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RICCI-YAMABE SOLITONS ON SUBMANIFOLDS OF SOME INDEFINITE ALMOST CONTACT MANIFOLDS

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ABSTRACT. In this paper, we study Ricci-Yamabe soliton on invariant and anti-invariant submanifolds of indefinite Sasakian manifolds, indefinite Kenmotsu manifolds and indefinite trans-Sasakian manifolds concerning Riemannian connection and quarter symmetric metric connection.

1. INTRODUCTION

In the year 2019, Crasmareanu and Guler [4] confer the exploration of another geometric flow and that is a scalar blend of Yamabe and Ricci flow under the name called Ricci-Yamabe flow (RYF). The (RYF) is defined as follows [4]:

$$\frac{\partial}{\partial \mathrm{t}} g(\mathrm{t}) = -2p\ Ric(\mathrm{t}) + q\, r(\mathrm{t}) g(\mathrm{t}),\ \ go = g(0).$$

A solution to the (RYF) is called Ricci-Yamabe soliton, denoted as (RYS) and its (g, V, λ, p, q) on a Reimannian manifold (M, g) such that

$$\mathfrak{L}_V g + 2p\ S + (2\lambda - qr)g = 0. \tag{1.1}$$

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Here r is a scalar curvature , S is a Ricci tensor, $\mathfrak{L}_V$ denotes the Lie-derivative along the vector field and p, q are the scalars. The (RYS) is said to be steady, shrinking and expanding accordingly as λ is zero,negative and positive respectively. In this way, condition (1.1) is called (RYS) of (p,q)-type, which is a speculation of Yamabe and Ricci solitons. It notes us that (RYS) of type $(0,q)$ and $(p,0)$-type are q-Yamabe soliton and p-Ricci soliton separately.

S. Golab [3] characterized and examined quarter symmetric linear connection on a differentiable manifold. A straight association $\bar{\nabla}$ is an n-dimensional Reimannian manifold and is known as a quarter symmetric connection [3] if twist tensor T is of the structure

$$T(U_1,U_2)=\bar{\nabla}_{U_1}U_2-\bar{\nabla}_{U_2}U_1-[U_1,U_2]=\eta(U_2)\varphi U_1-\eta(U_1)\varphi U_2, \tag{1.2}$$

where η is a 1-form and φ is a tensor of type $(1,1)$. If a quarter symmetric linear connection $\bar{\nabla}$ fulfils the condition $(\bar{\nabla}_{U_1}g)(U_2,U_3)=0$, for all $U_1,U_2,U_3\in\chi(M)$, where $\chi(M)$ is a Lie algebra of vector fields on the manifold M, at that point $\bar{\nabla}$ is known as a quarter symmetric metric connection and is noted as $(QSMC)$. Somashekhara et al. [7], studied some results on invariant sub-manifolds of LP-Sasakian manifolds endowed with semi-symmetric metric conection. Also, they have obtained a condition for totally geodesic by using certain geometrical conditions. In [8], the authors studied the C-Bochner curvature tensor under D-homothetic deformation in LP-Sasakian manifolds.

2. Preliminaries

A $(2n+1)$-dimension semi-Riemannian manifold $(\bar{M},\bar{g})$ is called an indefinite almost contact manifold if it reaches an indefinite almost contact structure (φ,ξ,η), where φ is a tensor field of type (1,1), ξ is a vector field and η is a 1-form fulfilling for all vector fields U_1,U_2 on $\bar{M}$ [1].

$$\varphi^2U_1=-U_1+\eta(U_1)\xi,\eta\circ\varphi=0,\varphi\xi=0,\eta(\xi)=1,$$

$$\bar{g}(\varphi U_1,\varphi U_2)=\bar{g}(U_1,U_2)-\varepsilon\eta(U_1)\eta(U_2),$$

$$\bar{g}(U_1,\xi)=\varepsilon\eta(U_1),\bar{g}(\varphi U_1,U_2)=-\bar{g}(U_1,\varphi U_2).$$

Here $\varepsilon=\bar{g}(\xi,\xi)=\pm1$ and $\bar{\nabla}$ is the Levi-Civita connection for a semi-Riemannian metric $\bar{g}$.

