DEGREE EXPONENT SUBTRACTION ENERGY

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ABSTRACT. The ordinary energy of a graph $G$ is defined as the sum of the absolute eigenvalues of its adjacency matrix. In this paper we introduced a degree exponent subtraction matrix and we investigate its bounds for spectral radius of spectrum, partial sum of absolute eigenvalues and energy.

1. INTRODUCTION

The energy of a graph $G$ is closely related with the total $\pi$-electron energy of molecules [2, 4]. This has motivated researchers to introduce different matrices associated with the graph and study their energies such as, Laplacian energy [1, 5], distance energy [7], degree sum energy [11], degree subtraction energy [10], degree exponent energy [9] etc. In this paper, we introduce degree exponent subtraction matrix and obtain its bounds for spectral radius, partial sum of absolute eigenvalues and energy.

Let $G$ be a simple, finite, undirected, nontrivial graph of order $n$ and size $m$. Let $V(G) = \{v_1, v_2, \ldots, v_j, \ldots, v_n\}$ be a vertex set. Let $d_j = \text{deg}_G(v_j)$ be the degree of a vertex $v_j$ in $G$. Let eigenvalues of adjacency matrix [6] be $\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots, \lambda_n$. Then the energy $\epsilon(G)$ of $G$ is defined as $\epsilon(G) = \sum_{j=1}^{n} |\lambda_j|$ [3]. The adjacency eigenvalues of a complete graph $K_n$ are $n - 1$
and $-1$ $(n-1)$ times. The adjacency eigenvalues of a complete bipartite graph $K_{p,q}$ are $\sqrt{pq}$, $0$ $(p+q-2)$ times and $-\sqrt{pq}$.

The degree exponent subtraction matrix (DES) of a graph $G$ is $n \times n$ matrix, defined as $DES(G) = [des_{jk}]$ where

$$des_{jk} = \begin{cases} d^d_j - d^d_k & j \neq k \\ 0 & \text{otherwise} \end{cases}.$$ 

Characteristic polynomial of $DES(G)$ is defined as

$$\phi(G, y) = \text{det} \ (yI_n - DES(G)),$$

where $I_n$ is unit matrix of order $n$. The roots of $\phi(G : y) = 0$ are called DES-eigenvalues which are labeled as $y_1, y_2, \ldots, y_j, \ldots, y_n$. The energy of degree exponent subtraction matrix of $G$ is defined as

$$DESE(G) = \sum_{j=1}^{n} |y_j|.$$ 

Example 1.

Graph and its DES matrix

Characteristic polynomial of above matrix is

$$\phi(G, y) = y^5 + 306y^3 + 392y$$

$$\text{spec}(DES(G)) = \begin{pmatrix} 0 & 17.4560 & -17.4560i & 1.1342i & -1.1342i \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

where $i = \sqrt{-1},$

$$DESE(G) \approx 37.1804.$$
2. Preliminaries

**Theorem 2.1.** Cauchy-Schwarz inequality [12] states that if \((a_1, a_2, \ldots, a_n)\) and 
\((b_1, b_2, \ldots, b_n)\) are \(n\) real vector, then
\[
\left( \sum_{j=1}^{n} a_j b_j \right)^2 \leq \left( \sum_{j=1}^{n} a_j^2 \right) \left( \sum_{j=1}^{n} b_j^2 \right).
\]

**Theorem 2.2.** Ozeki’s inequality [8], if \(a_j\) and \(b_j\) are nonnegative real numbers, then
\[
\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 - \left( \sum_{j=1}^{n} a_j b_j \right)^2 \leq \frac{n^2}{4} \left( M_1 M_2 - m_1 m_2 \right)^2
\]
where \(M_1 = \max_{1 \leq j \leq n}(a_j)\), \(M_2 = \max_{1 \leq j \leq n}(b_j)\), \(m_1 = \min_{1 \leq j \leq n}(a_j)\), \(m_2 = \min_{1 \leq j \leq n}(b_j)\).

**Theorem 2.3.** Polya-Szego inequality [8], if \(a_j\) and \(b_j\) are non-negative real numbers, then
\[
\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 \leq \frac{1}{4} \left( \sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2 \left( \sum_{j=1}^{n} a_j b_j \right)^2,
\]
where \(M_1, M_2\) and \(m_1, m_2\) are defined similarly to theorem (2.2).

