graph of R. Taking the modified concept of the nil graph defined by A.H.Li and Q.S.Li [4], we determine the independence number of $\Gamma_N(\mathbb{Z}_n)$, for some particular values of n. Throughout the paper the size of a set A we mean the cardinality of the set and it is usually denoted by $|A|$.

Other valuable references are [2,3,6].

2. Preliminaries and Definitions

Definition 2.1. *An independent set in a graph G is a subset I of the vertex set of G such that no two vertices of I are adjacent. The independence number of G, denoted by $Indep(G)$, is defined as the cardinality of a maximum independent set of G.*

Definition 2.2. *The vertex connectivity $\kappa = \kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph. A vertex cut of a graph G is a subset S of the vertex set of G such that $G - S$ is disconnected.*

Definition 2.3. *Let R be a non reduced commutative ring and $N(R)$ be the set of all nil- elements of R, i.e, $N(R) = \{x \in R \mid x^2 = 0\}$. Let $\mathcal{Z}_N(R)^* = \{x \in R^* \mid xy \in N(R)\ for\ some\ y\ in\ R^* = R - \{0\}\}$. The Nil Graph of a ring R can be defined as the undirected graph $\Gamma_N(R)$ with vertex set $\mathcal{Z}_N(R)^*$ where two vertices x and y are adjacent if and only if $xy \in N(R)$ (or equivalently $yx \in N(R)$).*

Example 1. *Consider $R = \mathbb{Z}_8$. Here $N(\mathbb{Z}_8) = \{0, 4\}$.*

$V(\Gamma_N(\mathbb{Z}_8)) = \mathcal{Z}_N(\mathbb{Z}_8)^* = \{1, 2, 3, ...7\}$,

$E(\Gamma_N(\mathbb{Z}_8)) = \{(2, 4), (2, 6), (4, 6), (1, 4), (3, 4), (5, 4), (7, 4)\}$.

$V(\Gamma_N(\mathbb{Z}_8))$ and $E(\Gamma_N(\mathbb{Z}_8))$ respectively denote the vertex set and the edge set of the graph $\Gamma_N(\mathbb{Z}_8)$. The Figure 1 shows the nil graph $\Gamma_N(\mathbb{Z}_8)$.

3. Independence number of $\Gamma_N(\mathbb{Z}_n)$:

In this section we determine the independence number of $\Gamma_N(\mathbb{Z}_n)$, for $n = p^q$, where p is a prime and q is any positive integer.

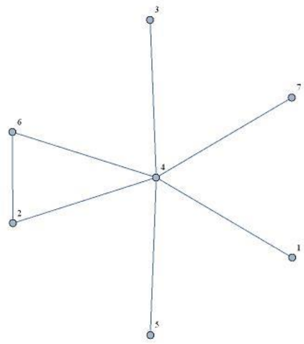

FIGURE 1. $\Gamma_N(\mathbb{Z}_8)$

Theorem 3.1. *Let p be any prime number and q be any odd positive integer, then*
(a) $Indep(\Gamma_N(\mathbb{Z}_{p^q})) = p^{4n+1} - p^{3n}, if q = 4n+1, n = 1, 2, ...$
(b) $Indep(\Gamma_N(\mathbb{Z}_{p^q})) = p^{4n+3} - p^{3n+2} + 1, if q = 4n+3, n = 0, 1, 2...$.

Proof. (a) If $q = 4n+1$, then the vertex set of the graph $\Gamma_N(\mathbb{Z}_{p^q})$ can be partitioned into $V_k = \{x = mp^k : p \nmid m, 1 \leq m \leq p^{4n+1-k} - 1\}$, for $0 \leq k \leq 4n$.

Let $x = m_1 p^{k_1} \in V_{k_1}$ and $y = m_2 p^{k_2} \in V_{k_2}$, then $xy = m_1 p^{k_1} m_2 p^{k_2} = m_1 m_2 p^{k_1+k_2}$.