An indefinite almost contact metric structure $(\varphi, \xi, \eta, \bar{g})$ is called an indefinite Sasakian structure is for all vector fields U_3, W on $\bar{M}$,

$$(\bar{\nabla}_{U_3}\varphi)W = \varepsilon\eta(W)U_3 - \bar{g}(U_3, W)\xi, \bar{\nabla}_{U_3}\xi = -\varepsilon\varphi U_3. \tag{2.1}$$

An indefinite almost contact metric structure $(\varphi, \xi, \eta, \bar{g})$ is called an indefinite trans-Sasakian structure of type (α, β) [5,6], if

$$\begin{aligned}(\bar{\nabla}_{U_3}\varphi)W_1 &= \alpha[\bar{g}(U_3, W_1)\xi - \varepsilon\eta(W_1)U_3] + \beta[\bar{g}(\varphi U_3, W_1)\xi - \varepsilon\eta(W_1)\varphi U_3],\\ \bar{\nabla}_{U_3}\xi &= -\varepsilon\alpha\varphi U_3 + \varepsilon\beta[U_3 - \eta(U_3)\xi].\end{aligned} \tag{2.2}$$

For smooth functions α, β on $\bar{M}$ and for all vector fields U_3, W_1 on $\bar{M}$. ε-Kenmotsu manifolds with indefinite metric by giving a case of $\alpha = 0, \beta = 1$, at that point indefinite almost contact metric structure $(\varphi, \xi, \eta, \bar{g})$ is said to be an indefinite Kenmotsu structure [2]. Hence, the structure conditions become:

$$(\bar{\nabla}_{U_3}\varphi)W_1 = [\bar{g}(\varphi U_3, W_1)\xi - \varepsilon\eta(W_1)\varphi U_3], \bar{\nabla}_{U_3}\xi = \varepsilon U_3 - \varepsilon\eta(U_3)\xi. \tag{2.3}$$

Make M be a submanifold of dimension m of a manifold $\bar{M}(m < n)$ with actuated metric g. Likewise let ∇ and $\nabla^{\perp}$ be the incited connection on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M individually. At that point the Weingarten and Gauss formulae are stated as:

$$\begin{aligned}\bar{\nabla}_{U_1}U_2 &= \nabla_{U_1}U_2 + h(U_1, U_2),\\ \bar{\nabla}_{U_1}V &= -A_V U_1 + \nabla^{\perp}_{U_1}V,\end{aligned} \tag{2.4}$$

for all $U_1, U_2 \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where h and A_V are second fundamental form and the shape operator (corresponding to the normal vector field V) respectively for the immersion of M into $\bar{M}$. The second fundamental form h and the shape operator A_V are related by [9]

$$g(h(U_1, U_2), V) = g(A_V U_1, U_2),$$

for any $U_1, U_2 \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. The mean curvature vector L on M is given by

$$L = \frac{1}{m}\Sigma_{i=1}^{m} g(e_i, e_i),\ \{e_i\}_{i=1}^{m}$$

is a local orthonormal frame of vector fields on M.

A submanifold M of a manifold $\bar{M}$ is called totally umbilical if

$$h(U_1, U_2) = g(U_1, U_2)L, \tag{2.5}$$

for $U_1, U_2 \in TM$. Moreover if $h(U_1, U_2) = 0$. Also, M is called totally geodesic and if $L = 0$, then M is minimal in $\bar{M}$.

A submanifold M of a manifold $\bar{M}$ is called invariant (anti-invariant) if ϕU_1 is tangent (normal) to M for every vector field U_1 tangent to M, that is, $\varphi(TM) \subset TM(\varphi(TM) \subset T^{\perp}M)$ at each pointed M. Throughout this paper, we denote invariant submanifolds as (ISM).

