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POLAR COORDINATE FORMS FOR THE REGULARITY OF BICOMPLEX-VALUED FUNCTIONS IN CLIFFORD ANALYSIS

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ABSTRACT. In this study, using forms of conjugations, we provide some algebraic properties of bicomplex numbers. We studied an analogous Cauchy-Riemann system with a bicomplex number system. In addition, we investigated the definition and properties of polar coordinate forms for bicomplex numbers and Cauchy-Riemann systems that are equivalent in a bicomplex setting in Clifford analysis.

1. INTRODUCTION

Let p be a bicomplex number, introduced by Segre [15] in 1892, consisting of complex coefficients and $\mathbb{C}_B$ be the set $\mathbb{C}_B$ of bicomplex numbers such that

$$\mathbb{C}_B := \{p = z_1 + z_2 j \mid z_1 = x_0 + x_1 i,\ z_2 = x_2 + x_3 i \in \mathbb{C},\ x_r \in \mathbb{R}\ (r = 0,1,2,3)\},$$

where i is an imaginary unit $\sqrt{-1}$ defined in complex analysis and j is another imaginary unit satisfying

$$ij = ji,\ i^2 = j^2 = -1.$$

The set $\mathbb{C}_B$ is identical to $\mathbb{C}^2$.

In 19th century, Hamilton and Cayley considered for the extended the field of complex numbers. Hamilton [4] introduced the quaternions, a skew field in real-dimension four, while Cockle [2] introduced a commutative four dimensional real algebra, which was referred to bicomplex numbers in 1849 by Segre [15]. In

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recent years, Price [12] has been interested in the study of bicomplex numbers and holomorphic functions defined on a bicomplex variable.

In the present paper (see [5, 8, 9, 13]), an attempt has been made to discuss the extensive algebraic properties of bicomplex numbers. Properties of three types of conjugation of bicomplex Numbers have been determined and established some relation between them. The invertible and non-invertible elements of bicomplex numbers have been investigated. Alpay et al. [1] have provided the foundations for a study of modules of bicomplex holomorphic functions with bicomplex scalars. Based on the algebraic properties of bicomplex numbers, bicomplex-valued functions defined on bicomplex numbers have been proposed and the results have been obtained to deal with an analytic approach to functions of bicomplex numbers (see [14, 16, 17]).

Recent works (see [3, 10, 11, 18]) have studied any advanced function theory for bicomplex function spaces and analytic functionals on bicomplex holomorphic spaces . These results have been expected by considering only the $\mathbb{C}$–linear structure. The novel ideas of bicomplex functional analysis have been studied. In particular, the general ideas have been developed for a bicomplex functional analysis can be directly employed to generalize the classical complex analysis to the bicomplex setting. Kim and Shon [6] have researched elementary functions and the analogous Cauchy-Riemann system in bicomplex number systems. Kim and Shon [7] have investigated the definition and properties of regular functions with values in bicomplex settings in Clifford analysis.

In this paper, we investigate the differential operator of a function with a bicomplex number as a variable. From the commutativity of the product for bicomplex numbers, we give a corresponding Cauchy-Riemann equation for bicomplex-valued functions of a bicomplex variable. In addition, the polar coordinate form is applied to the corresponding Cauchy-Riemann equation and proposed each notion of regularity and harmonicity of bicomplex-valued functions.

2. Preliminaries

Consider a bicomplex number $p = z_1 + z_2 j$, where $z_1 = x_0 + x_1 i,\ z_2 = x_2 + x_3 i \in \mathbb{C}$ and $x_r \in \mathbb{R}$ $(r = 0, 1, 2, 3)$. Since i and j have the commutation rule each other over $\mathbb{C}_B$, for two bicomplex numbers $p = z_1 + z_2 j$ and $q = w_1 + w_2 j$, the product for bicomplex numbers is $pq = (z_1 w_1 - z_2 w_2) + (z_1 w_2 + z_2 w_1) j$. The conjugate p^* of p is given by $p^* = z_1 - z_2 j$, and the modulus $M(p)$ is given by $M(p) := pp^* = z_1^2 + z_2^2 \in \mathbb{C}$ which is a complex number. The inverse p^{-1} of p is given by $p^{-1} = \frac{p^*}{pp^*}$ and so $pp^* = 1$.