Now $(xy)^2 = (m_1 m_2)^2 p^{2(k_1+k_2)}$. Then $xy \in N(\mathbb{Z}_{p^q})$ if $k_1 + k_2 \geq 2n+1$.

Therefore elements of V_{k_1} are adjacent to elements of V_{k_2} if $k_1 + k_2 \geq 2n+1$. Again for $k_1 + k_2 < 2n+1$, no two elements of V_k are adjacent to each other.

The set $I = \bigcup_{k=0}^{n} V_k$ is an independent set since any two elements of these sets are not adjacent among themselves. Again the set $\bigcup_{k=n+1}^{4n} V_k$ forms a complete subgraph of $\Gamma_N(\mathbb{Z}_{p^q})$.Therefore we get I is an independent set with maximum cardinality. Hence the size of I, which is the independence number, is equal to $|I| = \sum_{k=0}^{n}(p^{4n+1-k} - p^{4n-k}) = p^{4n+1} - p^{3n}$.

(b) If $q = 4n+3$, then the vertex set of the graph $\Gamma_N(\mathbb{Z}_{p^q})$ can be partitioned into $V_k = \{x = mp^k : p \nmid m, 1 \leq m \leq p^{4n+3-k} - 1\}$, for $0 \leq k \leq 4n+2$.

Let $x = m_1 p^{k_1} \in V_{k_1}$ and $y = m_2 p^{k_2} \in V_{k_2}$, then $xy = m_1 p^{k_1} m_2 p^{k_2} = m_1 m_2 p^{k_1+k_2}$.

Now $(xy)^2 = (m_1 m_2)^2 p^{2(k_1+k_2)}$. Then $xy \in N(\mathbb{Z}_{p^q})$ if $k_1 + k_2 \geq 2n+2$.

Therefore elements of V_{k_1} are adjacent to elements of V_{k_2} if $k_1 + k_2 \geq 2n+2$. Again for $k_1 + k_2 < 2n+2$, no two elements of V_k are adjacent to each other.

Taking $I = \bigcup_{k=0}^{n} V_k \cup \{a\}$, where $a \in V_{n+1}$, is an independent set of $\Gamma_N(\mathbb{Z}_{p^q})$ since the elements of the set $\bigcup_{k=0}^{n} V_k$ are not adjacent among themselves and also not adjacent to any element of V_{n+1}. The set $\bigcup_{k=n+1}^{4n+2} V_k$ forms a complete subgraph of $\Gamma_N(\mathbb{Z}_{p^q})$, so we get I is an independent set with maximum cardinality. The size of I, which is the independence number, is equal to $|I| = 1 + \sum_{k=0}^{n}(p^{4n+3-k} - p^{4n+2-k}) = p^{4n+3} - p^{3n+2} + 1$. □

Theorem 3.2. *Let p be any prime number and q be an even positive integer, then*

(a) $Indep(\Gamma_N(\mathbb{Z}_{p^q})) = p^{4n} - p^{3n} + 1, if q = 4n, n = 1, 2, ...$

(b) $Indep(\Gamma_N(\mathbb{Z}_{p^q})) = p^{4n+2} - p^{3n+1}, if q = 4n + 2, n = 0, 1, 2, ...$

Proof. (a) If $q = 4n$, then the vertex set of the graph $\Gamma_N(\mathbb{Z}_{p^q})$ can be partitioned into $V_k = \{x = mp^k : p \nmid m, 1 \leq m \leq p^{4n-k} - 1\}$, $0 \leq k \leq 4n - 1$.

Let $x = m_1p^{k_1} \in V_{k_1}$ and $y = m_2p^{k_2} \in V_{k_2}$, then $xy = m_1p^{k_1}m_2p^{k_2} = m_1m_2p^{k_1+k_2}$. Now $(xy)^2 = (m_1m_2)^2p^{2(k_1+k_2)}$. Then $xy \in N(\mathbb{Z}_{p^q})$ if $k_1 + k_2 \geq 2n$.