Let $\bar{\bar{\nabla}}$ be a linear connection and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ such that

$$\bar{\bar{\nabla}}_{U_1} U_2 = \bar{\nabla}_{U_1} U_2 + H(U_1, U_2),$$

where H is a (1,1) type tensor and $U_1, U_2 \in \Gamma(T\bar{M})$. For $\bar{\bar{\nabla}}$ to be a quarter symmetric metric connection $(QSMC)$ on $\bar{M}$, we have

$$H(U_1, U_2) = \frac{1}{2}[T(U_1, U_2) + T'(U_1, U_2) + T'(U_2, U_1)],$$

where

$$g(T'(U_1, U_2), U_3) = g(T(U_3, U_1), U_2). \tag{2.6}$$

From (1.2) and (2.6), we get:

$$T'(U_1, U_2) = \eta(U_1)\varphi U_2 - g(U_2, \varphi U_1)\xi,$$

$$H(U_1, U_2) = \eta(U_2)\varphi U_1 - g(U_2, \varphi U_1)\xi.$$

Thus, a $(QSMC)$ $\bar{\bar{\nabla}}$ in a manifold $\bar{M}$ is specified by

$$\bar{\bar{\nabla}}_{U_1} U_2 = \bar{\nabla}_{U_1} U_2 + \eta(U_2)\varphi U_1 - g(\varphi U_1, U_2)\xi. \tag{2.7}$$

3. (RYS) on Submanifolds of Indefinite Sasakian Manifolds in respect of Riemannian Connection

Suppose (g, ξ, λ, p, q) be a (RYS) on a submanifold M of an indefinite Sasakian manifold $\bar{M}$. We now have:

$$(\mathfrak{L}_\xi g)(U_1, U_2) + 2pS(U_1, U_2) + (2\lambda - qr)g(U_1, U_2) = 0. \tag{3.1}$$

From (2.1) and (2.4), we get:

$$-\varepsilon\varphi U_1 = \bar{\nabla}_{U_1}\xi = \nabla_{U_1}\xi + h(U_1, \xi). \tag{3.2}$$

Whether M is invariant in $\bar{M}$, in that case $\varphi U_1 \in TM$, hence equating tangential as well as normal component of (3.2), we get

$$\nabla_{U_1}\xi = -\varepsilon\varphi U_1, h(U_1, \xi) = 0. \tag{3.3}$$

Using (3.3) we get

$$(\mathfrak{L}_\xi g)(U_1, U_2) = -\varepsilon[g(\varphi U_1, U_2) + g(U_1, \varphi U_2)] = 0. \tag{3.4}$$

In view of (3.1) and (3.4) yields

$$S(U_1, U_2) = (\frac{qr - 2\lambda}{2p})g(U_1, U_2). \tag{3.5}$$

It suggests this M is Einstein. Additionally from (2.5) and (3.3) it obtains $\eta(U_1)L = 0$ i,e., $L = 0$, where as $\eta(U_1) \neq 0$. Consequently, M is minimal in $\bar{M}$. Thus it is following that:

Theorem 3.1. *If (g, ξ, λ, p, q) is a (RYS) on an (ISM) M of an indefinite Sasakian manifold $\bar{M}$, then M is minimal in $\bar{M}$ and also M is Einstein.*

Also from (3.3), we get:

$$S(U_1, \xi) = -\varphi U_1 + (n-1)\eta(U_1)\xi. \tag{3.6}$$

In view of (3.5) and (3.6), we come into $\lambda = \frac{qr - 2p(n-1)}{2}$. Thus we express that

Theorem 3.2. *A (RYS) on an (ISM) M of an indefinite Sasakian manifold $\bar{M}$ is shrinking or expanding or steady accordingly as:*

$$qr - 2p(n-1) < 0 \quad \textit{or} \quad qr - 2p(n-1) > 0 \quad \textit{or} \quad qr = 2p(n-1).$$

If $p = 0$ then $\lambda = \dfrac{qr}{2}$. Thus, we can state

Corollary 3.1. *A q-Yamabe soliton on an (ISM) M of an indefinite Sasakian manifold $\bar{M}$ is shrinking or expanding or steady accordingly as:*

$$qr < 0 \quad \textit{or} \quad qr > 0 \quad \textit{or} \quad qr = 0.$$

If $q = 0$ then $\lambda = p(1-n)$. We now have:

Corollary 3.2. *A p-Ricci soliton on an (ISM) M of an indefinite Sasakian manifold $\bar{M}$ is shrinking or expanding or steady accordingly as:*

$$p(1-n) < 0 \quad \textit{or} \quad p(1-n) > 0 \quad \textit{or} \quad p(1-n) = 0.$$