**Theorem 2.4.** [8] If \(a_j\) and \(b_j\) are non-negative real numbers, then
\[
\left| \sum_{j=1}^{n} a_j b_j - \sum_{j=1}^{n} a_j \sum_{j=1}^{n} b_j \right| \leq \alpha(n) (S - s)(T - t),
\]
where \(s, t, S\) and \(T\) are real constants such that \(s \leq a_j \leq S\) and \(t \leq b_j \leq T\) for each \(j, 1 \leq j \leq n\). \(\alpha(n) = n \left| \frac{n}{2} \right| \left( 1 - \frac{1}{n} \right) \left( \frac{n}{2} \right) \).

**Theorem 2.5.** [8] If \(a_j\) and \(b_j\) are non-negative real numbers, then
\[
\sum_{j=1}^{n} b_j^2 + C D \sum_{j=1}^{n} a_j^2 \leq (C + D) \sum_{j=1}^{n} a_i b_j,
\]
where \(C\) and \(D\) are real constants such that \(Ca_j \leq b_j \leq Da_j\) for each \(j = 1, 2, \ldots, n\).
3. BOUNDS FOR SPECTRAL RADIUS OF DES(G)

Lemma 3.1. If $G$ is a graph with $n$ vertices and $m$ edges then the eigenvalues $y_j, 1 \leq j \leq n$, of the $DES(G)$ satisfy the following relations

(i) $\sum_{j=1}^{n} y_j = 0$;  
(ii) $\sum_{j=1}^{n} y_j^2 = -2P$;  
(iii) $\sum_{j=1}^{n} |y_j|^2 = 2P$;

where $P = \sum_{1 \leq j < k \leq n} (d_j d_k - d_k d_j)^2$.

Proof. Since $\sum_{j=1}^{n} y_j = \text{trace}(DES(G)) = 0$,

$\sum_{j=1}^{n} y_j^2 = \text{trace}(DES(G)^2) = -2 \sum_{1 \leq j < k \leq n} (d_j d_k - d_k d_j)^2 = -2P$.

Next, having in mind that the eigenvalues $y_j$ are purely imaginary or zeros, it follows that

$\sum_{j=1}^{n} |y_j|^2 = 2 \sum_{1 \leq j < k \leq n} (d_j d_k - d_k d_j)^2 = 2P$.

□

Lemma 3.2. Let $G$ be a graph with $n$ vertices, $m_1$ edges. Let $\lambda_j$ be adjacency eigenvalues of $G$ such as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $H$ be another graph with $n$ vertices, $m_2$ edges with vertex degree $d_j, j = 1, 2, \ldots, n$. Let $y_j$ be $DES(H)$-eigenvalues of $H$ which are given as $|y_1| \geq |y_2| \geq \cdots \geq |y_n|$. Then

$\sum_{j=1}^{n} (\lambda_j |y_j|) \leq \sqrt{4m_1P}$.

Proof. Substitute $a_j = \lambda_j$ and $b_j = |y_j|$ in Theorem 2.1, we get

$$\left( \sum_{j=1}^{n} (\lambda_j |y_j|) \right)^2 \leq \left( \sum_{j=1}^{n} \lambda_j^2 \right) \left( \sum_{j=1}^{n} |y_j|^2 \right).$$

Since $\sum_{j=1}^{n} |y_j|^2 = 2P$ and $\sum_{j=1}^{n} \lambda_j^2 = 2m_1$. On substituting and simplifying equation (3.1) we get required result. □

Theorem 3.1. Let $G$ be a graph with $n$ vertices and $m$ edges with vertex degree $d_j, j = 1, 2, \ldots, n$. Let absolute $DES$-eigenvalues be $|y_1| \geq |y_2| \geq \cdots \geq |y_n|$. Then

$$|y_1| \leq \sqrt{\frac{2a}{a-1}P} + \frac{1}{a-1} \sum_{j=2}^{a} |y_{n-a+j}| \quad (2 \leq a \leq n).$$
Proof. Let \(|y_1|, |y_2|, \cdots, |y_{n-a+1}|, |y_{n-a+2}|, \cdots, |y_n|\) be the absolute DES-eigenvalues of \(G\). Let \(H = K_a \cup K_{n-a}\). Then adjacency eigenvalues of \(H\) are \(a - 1, 0\) \((n - a\ times)\) and \(-1\) \((a - 1\ times)\). The number of edges of \(H\), \(m_1 = \frac{a(a-1)}{2}\). By using Lemma 3.2, we get

\[
\sum_{j=1}^{n} (\lambda_j |y_j|) \leq \sqrt{4m_1P}
\]

\[
(a - 1)|y_1| - \sum_{j=n-a+2}^{n} (|y_j|) \leq \sqrt{2Pa(a - 1)}
\]

(3.3) \(|y_1| \leq \sqrt{\frac{2Pa}{a - 1}} + \frac{1}{a - 1} \sum_{j=n-a+2}^{n} (|y_j|)
\]

from equation (3.3) we get our required result (3.2).  