Let $f : \mathbb{C}_B \to \mathbb{C}_B$ be a bicomplex function, denoted by

$$f(p) = u(z_1, z_2) + v(z_1, z_2)j,$$

where $u, v : \mathbb{C}^2 \to \mathbb{C}$ are complex-valued functions. As the definition of derivative in complex analysis, we consider the definition of the derivative for a bicomplex function f of a bicomplex variable

Definition 2.1. *Let Ω be an open domain in $\mathbb{C}_B$ and let $f : \Omega \to \mathbb{C}_B$ be real differentiable. The function f is said to be hyperholomorphic if for any $p \in \mathbb{C}_B$ the limit*

$$\frac{df}{dp} = \lim_{h \to 0} h^{-1}\{f(p+h) - f(p)\}$$

exists. The limit $\frac{df}{dp}$ is called the bicomplex-derivative for bicomplex numbers.

Definition 2.2. *Let Ω be an open domain in $\mathbb{C}_B$ and let $f = u + vj$ be a bicomplex-valued function. The function f is called hyperholomorphic in $\Omega \subset \mathbb{C}_B$ if*

(i) *u and v are continuously differentiable function on Ω, and*

(ii) *$DF = 0$ in Ω, where D is the differential operator such that*

$$D = \frac{1}{2}\Big(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} j\Big).$$

The equation $Df = 0$ of the definition of a hyperholomorphic function f is equivalent to the following system of complex differential equations:

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \quad \frac{\partial f_1}{\partial z_2} = -\frac{\partial f_2}{\partial z_1},$$

and is called the bicomplex Cauchy-Riemann system on $\mathbb{C}_B$.

3. Hyperholomorphic functions of bicomplex numbers

Let Ω be an open set of $\mathbb{C}_B$. Consider a bicomplex function $f = u + vj : \Omega \subset \mathbb{C}_B \to \mathbb{C}_B$ such that

$$f(p) = u(z_1, z_2) + v(z_1, z_2)j,$$

where u and v are complex-valued functions.

Definition 3.1. *Let Ω be an open set of $\mathbb{C}_B$. A bicomplex function $f \in \mathcal{C}^1(U)$ is said to be bihyperholomorphic if f is invertible and f and f^{-1} are hyperholomorphic, where*

$$f^{-1} = \frac{u}{u^2+v^2} - \frac{v}{u^2+v^2}j.$$

Theorem 3.1. *Let Ω be a bounded open set in $\mathbb{C}_B$. If a function f is hyperholomorphic on Ω, then a function f^{-1} is hyperholomorphic on Ω.*

Proof. Since f^{-1} exists for a bicomplex function f, u and v are nonzero. By the definition of hyperholomorphy, we show that the equation $Df^{-1} = 0$ is satisfied. Consider the following process:

$$\begin{aligned} Df^{-1} &= \Big(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}j\Big)\Big(\frac{u}{u^2+v^2} - \frac{v}{u^2+v^2}j\Big) \\ &= \frac{1}{u^2+v^2}\frac{\partial u}{\partial z_1} - \frac{1}{(u^2+v^2)^2}\Big(2u^2\frac{\partial u}{\partial z_1} + 2uv\frac{\partial v}{\partial z_1}\Big) \\ &\quad + \frac{1}{u^2+v^2}\frac{\partial v}{\partial z_2} - \frac{1}{(u^2+v^2)^2}\Big(2uv\frac{\partial u}{\partial z_2} + 2v^2\frac{\partial v}{\partial z_2}\Big) \\ &\quad + \{\frac{-1}{u^2+v^2}\frac{\partial v}{\partial z_1} + \frac{1}{(u^2+v^2)^2}\Big(2uv\frac{\partial v}{\partial z_1} + 2v^2\frac{\partial v}{\partial z_1}\Big)\}j \\ &\quad + \{\frac{1}{u^2+v^2}\frac{\partial u}{\partial z_2} - \frac{1}{(u^2+v^2)^2}\Big(2u^2\frac{\partial u}{\partial z_2} + 2uv\frac{\partial u}{\partial z_2}\Big)\}j. \end{aligned}$$