If M is anti-invariant in $\bar{M}$, then for any $U_1 \in TM$, $\varphi U_1 \in T^{\perp}M$ and hence from (3.2), it becomes

$$\nabla_{U_1}\xi = 0, h(U_1,\xi) = -\varepsilon\varphi U_1.$$

Using (3.4) it gives $(\mathfrak{L}_{\xi}g)(U_1,U_2) = 0$. It suggests this ξ is a Killing vector field and consequently (3.1) capitulates

$$S(U_1,U_2) = (\frac{qr - 2\lambda}{2p})g(U_1,U_2),$$

which infers that M is Einstein. It is expressing as:

Theorem 3.3. *If (g,ξ,λ,p,q) is a (RYS) on an anti-(ISM) M of an indefinite Sasakian manifold $\bar{M}$,then ξ is a Killing vector field and M is Einstein.*

As well as, $\nabla_{U_1}\xi = 0 \Rightarrow R(U_1,U_2)\xi = 0 \Rightarrow S(U_1,\xi) = 0 \Rightarrow \lambda = \frac{qr}{2}$. We now have:

Theorem 3.4. *A (RYS) (g,ξ,λ,p,q) on an anti-(ISM) M of an indefinite Sasakian manifold $\bar{M}$ is expanding or shrinking or steady accordingly as $qr > 0$ or $qr < 0$ or $qr = 0$.*

4. (RYS) on Submanifolds of Indefinite Sasakian Manifolds with respect to $(QSMC)$

Let us contemplate that (g,ξ,λ,p,q) is a (RYS) on a submanifold M of an indefinite Sasakian manifold $\bar{M}$ with respect to $(QSMC)$, where $\bar{\nabla}$ is the actuated connection on M from the connection$\bar{\bar{\nabla}}$, then we obtain

$$(\bar{\mathfrak{L}}_{\xi}g)(U_1,U_2) + 2p\bar{S}(U_1,U_2) + (2\lambda - qr)g(U_1,U_2) = 0. \tag{4.1}$$

Let $\bar{h}$ be the second fundamental form of $\bar{M}$ regarding induced connection $\bar{\nabla}$. At that point we have

$$\bar{\bar{\nabla}}_{U_1} = \bar{\nabla}_{U_1}U_2 + \bar{h}(U_1,U_2), \tag{4.2}$$

and hence by virtue of (2.4), (2.7) we get,

$$\bar{\nabla}_{U_1}U_2 + \bar{h}(U_1,U_2) = \nabla_{U_1}U_2 + h(U_1,U_2) + \eta(U_2)\varphi U_1 - g(\varphi U_1,U_2)\xi. \tag{4.3}$$

If M is an (ISM) of $\bar{M}$, then $\varphi U_1 \in TM$ for any $U_1 \in TM$ along with consequently collating tangential parts from (4.2), it becomes

$$\bar{\nabla}_{U_1}U_2 = \nabla_{U_1}U_2 + \eta(U_2)\varphi U_1 - g(\varphi U_1,U_2)\xi, \tag{4.4}$$

which express that M admits $(QSMC)$. Also from (4.4), we obtain

$$\bar{\nabla}_{U_1}\xi = (-\varepsilon + 1)\varphi U_1, \tag{4.5}$$

and hence

$$(\bar{\mathfrak{L}}_\xi g)(U_1, U_2) = (-\varepsilon + 1)g(\varphi U_1, U_2) + g(U_1, \varphi U_2) = 0. \tag{4.6}$$

Hence, from (4.1) we get

$$\bar{S}(U_1, U_2) = (\frac{qr - 2\lambda}{2p})g(U_1, U_2). \tag{4.7}$$

Hence, we can declare that:

Theorem 4.1. *Let (g, ξ, λ, p, q) be a (RYS) on an (ISM) M of an indefinite Sasakian manifold $\bar{M}$, with respect to $(QSMC)$ $\bar{\bar{\nabla}}$. Then M is Einstein with respect to induced Riemannian connection.*

Also from (4.5), it becomes

$$\bar{S}(U_1, \xi) = -\varphi U_1 + (n-1)(\varepsilon - 1)\eta(U_1). \tag{4.8}$$

Compare (4.7) and (4.8), we obtain $\lambda = \frac{qr-(n-1)(\varepsilon-1)2p}{2}$.