Remark 3.1. Since DES\((G)\) is skew symmetric matrix therefore, \(|y_1| = |y_2|\). Hence \(|y_1|, |y_2|\) have the same upper bound.

Corollary 3.1. If \(G\) be a graph with \(n\) vertices, \(m\) edges having vertex degrees \(d_1, d_2, \cdots, d_n\), then

\[
|y_1| \leq \sqrt{\frac{2P(n-1)}{n}} + \frac{DESE(G)}{n}.
\]

Proof. Putting \(a = n\) in above equation (3.2), we get

\[
|y_1| \leq \sqrt{\frac{2nP}{n - 1}} + \frac{1}{n - 1} \sum_{j=2}^{n} |y_j|
\]

\[
\leq \sqrt{\frac{2nP}{n - 1}} + \frac{1}{n - 1} (DESE(G) - |y_1|).
\]

On simplifying we get result (3.4).  

Remark 3.2. The equality of (3.4) holds for regular graphs. As \(P = 0\) So \(|y_1| = 0\).
4. Bounds for partial sum of absolute eigenvalue of \( DES(G) \)

**Theorem 4.1.** If \( G \) is a graph with \( n \) vertices and \( m \) edges, with vertex degrees \( d_1, d_2, \ldots, d_n \) and \( DES \)-eigenvalues \( |y_1| \geq |y_2| \geq \cdots \geq |y_n| \), then

\[
\sum_{j=1}^{k} |y_j| \leq \sqrt{\frac{2k(a-1)P}{a}} + \frac{DESE(G)}{a} \quad 1 \leq k \leq n.
\]

**Proof.** Let \( |y_1|, |y_2|, \ldots, |y_k|, |y_{k+1}|, \ldots, |y_n| \) be the absolute \( DES \)-eigenvalues of \( G \). Let \( H \) be the union of \( k \) copies of complete graph \( K_a \), that is \( H = \bigcup_k K_a \) where \( ka = n \). The adjacency eigenvalues of \( H \) are \( a-1 \) (\( k \) times), \(-1\) (\( n-k \) times). The number of vertices and edges of \( H \) are \( n = ak \) and \( \frac{ka(a-1)}{2} \) respectively. Using Lemma 3.2, we get

\[
(a - 1) \sum_{j=1}^{k} |y_j| - \sum_{j=k+1}^{n} |y_j| \leq \sqrt{\frac{4ka(a-1)P}{2}}
\]

\[
a \sum_{j=1}^{k} |y_j| - \sum_{j=1}^{n} |y_j| \leq \sqrt{2ka(a-1)P}
\]

\[
a \sum_{j=1}^{k} |y_j| \leq \sqrt{2ka(a-1)P} + DESE(G)
\]

\[
\sum_{j=1}^{k} |y_j| \leq \sqrt{\frac{2k(a-1)P}{a}} + \frac{DESE(G)}{a}.
\]

Thus, we obtain the bound for the sum of \( k \) absolute \( DES \)-eigenvalues of a \( G \). If \( k = 1 \) we observe that the equation (4.1) gets reduced to equation (3.4). \( \square \)

**Theorem 4.2.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges with vertex degree \( d_j, j = 1, 2, \ldots, n \) and absolute \( DES \)-eigenvalues be \( |y_1| \geq |y_2| \geq \cdots \geq |y_n| \). Then

\[
\sum_{j=1}^{k} (|y_j| - |y_{n-k+j}|) \leq \sqrt{4kP}.
\]