Therefore any two elements $x \in V_{k_1}$ and $y \in V_{k_2}$ are adjacent if $k_1 + k_2 \geq 2n$. Again for $k_1 + k_2 < 2n$, no two elements of V_k are adjacent to each other.

The set $I = \bigcup_{k=0}^{n-1} V_k \cup \{a\}$, where $a \in V_n$, is an independent set since any two elements of these sets $\bigcup_{k=0}^{n-1} V_k$ are not adjacent among themselves and also not adjacent to any element of the set V_n. Again the set $\bigcup_{k=n}^{4n-1} V_k$ forms a complete subgraph of $\Gamma_N(\mathbb{Z}_{p^q})$, therefore we get I is an independent set with maximum cardinality. Therefore independence number is equal to $|I| = 1 + \sum_{k=0}^{n-1}(p^{4n-k} - p^{4n-1-k}) = p^{4n} - p^{3n} + 1$.

(b) If $q = 4n + 2$, then the vertex set of the graph $\Gamma_N(\mathbb{Z}_{p^q})$, can be partitioned into $V_k = \{x = mp^k : p \nmid m, 1 \leq m \leq p^{4n+2-k} - 1\}$, for $0 \leq k \leq 4n + 1$.

Let $x = m_1p^{k_1} \in V_{k_1}$ and $y = m_2p^{k_2} \in V_{k_2}$, then $xy = m_1p^{k_1}m_2p^{k_2} = m_1m_2p^{k_1+k_2}$.

Now $(xy)^2 = (m_1m_2)^2p^{2(k_1+k_2)}$. Then $xy \in N(\mathbb{Z}_{p^q})$ if $k_1 + k_2 \geq 2n + 1$.

Therefore any two elements $x \in V_{k_1}$ and $y \in V_{k_2}$ are adjacent if $k_1+k_2 \geq 2n+1$.

Again for $k_1 + k_2 < 2n + 1$, no two elements of V_k are adjacent to each other.

Taking $I = \bigcup_{k=0}^{n} V_k$ is an independent set of $\Gamma_N(\mathbb{Z}_{p^q})$ since the elements of the set $\bigcup_{k=0}^{n} V_k$ are not adjacent among themselves. The set $\bigcup_{k=n+1}^{4n+1} V_k$ forms a complete subgraph of $\Gamma_N(\mathbb{Z}_{p^q})$, so we get I is an independent set with maximum cardinality. The size of I, which is the independence number, is equal to $|I| = \sum_{k=0}^{n}(p^{4n+2-k} - p^{4n+1-k}) = p^{4n+2} - p^{3n+1}$. □

Corollary 3.1. *Let $\Gamma_N(\mathbb{Z}_{p^k})$ be the nil graph for the commutative ring $\mathbb{Z}_{p^k}$, where p is a prime number and $k \geq 2$ be any positive integer, then the independence number of $\Gamma_N(\mathbb{Z}_{p^k})$ is*

(i) $p(p - 1)$, *for $k = 2$;*

(ii) $p^3 - p^2 + 1$, *for* $k = 3$;
(iii) $p^4 - p^3 + 1$, *for* $k = 4$;
(iv) $p^5 - p^3$, *for* $k = 5$;
(v) $p^6 - p^4$, *for* $k = 6$.

Theorem 3.3. *If p and q are distinct primes with* $p < q$*, then* $Indep(\Gamma_N(\mathbb{Z}_{p^2q^2})) = p^2q^2 - p^2q$.