Theorem 4.2. *A (RYS) on an (ISM) M of an indefinite Sasakian manifold $\bar{M}$ is expanding or shrinking or steady accordingly as:*
$qr - (n-1)(\varepsilon - 1)2p > 0$ or $qr - (n-1)(\varepsilon - 1)2p < 0$ or $qr = (n-1)(\varepsilon - 1)2p$.

If $p = 0$ then $\lambda = \dfrac{qr}{2}$. Thus we can state

Corollary 4.1. *A q-Yamabe soliton on an (ISM) M of an indefinite Sasakian manifold $\bar{M}$ is expanding or shrinking or steady accordingly as:*

$$qr > 0 \quad \textit{or} \quad qr < 0 \quad \textit{or} \quad qr = 0.$$

If $q = 0$ then $\lambda = (1-n)(\varepsilon - 1)p$

Corollary 4.2. *A p-Ricci soliton on an (ISM) M of an indefinite Sasakian manifold $\bar{M}$ is expanding or shrinking or steady accordingly as:*

$$(1-n)(\varepsilon - 1)p > 0 \quad \textit{or} \quad (1-n)(\varepsilon - 1)p < 0 \quad \textit{or} \quad (1-n)(\varepsilon - 1)p = 0.$$

If M is an anti-(ISM) of $\bar{M}$ with respect to $(QSMC)$ then from (4.4), we have $\bar{\nabla}_{U_1}\xi = 0$ and hence $(\bar{\mathfrak{L}}_\xi g)(U_1, U_2) = 0$. We now obtain

$$\bar{S}(U_1, U_2) = (\frac{qr - 2\lambda}{2p})g(U_1, U_2).$$

Consequently we can express:

Theorem 4.3. *Let (g, ξ, λ, p, q) be a (RYS) on an anti-(ISM) M of an indefinite Sasakian manifold $\bar{M}$, with respect to $(QSMC)$ $\bar{\bar{\nabla}}$. Then M is Einstein with respect to induced Riemannian connection.*

Also, $\bar{\nabla}_{U_1}\xi = 0$. It implies $R(U_1, U_2)\xi = 0 \Rightarrow S(U_1, \xi) = 0$ then $\lambda = \frac{qr}{2}$.

Theorem 4.4. *A (RYS) (g, ξ, λ, p, q) on an anti-(ISM) M of an indefinite Sasakian manifold $\bar{M}$ is expanding or shrinking or steady accordingly as $qr > 0$ or $qr < 0$ or $qr = 0$.*

5. (RYS) ON SUBMANIFOLDS OF INDEFINITE KENMOTSU MANIFOLDS WITH RESPECT TO RIEMANNIAN CONNECTION

Let (g, ξ, λ, p, q) be a (RYS) on submanifold M of an indefinite Kenmotsu manifold $\bar{M}$ then we have

$$(\mathfrak{L}_\xi g)(U_1, U_2) + 2pS(U_1, U_2) + (2\lambda - qr)g(U_1, U_2) = 0. \tag{5.1}$$

From (2.3) and (2.4) we obtain,

$$\varepsilon[U_1 - \eta(U_1)\xi] = \bar{\nabla}_{U_1}\xi = \nabla_{U_1}\xi + h(U_1, \xi). \tag{5.2}$$

Comparing normal and tangential components of (5.2), we get:

$$\nabla_{U_1}\xi = \varepsilon[U_1 - \eta(U_1)\xi], h(U_1, \xi) = 0, \tag{5.3}$$

using (5.3) we get

$$(\mathfrak{L}_\xi g)(U_1, U_2) = 2\varepsilon[g(U_1, U_2) - \eta(U_1)\eta(U_2)]. \tag{5.4}$$

In view (5.4) and (5.1), we obtain

$$S(U_1, U_2) = (\frac{qr - 2\lambda - 2\varepsilon}{2p})g(U_1, U_2) + (\frac{\varepsilon}{p})\eta(U_1)\eta(U_2). \tag{5.5}$$

Also from (2.5) and (5.3) it gives $L = 0$ as $\eta(U_1) \neq 0$. Hence, we can state

Theorem 5.1. *If (g,ξ,λ,p,q) is (RYS) on a submanifold M of an indefinite Kenmotsu manifold $\bar{M}$. Then M is minimal in $\bar{M}$ and also M is η-Einstein.*

From (5.3), we get:

$$S(U_1,\xi) = [-\varepsilon(n-1) - \xi]\eta(U_1). \tag{5.6}$$

Assimilating (5.5) and (5.6), it yields $\lambda = \frac{qr+2p(\varepsilon(n-1)+\xi)}{2}$.