The above terms are arranged as follows:

$$\begin{aligned} Df^{-1} &= \frac{-u^2+v^2}{(u^2+v^2)^2}\Big(\frac{\partial u}{\partial z_1} - \frac{\partial v}{\partial z_2}\Big) - \frac{2uv}{(u^2+v^2)^2}\Big(\frac{\partial v}{\partial z_1} + \frac{\partial u}{\partial z_2}\Big) \\ &\quad + \{\frac{-u^2+v^2}{(u^2+v^2)^2}\Big(\frac{\partial u}{\partial z_2} + \frac{\partial v}{\partial z_1}\Big) + \frac{2uv}{(u^2+v^2)^2}\Big(\frac{\partial v}{\partial z_2} - \frac{\partial u}{\partial z_1}\Big)\}j. \end{aligned}$$

Applying the bicomplex Cauchy-Riemann system on $\mathbb{C}_B$, we have $Df^{-1} = 0$. Thus, the result is obtained. $\square$

4. Polar coordinate forms of bicomplex numbers

For the complex numbers z_1 and z_2 expressed as constituent elements in the form of the bicomplex number used in this paper, we consider the polar coordinate form for the bicomplex numbers as follows:

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1),$$

where $r_1 := \sqrt{x_0^2 + x_1^2}$ is the norm used in complex analysis and θ_1 is the angle between the vector $z_1 \in \mathbb{C}$ and the real axis with $0 \le \theta_1 \le 2\pi$. Also, z_2 can be written by

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

where $r_2 := \sqrt{x_2^2 + x_3^2}$ and θ_2 is the angle between the vector $z_2 \in \mathbb{C}$ and the real axis with $0 \le \theta_2 \le 2\pi$.

Suppose the first-order partial derivatives of u and v with respect to r and θ, and the chain rule for complex-valued functions can be used to write in terms of r and θ with respect to x and y. For $z_k = r_k(\cos\theta_1 + i\sin\theta_k) \in \mathbb{C}$, we have

$$\begin{aligned} dz_k &= \frac{\partial z_k}{\partial r_k} dr_k + \frac{\partial z_k}{\partial \theta_k} d\theta_k \\ &= (\cos\theta_k + i\sin\theta_k)dr_k + r_k(-\sin\theta_k + i\cos\theta_k)d\theta_k. \quad (k = 1, 2) \end{aligned}$$

Hence, we obtain for $k = 1, 2$,

$$\begin{aligned} \frac{\partial}{\partial z_k} &= (\cos\theta_k - i\sin\theta_k)\frac{\partial}{\partial r_k} + \frac{-i}{r_k}\frac{1}{(-i\sin\theta_k + \cos\theta_k)}\frac{\partial}{\partial \theta_k} \\ &= \exp(-i\theta_k)\frac{\partial}{\partial r_k} + \frac{-i}{r_k}\exp(-i\theta_k)\frac{\partial}{\partial \theta_k} \\ &= \exp(-i\theta_k)\Big(\frac{\partial}{\partial r_k} + \frac{-i}{r_k}\frac{\partial}{\partial \theta_k}\Big). \end{aligned} \tag{4.1}$$