**Proof.** Let \( |y_1|, |y_2|, \ldots, |y_k|, |y_{k+1}|, \ldots, |y_n| \) be the absolute \( DES \)-eigenvalues of \( G \). Let \( H \) be the union of \( k \) copies of complete bipartite graph \( K_{a,b} \), \( H = \bigcup_k K_{a,b} \) where \( n = ka \). Then adjacency eigenvalues of \( H \) are \( \sqrt{ab} \) of multiplicity \( k \), zero of multiplicity \( n-2k \) and \( -\sqrt{ab} \) of multiplicity \( k \). The number of edges of \( H \) is \( kab \). By using Lemma 3.2 we get
5. Bounds for Degree exponent subtraction energy

**Theorem 5.1.** If $G$ is a graph with $n$ vertices, then

$$\sqrt{2P} \leq DESE(G) \leq \sqrt{2nP}.$$  

**Proof.** Putting $a_j = 1$ and $b_j = |y_j|$ in Theorem 2.1 we get

$$\left( \sum_{j=1}^{n} |y_j| \right)^2 \leq n \sum_{j=1}^{n} |y_j|^2 \leq 2nP \leq \sqrt{2nP},$$

which is the upper bound for $DES(G)$ and we know that

$$\left( \sum_{j=1}^{n} |y_j| \right)^2 \geq \sum_{j=1}^{n} |y_j|^2 \geq 2P \geq \sqrt{2P}.$$

Therefore, we get

$$\sqrt{2nP} \leq DESE(G) \geq \sqrt{2P}.$$  

□
Theorem 5.2. Let $G$ be a simple connected graph and let $\det(DES(G))$ be the absolute value of the determinant of degree exponent subtraction matrix. Then

$$\sqrt{2P + n(n - 1)(\det(DES(G)))^{\frac{2}{n}}} \leq DESE(G) \leq \sqrt{2nP}.$$ 

Proof. We know that

$$\left(\sum_{j=1}^{n} |y_j|\right)^2 = \sum_{j=1}^{n} |y_j|^2 + 2 \sum_{j<k} |y_j||y_k|$$

$$DESE(G)^2 = 2P + 2 \sum_{j<k} |y_j||y_k|$$

(5.1)

$$= 2P + \sum_{j \neq k} |y_j||y_k|.$$ 

We know that the arithmetic mean is always greater than or equal to the geometric mean.

$$\frac{1}{n(n-1)} \sum_{j \neq k} |y_j||y_k| \geq \left(\prod_{j \neq k} |y_j||y_k|\right)^{\frac{1}{n(n-1)}}$$

$$= \left(\prod_{j \neq k} |y_j|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}$$

$$= \left(\prod_{j \neq k} |y_j|^{\frac{2}{n}}\right)^{\frac{n}{n(n-1)}}$$

$$= (DESE(G))^{\frac{2}{n}}$$

(5.2)

$$\sum_{j \neq k} |y_j||y_k| \leq n(n - 1)(DESE(G))^{\frac{2}{n}}$$

from equation (5.1) and equation (5.2), we get

$$DESE(G) \geq \sqrt{2P + n(n - 1)(DESE(G))^{\frac{2}{n}}}.$$
Let a non-negative quantity $Z$. Which is given as
\[ Z = \sum_{j=1}^{n} \sum_{k=1}^{n} (|y_j| - |y_k|)^2 \]
\[ Z = n \sum_{j=1}^{n} |y_j|^2 + n \sum_{k=1}^{n} |y_k|^2 - 2 \left( \sum_{j=1}^{n} |y_j| \right) \left( \sum_{k=1}^{n} |y_k| \right) \]
\[ Z = 4nP - 2(\text{DESE}(G))^2. \]
Since $Z \geq 0$,
\[ (5.4) \quad \text{DESE}(G) \leq \sqrt{2nP}, \]
from equation (5.3) and equation (5.4) we get required result. \[ \blacksquare \]

**Theorem 5.3.** If $G$ is a graph of order $n$, Let $|y_1| \geq |y_2| \geq \cdots \geq |y_n|$ are the absolute eigenvalues of $\text{DES}(G)$. Then,
\[ \text{DESE}(G) \geq \sqrt{2nP - \frac{n^2}{4} (|y_1| - |y_n|)^2}, \]
where $|y_1|$ and $|y_n|$ are maximum and minimum of the absolute value of $y_j$'s.