Theorem 5.2. *A (RYS) on an (ISM) M of an indefinite Kenmotsu manifold $\bar{M}$ is expanding or shrnking or steady accordingly as:*

$$qr > 2p(\varepsilon(n-1)+\xi) \quad \textit{or} \quad qr < 2p(\varepsilon(n-1)+\xi) \quad \textit{or} \quad qr = 2p(\varepsilon(n-1)+\xi).$$

If $p=0$ then $\lambda = \frac{qr}{2}$. Thus we have

Corollary 5.1. *A q-Yamabe soliton on an (ISM) M of an indefinite Kenmotsu manifold $\bar{M}$ is expanding or shrinking or steady accordingly as $qr > 0$ or $qr < 0$ or $qr = 0$*

If $q=0$ then $\lambda = p[(n-1)\varepsilon + \xi]$. Thus we have

Corollary 5.2. *A p-Ricci soliton on an (ISM) M of an indefinite Kenmotsu manifold $\bar{M}$ is expanding or shrinking or steady accordingly as*

$$p[(n-1)\varepsilon+\xi] > 0 \quad \textit{or} \quad p[(n-1)\varepsilon+\xi] < 0 \quad \textit{or} \quad p[(n-1)\varepsilon+\xi] = 0.$$

6. (RYS) on Submanifolds of Indefinite Kenmotsu Manifolds with respect to $(QSMC)$

Let us assume that (g,ξ,λ,p,q) is a (RYS) on a submanifold M of an indefinite Kenmotsu manifold $\bar{M}$ with respect to $(QSMC)$, where $\bar{\nabla}$ is the induced connection on M from the connection $\bar{\bar{\nabla}}$. Also let $\bar{h}$ be the second fundamental form of $\bar{M}$ with respect to induced connection $\bar{\nabla}$. Then we can consider the equations (4.1), (4.2), (4.3).

In case M is an (ISM) $\bar{M}$, then we have the equation (4.4) which implies that M accord $(QSMC)$. From $\bar{\nabla}_{U_1}\xi = \varepsilon[U_1 - \eta(U_1)\xi] + \varphi U_1$ and hence

$$(\bar{\mathfrak{L}}_\xi g)(U_1,U_2) = 2\varepsilon[g(U_1,U_2) - \eta(U_1)\eta(U_2)]. \tag{6.1}$$

Using (6.1) in (4.1), we get: $\bar{S}(U_1,U_2) = (\frac{qr-2\lambda-2\varepsilon}{2p})g(U_1,U_2) + (\frac{\varepsilon}{p})\eta(U_1)\eta(U_2)$.

Hence, it follows that

Theorem 6.1. *Let (g, ξ, λ, p, q) be a (RYS) on an (ISM) M of an indefinite Kenmotsu manifold $\bar{M}$ with respect to $(QSMC)$ $\bar{\tilde{\nabla}}$. Then M is η-Einstein in respect of induced Riemannian connection.*

Again, if M is an anti-(ISM) of $\bar{M}$ in respect of $(QSMC)$ then from (4.4) we obtain

$$\bar{\nabla}_{U_1}\xi = \varepsilon[U_1 - \eta(U_1)\xi],$$

$$(\bar{\mathfrak{L}}_\xi g)(U_1, U_2) = 2\varepsilon[g(U_1, U_2) - \eta(U_1)\eta(U_2)]. \tag{6.2}$$

Hence, using (6.2) in (4.1), we have: $\bar{S}(U_1, U_2) = (\frac{qr-2\lambda-2\varepsilon}{2p})g(U_1, U_2) + (\frac{\varepsilon}{p})\eta(U_1)\eta(U_2)$.