In detail, the differential operator

$$D = \frac{1}{2}\Big(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} j\Big)$$

can be written by

$$D = \frac{1}{2}\exp(-i\theta_1)\Big(\frac{\partial}{\partial r_1} - \frac{i}{r_1}\frac{\partial}{\partial \theta_1}\Big) + \frac{1}{2}\exp(-i\theta_2)\Big(\frac{\partial}{\partial r_2} - \frac{i}{r_2}\frac{\partial}{\partial \theta_2}\Big))j,$$

and more specifically, $D = \frac{1}{2}\Big(\partial_1 + \partial_2\Big)$, where

$$\partial_1 := \Big(\cos\theta_1\frac{\partial}{\partial r_1} - \frac{1}{r_1}\sin\theta_1\frac{\partial}{\partial \theta_1}\Big) - \Big(\sin\theta_1\frac{\partial}{\partial r_1} + \frac{1}{r_1}\cos\theta_1\frac{\partial}{\partial \theta_1}\Big)i$$

and

$$\partial_2 := \Big(\cos\theta_2\frac{\partial}{\partial r_2} - \frac{1}{r_2}\sin\theta_2\frac{\partial}{\partial \theta_2}\Big) - \Big(\sin\theta_2\frac{\partial}{\partial r_2} + \frac{1}{r_2}\cos\theta_2\frac{\partial}{\partial \theta_2}\Big)i.$$

Lemma 4.1. *If the partial derivatives of u and v with respect to z_1 and z_2 also satisfy the bicomplex Cauchy-Riemann equations, then the Cauchy-Riemann equation can be also expressed as follows:*

$$\cos\theta_1\frac{\partial u}{\partial r_1}-\frac{1}{r_1}\sin\theta_1\frac{\partial u}{\partial \theta_1}=\cos\theta_2\frac{\partial v}{\partial r_2}-\frac{1}{r_2}\sin\theta_2\frac{\partial v}{\partial \theta_2}$$

and

$$\frac{1}{r_1}\cos\theta_1\frac{\partial u}{\partial \theta_1}+\sin\theta_1\frac{\partial u}{\partial r_1}=\frac{1}{r_2}\cos\theta_2\frac{\partial v}{\partial \theta_2}+\sin\theta_2\frac{\partial v}{\partial r_2}.$$

Proof. The components z_1 and z_2 of a bicomplex variable p can be expressed by a polar coordinate form with r_k and θ_k $(k=1,2)$. From the equation (4.1), we have

$$\begin{aligned}
\frac{\partial u}{\partial z_1}&=\exp(-i\theta_1)\Big(\frac{\partial u}{\partial r_1}+\frac{-i}{r_1}\frac{\partial u}{\partial \theta_1}\Big),\ \frac{\partial u}{\partial z_2}=\exp(-i\theta_2)\Big(\frac{\partial u}{\partial r_2}+\frac{-i}{r_2}\frac{\partial u}{\partial \theta_2}\Big),\\
\frac{\partial v}{\partial z_1}&=\exp(-i\theta_1)\Big(\frac{\partial v}{\partial r_1}+\frac{-i}{r_1}\frac{\partial v}{\partial \theta_1}\Big),\ \frac{\partial v}{\partial z_2}=\exp(-i\theta_2)\Big(\frac{\partial v}{\partial r_2}+\frac{-i}{r_2}\frac{\partial v}{\partial \theta_2}\Big).
\end{aligned}$$

By replacing the bicomplex Cauchy-Riemann equations with r_k and θ_k $(k=1,2)$.

$$\begin{aligned}
&\exp(-i\theta_1)\frac{\partial u}{\partial r_1}=\exp(-i\theta_2)\frac{\partial v}{\partial r_2},\ \exp(-i\theta_1)\frac{\partial v}{\partial r_1}=-\exp(-i\theta_2)\frac{\partial u}{\partial r_2},\\
&\frac{1}{r_1}\exp(-i\theta_1)\frac{\partial u}{\partial \theta_1}=\frac{1}{r_2}\exp(-i\theta_2)\frac{\partial v}{\partial \theta_2},\ \frac{1}{r_1}\exp(-i\theta_1)\frac{\partial v}{\partial \theta_1}=-\frac{1}{r_2}\exp(-i\theta_2)\frac{\partial u}{\partial \theta_2}.
\end{aligned}$$