Proof. The vertex of the graph can be partitioned into the following sets

$$V_1 = \{mpq : p \nmid m, q \nmid m, 1 \leq m \leq pq - 1\}$$
$$V_2 = \{mp : q \nmid m; 1 \leq m \leq pq^2 - 1\}$$
$$V_3 = \{mq : p \nmid m; 1 \leq m \leq p^2q - 1\}$$
$$V_4 = \{m : p \nmid m, q \nmid m; 1 \leq m \leq p^2q^2 - 1\}$$

Any two vertices x and y of $\Gamma_N(\mathbb{Z}_{p^2q^2})$ are adjacent to each other if $xy \in N(\mathbb{Z}_{p^2q^2})$ i.e, if $pq \mid xy$.

The set $H = V_1 \bigcup \{a\} \bigcup \{b\}$, where $a \in V_2$ and $b \in V_3$, is a complete subgraph of $\Gamma_N(\mathbb{Z}_{p^2q^2})$. The set $I = V_2 \bigcup V_4$ is an independent set in $\Gamma_N(\mathbb{Z}_{p^2q^2})$. We will show that I has a maximum cardinality among all the independent sets in $\Gamma_N(\mathbb{Z}_{p^2q^2})$. Assume that I' is any independent set in $\Gamma_N(\mathbb{Z}_{p^2q^2})$. If $I' \bigcap H = \phi$, then I' is a subset of V_4. Since $|V_4| < |I|$, therefore the size of I' is less than the size of I. If I' contains at least an element of H, say c and $c \in V_2$ or $c \in V_3$, then I' is a subset of either $V_2 \bigcup V_4$ or $V_3 \bigcup V_4$. We have $|V_3| < |V_2|$ since $p < q$. Hence $I = V_2 \bigcup V_4$ is an independent set with maximum cardinality among all independent set in $\Gamma_N(\mathbb{Z}_{p^2q^2})$. The size of I is the independence number of $\Gamma_N(\mathbb{Z}_{p^2q^2})$. Therefore

$$\begin{aligned}|I| = |V_2 \bigcup V_4| = |V_2| + |V_4| = \{(q^2p - 1) - (pq - 1)\}\\ + \{(p^2q^2 - 1) - (q^2p - 1) - (p^2q - 1) + (pq - 1)\}\\ = p^2q^2 - p^2q.\end{aligned}$$

□

4. CONCLUSION

From the above discussions we can make the following conclusions.

Between any two non nil-elements of the nil graph of $\mathbb{Z}_n$, there exists a path through the non zero nil-elements of the graph. Thus the set of all nil-elements of the graph forms a vertex cut of the graph.

Let A be another vertex cut of the graph such that $|A| < |N(\mathbb{Z}_n)^*|$ and $A \nsubseteq N(\mathbb{Z}_n)^*$. Now we can consider the following cases:

Case 1: If $A \bigcap N(\mathbb{Z}_n)^* \neq \phi$, then $\Gamma_N(\mathbb{Z}_n) \setminus A$ contains at least one non zero nil element through which all the non nil-elements are connected. So the graph cannot be disconnected by the removal of the elements of A. Hence A cannot be a vertex cut.

Case 2: If $A \bigcap N(\mathbb{Z}_n)^* = \phi$, then $\Gamma_N(\mathbb{Z}_n) \setminus A$ is connected because any two non nil-elements of the graph are connected through a non zero nil element.

So in any case the graph $\Gamma_N(\mathbb{Z}_n) \setminus A$ is connected and there exists no vertex cut lesser than the set of all non zero nil-elements of the graph. Therefore we can conclude that the connectivity of the graph $\Gamma_N(\mathbb{Z}_n)$ is equal to the cardinality of the set of the non zero nil-elements of the ring, i.e, $\kappa = |N(\mathbb{Z}_n)^*|$.

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DEPARTMENT OF MATHEMATICS
THE ASSAM ROYAL GLOBAL UNIVERSITY
BETKUCHI, GUWAHATI-35, INDIA
Email address: shazida.begum17@gmail.com

DEPARTMENT OF MATHEMATICS
GAUHATI UNIVERSITY
GUWAHATI-781014, INDIA
Email address: kuntalapatra@gmail.com