Therefore, we state that

Theorem 6.2. *Let (g, ξ, λ, p, q) be a (RYS) on an anti-(ISM) M of an indefinite Kenmotsu manifold $\bar{M}$ with respect to $(QSMC)$ $\bar{\tilde{\nabla}}$. Then M is η-Einstein with respect to induced Riemannian connection.*

7. (RYS) ON SUBMANIFOLDS OF INDEFINITE TRANS-SASAKIAN MANIFOLDS WITH RESPECT TO RIEMANNIAN CONNECTION

Let us consider that (g, ξ, λ, p, q) is a (RYS) on a submanifold M of an indefinite trans-Sasakian manifold $\bar{M}$. Then we get

$$(\mathfrak{L}_\xi g)(U_1, U_2) + 2pS(U_1, U_2) + (2\lambda - qr)g(U_1, U_2) = 0. \tag{7.1}$$

From (2.2) and (2.4) we have

$$-\varepsilon\alpha\varphi U_1 + \beta[U_1 - \eta(U_1)\xi] = \tilde{\nabla}_{U_1}\xi = \nabla_{U_1}\xi + h(U_1, \xi). \tag{7.2}$$

In the event that M is invariant in $\bar{M}$, at that point $\varphi U_1 \in TM$ consequently likening normal and tangential components of (7.2) we acquire

$$\nabla_{U_1}\xi = -\varepsilon\alpha\varphi U_1 + \delta\beta\varphi^2 U_1, h(U_1, \xi) = 0, \tag{7.3}$$

using (7.3) we have,

$$(\mathfrak{L}_\xi g)(U_1, U_2) = 2\beta\delta[g(U_1, U_2) + \varepsilon\eta(U_1)\eta(U_2)].$$

Using this to (7.1) we get

$$S(U_1, U_2) = (\frac{qr - 2\lambda - 2\beta\delta}{2p})g(U_1, U_2) + (\frac{-\beta}{p})\eta(U_1)\eta(U_2). \tag{7.4}$$

It implicit that M is η-Einstein. As well as from (2.5) along with (7.3) it acquires $\eta(U_1)L = 0$. That is, $L = 0$, $\eta(U_1) \neq 0$. Consequently, M is minimal in $\bar{M}$. Therefore, it follows that

Theorem 7.1. *If (g, ξ, λ, p, q) is (RYS) on an (ISM) M of an indefinite trans-Sasakian manifold $\bar{M}$. Then M is η-Einstein and also M is minimal in $\bar{M}$.*

Again, if M is anti invariant in $\bar{M}$, then for any $X \in TM$, $\varphi X \in T^{\perp}M$ and hence from (7.2), we have: $\nabla_{U_1}\xi = \varepsilon[\beta U_1 - \eta(U_1)\xi], h(U_1, \xi) = -\varepsilon\alpha\varphi U_1$.

Hence, $(\mathfrak{L}_\xi g)(U_1, U_2) = 2\beta\delta[g(U_1, U_2) + \varepsilon\eta(U_1)\eta(U_2)]$, so we obtain $S(U_1, U_2) = (\frac{qr-2\lambda-2\beta\delta}{2p})g(U_1, U_2) + (\frac{-\beta}{p})\eta(U_1)\eta(U_2)$.

Hence, it can be expressed

Theorem 7.2. *If (g, ξ, λ, p, q) is (RYS) on an anti-(ISM) M of an indefinite trans-Sasakian manifold $\bar{M}$. Then M is η-Einstein.*

Also from (7.3), we obtain

$$S(\xi, \xi) = (n-1)\varepsilon\alpha^2 - \beta^2\delta - 2n\xi\beta. \tag{7.5}$$

Comparing (7.4) and (7.5), we get $\lambda = \frac{qr-2\beta(\delta-1)-2p[(n-1)\varepsilon\alpha^2-\beta^2\delta-2n\xi\beta]}{2}$.

Theorem 7.3. *A (RYS) on an (ISM) or anti-(ISM) M of an indefinite trans-Sasakian manifold $\bar{M}$ is expanding or shrinking or steady accordingly as:*

$$\frac{qr - 2\beta(\delta-1) - 2p[(n-1)\varepsilon\alpha^2 - \beta^2\delta - 2n\xi\beta]}{2} > 0,$$

or

$$\frac{qr - 2\beta(\delta-1) - 2p[(n-1)\varepsilon\alpha^2 - \beta^2\delta - 2n\xi\beta]}{2} < 0,$$

or

$$qr - 2\beta(\delta-1) - 2p[(n-1)\varepsilon\alpha^2 - \beta^2\delta - 2n\xi\beta] = 0.$$