If the above relations are classified into real and imaginary parts, and the above relations are reorganized by applying the definition of equality of complex numbers, they are as follows:

$$\begin{aligned}
\cos\theta_1\frac{\partial u}{\partial r_1}-\frac{1}{r_1}\sin\theta_1\frac{\partial u}{\partial \theta_1}&=\cos\theta_2\frac{\partial v}{\partial r_2}-\frac{1}{r_2}\sin\theta_2\frac{\partial v}{\partial \theta_2},\\
\cos\theta_1\frac{\partial v}{\partial r_1}-\frac{1}{r_1}\sin\theta_1\frac{\partial v}{\partial \theta_1}&=-\cos\theta_2\frac{\partial u}{\partial r_2}+\frac{1}{r_2}\sin\theta_2\frac{\partial u}{\partial \theta_2},\\
\frac{1}{r_1}\cos\theta_1\frac{\partial u}{\partial \theta_1}+\sin\theta_1\frac{\partial u}{\partial r_1}&=\frac{1}{r_2}\cos\theta_2\frac{\partial v}{\partial \theta_2}+\sin\theta_2\frac{\partial v}{\partial r_2},\\
\frac{1}{r_1}\cos\theta_1\frac{\partial v}{\partial \theta_1}+\sin\theta_1\frac{\partial v}{\partial r_1}&=-\frac{1}{r_2}\cos\theta_2\frac{\partial u}{\partial \theta_2}-\sin\theta_2\frac{\partial u}{\partial r_2}.
\end{aligned}$$

In this case, the following equations are satisfied

$$\begin{aligned}
\cos\theta_1\frac{\partial u}{\partial r_1}-\frac{1}{r_1}\sin\theta_1\frac{\partial u}{\partial\theta_1} &= \frac{1}{r_1}\cos\theta_1\frac{\partial v}{\partial\theta_1}+\sin\theta_1\frac{\partial v}{\partial r_1},\\
\cos\theta_2\frac{\partial v}{\partial r_2}-\frac{1}{r_2}\sin\theta_2\frac{\partial v}{\partial\theta_2} &= -\frac{1}{r_2}\cos\theta_2\frac{\partial u}{\partial\theta_2}-\sin\theta_2\frac{\partial u}{\partial r_2},\\
-\cos\theta_1\frac{\partial v}{\partial r_1}+\frac{1}{r_1}\sin\theta_1\frac{\partial v}{\partial\theta_1} &= \frac{1}{r_1}\cos\theta_1\frac{\partial u}{\partial\theta_1}+\sin\theta_1\frac{\partial u}{\partial r_1},\\
\cos\theta_2\frac{\partial u}{\partial r_2}-\frac{1}{r_2}\sin\theta_2\frac{\partial u}{\partial\theta_2} &= \frac{1}{r_2}\cos\theta_2\frac{\partial v}{\partial\theta_2}+\sin\theta_2\frac{\partial v}{\partial r_2}.
\end{aligned}$$

by the definition of the partial derivative of the function expressed in the form of polar coordinates for the complex numbers and the expression of Cauchy-Riemann equations for the polar coordinates in the complex analysis, so the above equations are summarized as follows:

$$\cos\theta_1\frac{\partial u}{\partial r_1}-\frac{1}{r_1}\sin\theta_1\frac{\partial u}{\partial\theta_1}=\cos\theta_2\frac{\partial v}{\partial r_2}-\frac{1}{r_2}\sin\theta_2\frac{\partial v}{\partial\theta_2}$$

and

$$\frac{1}{r_1}\cos\theta_1\frac{\partial u}{\partial\theta_1}+\sin\theta_1\frac{\partial u}{\partial r_1}=\frac{1}{r_2}\cos\theta_2\frac{\partial v}{\partial\theta_2}+\sin\theta_2\frac{\partial v}{\partial r_2}.$$