**Proof.** In inequality (2.2) let us take $a_j = 1$ and $b_j = |y_j|$, then we get
\[ n \sum_{j=1}^{n} |y_j|^2 - \left( \sum_{j=1}^{n} |y_j| \right)^2 \leq \frac{n^2}{4} (|y_1| - |y_n|)^2 \]
\[ 2nP - (\text{DESE}(G))^2 \leq \frac{n^2}{4} (|y_1| - |y_n|)^2 \]
\[ \text{DESE}(G) \geq \sqrt{2nP - \frac{n^2}{4} (|y_1| - |y_n|)^2}. \]
\[ \blacksquare \]

**Corollary 5.1.** If $G$ is graph with odd order, then
\[ \text{DESE}(G) \geq \sqrt{2nP - \frac{n^2}{4} (|y_1|)^2}. \]

**Theorem 5.4.** If $G$ be a graph of order $n$ and $|y_1| \geq |y_2| \geq \cdots \geq |y_n| > 0$ are the absolute eigenvalues of $\text{DES}(G)$, then
\[ \text{DESE}(G) \geq \frac{2\sqrt{2nP|y_1||y_n|}}{|y_1| + |y_n|}. \]
Proof. Suppose $a_j = 1$ and $b_j = |y_j|$ then from inequality Theorem 2.3, we get
\begin{equation}
\sum_{j=1}^{n} 1^2 \sum_{j=1}^{n} |y_j|^2 \leq \frac{1}{4} \left( \sqrt{\frac{|y_1|}{|y_n|}} + \sqrt{\frac{|y_n|}{|y_1|}} \right)^2 \left( \sum_{j=1}^{n} |y_j| \right)^2.
\end{equation}
Since $\sum_{j=1}^{n} 1^2 = n$, $\sum_{j=1}^{n} |y_j|^2 = 2P$ and $\sum_{j=1}^{n} |y_j| = DESE(G)$. On simplifying equation (5.5) we get required result.

Theorem 5.5. Let $G$ be a graph of order $n$. Let $|y_1| \geq |y_2| \geq \cdots \geq |y_n|$ are the absolute eigenvalues of $DES(G)$. Then
\begin{equation}
DESE(G) \geq \sqrt{2nP - \alpha(n) (|y_1| - |y_n|)^2},
\end{equation}
where $\alpha(n) = n \left[ \frac{n}{2} \right] (1 - \frac{1}{n} \left[ \frac{n}{2} \right])$.

Proof. Suppose $a_j = b_j = |y_j|, s = t = |y_n|, S = T = |y_1|$ then from inequality Theorem 2.4, we get
\begin{equation}
\left| \sum_{j=1}^{n} |y_j|^2 - \left( \sum_{j=1}^{n} |y_j| \right) \left( \sum_{j=1}^{n} |y_j| \right) \right| \leq \alpha(n) (|y_1| - |y_n|)^2.
\end{equation}
Since $\sum_{j=1}^{n} |y_j|^2 = 2P$ and $\sum_{j=1}^{n} |y_j| = DESE(G)$. On simplifying equation (5.6) we get required result.

Corollary 5.2. If $G$ is graph with odd order, then
\begin{equation}
DESE(G) \geq \sqrt{2nP - \alpha(n) (|y_1|)^2}.
\end{equation}

Remark 5.1. Since $\alpha(n) \leq \frac{n^2}{4}$, the lower bound Theorem 2.2 is more sharper than the lower bound Theorem 2.4.

Theorem 5.6. Let $G$ be a graph of order $n$. Let $|y_1| \geq |y_2| \geq \cdots \geq |y_n|$ are the absolute eigenvalues of $DES(G)$. Then
\begin{equation}
DESE(G) \geq \frac{|y_1| |y_n| n + 2P}{|y_1| + |y_n|}.
\end{equation}

Proof. Substituting $a_j = 1, b_j = |y_j|, C = |y_n|, D = |y_1|$ then from inequality (2.5), we get
\begin{equation}
\sum_{j=1}^{n} |y_j|^2 + |y_n| |y_1| \sum_{j=1}^{n} 1^2 \leq (|y_n| + |y_1|) \left( \sum_{j=1}^{n} |y_j| \right).
\end{equation}
Since $\sum_{j=1}^{n} |y_j|^2 = 2P$ and $\sum_{j=1}^{n} |y_j| = DESE(G)$. On simplifying equation (5.7) we get required result.
In this paper we have introduced a new matrix called degree exponent subtraction in which we found bounds for spectral radius of spectrum and partial sum of absolute eigenvalues of $DES(G)$, Here we got sharper lower bound of $DESE(G)$ even though we have skew-symmetric matrix.

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