If $q = 0$ then $\lambda = \frac{-2\beta(\delta-1)-2p[(n-1)\varepsilon\alpha^2-\beta^2\delta-2n\xi\beta]}{2}$. Thus we have

Corollary 7.1. *A p- Ricci soliton on an (ISM) or anti-(ISM) M of an indefinite trans-Sasakian manifold $\bar{M}$ is expanding or shrinking or steady accordingly as:*

$$\frac{-2p[(n-1)\varepsilon\alpha^2 - \beta^2\delta - 2n\xi\beta]}{2} > 0,$$

or

$$\frac{-2\beta(\delta-1) - 2p[(n-1)\varepsilon\alpha^2 - \beta^2\delta - 2n\xi\beta]}{2} < 0,$$

or

$$-2\beta(\delta-1)-2p[(n-1)\varepsilon\alpha^2-\beta^2\delta-2n\xi\beta]=0.$$

If $p=0$ then $\lambda=qr-2\beta(\delta-1)$. Thus, we have:

Corollary 7.2. *A q-Yamabe soliton on an (ISM) or anti-(ISM) M of an indefinite trans-Sasakian manifold $\bar{M}$ is expanding or shrinking or steady accordingly as*

$$qr>2\beta(\delta-1) \quad \text{or} \quad qr<2\beta(\delta-1) \quad \text{or} \quad qr=2\beta(\delta-1).$$

8. (RYS) ON SUBMANIFOLDS OF INDEFINITE TRANS-SASAKIAN MANIFOLDS WITH RESPECT TO $(QSMC)$

Let us consider that (g,ξ,λ,p,q) is a (RYS) on a submanifold M of an indefinite trans-Sasakian manifold $\bar{M}$ with respect of $(QSMC)$, where $\bar{\nabla}$ is the induced connection on M from the connection $\bar{\bar{\nabla}}$. Further, let $\bar{h}$ be the second fundamental form of $\bar{M}$ with respect to induced connection $\bar{\nabla}$. Then, we can consider the equations (4.1), (4.2), (4.3).

If M is an (ISM) $\bar{M}$ then we have the equation (4.4) which implies that M concur $(QSMC)$. As well as from $\bar{\nabla}_{U_1}\xi=(-\varepsilon\alpha+1)\varphi U_1+\beta\delta[U_1-\eta(U_1)\xi]$, and hence

$$(\bar{\mathfrak{L}}_\xi g)(U_1,U_2)=2\beta\delta[g(U_1,U_2)-\varepsilon\eta(U_1)\eta(U_2)]. \tag{8.1}$$

In view of (4.1) and (8.1), we get:

$$\bar{S}(U_1,U_2)=(\frac{qr-2\lambda-2\beta\delta}{2p})g(U_1,U_2)+(\frac{-\beta}{p})\eta(U_1)\eta(U_2).$$

Hence, it can be declared as:

Theorem 8.1. *Let (g,ξ,λ,p,q) be a (RYS) on an (ISM) M of an indefinite trans-Sasakian manifold $\bar{M}$ with respect to $(QSMC)$ $\bar{\bar{\nabla}}$. Then M is η-Einstein with respect to induced Riemannian connection.*

Again, If M is an anti-(ISM) of $\bar{M}$ with respect to $(QSMC)$ then from (4.4) we obtain: $\bar{\nabla}_{U_1}\xi=\varepsilon[\beta U_1-\eta(U_1)\xi]$, it implies that

$$(\bar{\mathfrak{L}}_\xi g)(U_1,U_2)=2\beta\delta[g(U_1,U_2)-\varepsilon\eta(U_1)\eta(U_2)].$$

Hence from (4.1), we have: $\bar{S}(U_1,U_2)=(\frac{qr-2\lambda-2\beta\delta}{2p})g(U_1,U_2)+(\frac{-\beta}{p})\eta(U_1)\eta(U_2)$. We now state that:

Theorem 8.2. *Let (g, ξ, λ, p, q) be (RYS) on an anti-(ISM) M of an indefinite trans-Sasakian manifold $\bar{M}$ with respect to $(QSMC)$ $\bar{\tilde{\nabla}}$. Then M is η-Einstein with respect to induced Riemannian connection.*

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