Therefore, the bicomplex Cauchy-Riemann equations corresponding to r_k and θ_k $(k=1,2)$ in polar coordinates is represented by

$$\begin{cases}
\cos\theta_1\dfrac{\partial u}{\partial r_1}-\dfrac{1}{r_1}\sin\theta_1\dfrac{\partial u}{\partial\theta_1}=\cos\theta_2\dfrac{\partial v}{\partial r_2}-\dfrac{1}{r_2}\sin\theta_2\dfrac{\partial v}{\partial\theta_2},\\
\dfrac{1}{r_1}\cos\theta_1\dfrac{\partial u}{\partial\theta_1}+\sin\theta_1\dfrac{\partial u}{\partial r_1}=\dfrac{1}{r_2}\cos\theta_2\dfrac{\partial v}{\partial\theta_2}+\sin\theta_2\dfrac{\partial v}{\partial r_2}.
\end{cases}$$

□

Considering the differential operator D previously defined, Laplacian operator is presented as follows:

$$\Delta_{\mathbb{C}_B}:=\frac{1}{4}\Big(\frac{\partial^2}{\partial z_1^2}+\frac{\partial^2}{\partial z_2^2}\Big).$$

Definition 4.1. *Let Ω be a bounded open set in $\mathbb{C}_B$ and $F:\Omega\longrightarrow\mathbb{C}_B$ be a $\mathbb{C}_B$ function with values in $\mathbb{C}_B$. A function F is said to be a bicomplex harmonic function if it satisfies the differential equation*

$$\triangle_{\mathbb{C}_B}F=0 \text{ on } \Omega.$$

This function is said to be bicomplex harmonic if all its components u_k $(k = 0, 1, 2, 3)$ of F are bicomplex harmonic on Ω.

Remark 4.1. *If the function $f = u + Jv$ is bicomplex hyperholomorphic on an open set Ω in $\mathbb{C}_B$, then, the functions u and v are bicomplex harmonic on Ω.*

Theorem 4.1. *Let Ω be a bounded open set in $\mathbb{C}_B$, and let F be a biccomplex harmonic function on Ω. Then, the following equation*

$$\sum_{k=1}^{2}\Big(\frac{\partial^2 f}{\partial r_k^2} - \frac{1}{r_k^2}\frac{\partial^2 f}{\partial \theta_k^2}\Big) = 0$$

is satisfied.

Proof. Since f is a bicomplex harmonic function on Ω, its components u and v satisfy

$$\frac{\partial^2 u}{\partial z_1^2} + \frac{\partial^2 u}{\partial z_2^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial z_1^2} + \frac{\partial^2 v}{\partial z_2^2} = 0.$$

Each term can be written as the polar coordinate form such that

$$\begin{aligned}
\frac{\partial^2 u}{\partial z_k^2} &= \exp(-i2\theta_k)\Big(\frac{\partial^2 u}{\partial r_k^2} - i\frac{2}{r_k}\frac{\partial^2 u}{\partial r_k \partial \theta_k} - \frac{1}{r_k^2}\frac{\partial^2 u}{\partial \theta_k^2}\Big) \\
&= \Big(\frac{\partial^2 u}{\partial r_k^2}\cos 2\theta_k - \frac{1}{r_k^2}\frac{\partial^2 u}{\partial \theta_k^2}\cos 2\theta_k - \frac{2}{r_k}\sin 2\theta_k \frac{\partial^2 u}{\partial r_k \partial \theta_k}\Big) \\
&\quad + i\Big(\frac{1}{r_k^2}\frac{\partial^2 u}{\partial \theta_k^2}\sin 2\theta_k - \frac{\partial^2 u}{\partial r_k^2}\sin 2\theta_k - \frac{2}{r_k}\cos 2\theta_k \frac{\partial^2 u}{\partial r_k \partial \theta_k}\Big).
\end{aligned}$$

Since the equation

$$\frac{\partial^2 u}{\partial z_1^2} + \frac{\partial^2 u}{\partial z_2^2} = 0$$

is satisfied, the equations

$$\sum_{k=1}^{2}\Big(\frac{\partial^2 u}{\partial r_k^2} - \frac{1}{r_k^2}\frac{\partial^2 u}{\partial \theta_k^2}\Big)\cos 2\theta_k - \frac{2}{r_k}\sin 2\theta_k \frac{\partial^2 u}{\partial r_k \partial \theta_k} = 0, \tag{4.2}$$

$$\sum_{k=1}^{2}\Big(\frac{1}{r_k^2}\frac{\partial^2 u}{\partial \theta_k^2} - \frac{\partial^2 u}{\partial r_k^2}\Big)\sin 2\theta_k - \frac{2}{r_k}\cos 2\theta_k \frac{\partial^2 u}{\partial r_k \partial \theta_k} = 0 \tag{4.3}$$

should be satisfied. Multiplying $\cos 2\theta_k$ to the equation (4.2) and $-\sin 2\theta_k$ to (4.3), we have

$$\sum_{k=1}^{2}\Big(\frac{\partial^2 u}{\partial r_k^2}-\frac{1}{r_k^2}\frac{\partial^2 u}{\partial \theta_k^2}\Big)\cos^2 2\theta_k-\frac{2}{r_k}\cos 2\theta_k \sin 2\theta_k \frac{\partial^2 u}{\partial r_k \partial \theta_k}=0, \tag{4.4}$$

and

$$\sum_{k=1}^{2}\Big(\frac{\partial^2 u}{\partial r_k^2}-\frac{1}{r_k^2}\frac{\partial^2 u}{\partial \theta_k^2}\Big)\sin^2 2\theta_k+\frac{2}{r_k}\sin 2\theta_k \cos 2\theta_k \frac{\partial^2 u}{\partial r_k \partial \theta_k}=0. \tag{4.5}$$

Adding the equation (4.4) and (4.5), we obtain

$$\sum_{k=1}^{2}\Big(\frac{\partial^2 u}{\partial r_k^2}-\frac{1}{r_k^2}\frac{\partial^2 u}{\partial \theta_k^2}\Big)=0.$$

Through a similar process with respect to v, we also obtain

$$\sum_{k=1}^{2}\Big(\frac{\partial^2 v}{\partial r_k^2}-\frac{1}{r_k^2}\frac{\partial^2 v}{\partial \theta_k^2}\Big)=0.$$

□

5. CONCLUSION

In this paper, we investigate the differential operator of a function with a bicomplex number as a variable. A bicomplex number is an extension of a complex number and, unlike quaternions, preserves the commutative product between each basis of a bicomplex number. From the commutativity, unlike the quaternion, the defined Cauchy-Riemann equation for bicomplex-valued functions of a bicomplex variable was investigated. In addition, the polar coordinate form of the bicomplex numbers is presented by using the complex number expressed in the polar coordinate form. Since variables can be expressed in the form of polar coordinates, a function written in the form of polar coordinates is presented and Cauchy-Riemann equation for bicomplex-valued functions is proposed. Furthermore, the criterion for the harmonic function is expressed in the form of polar coordinates for the bicomplex numbers.

The regularity of bicomplex-valued functions can be defined as a general complex number, but a bicomplex number is a direct extension of a complex number. Thus, it is a natural process to consider whether the expression method used for complex numbers can be applied. The expression of the bicomplex number will be as diverse as the expression of the complex number. It is possible to propose a corresponding polar coordinates form of the Cauchy-Riemann equations, the regularity and harmonicity of a bicomplex-valued function of a bicomplex variable.

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