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ASYMPTOTIC PROPERTIES OF THIRD-ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH VARIABLE DELAY ARGUMENTS

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ABSTRACT. The present paper focuses on the oscillation and asymptotic properties of the third-order nonlinear neutral differential equations with variable delay arguments. By applying the Riccati transformation and the integral averaging technique, we give an analytical method for the estimation of Riccati differential inequality to establish several oscillation criteria for the discussed equation, which show that any solution either oscillates or converges to zero. We give several theorems and related examples to prove the significance of new theorems.

1. INTRODUCTION

Consider third-order nonlinear neutral differential equations with variable delayed arguments

$$\left(a(t)\Big(\Big[y(t)+A(t)y(t-\eta(t))\Big]''\Big)^{\lambda}\right)' + \sum_{j=1}^{m} B_j(t) f_j(y(t-\sigma_j(t))) = 0, \tag{1.1}$$

$\lambda \geq 1$, where $m \geq 1$ is an integer. Further, assume that

(H_1) $a(t)$, $A(t) \in C\big([t_0,+\infty),\mathbb{R}^+\big)$ and $B_j(t)$, $\sigma_j(t) \in C\big([t_0,+\infty),\mathbb{R}^+\big)$ for $j = 1,2,...,m$.

(H_2) $a'(t) \geq 0$, $0 \leq A(t) \leq A_0 < 1$, $\lim_{t\to+\infty}(t-\eta(t)) = \infty$ and $\lim_{t\to+\infty}(t-\sigma_j(t)) = \infty$ for $j = 1,2,...,m$.

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(H_3) $f_j(v) \in C(\mathbb{R},\mathbb{R})$, there exists a constants $\xi_j > 0$ such that $f_j(v)/v^\lambda \geq \xi_j$ for $v \neq 0$ and $j = 1, 2, ..., m$.

We shall considering the case,

$$R[t_0, t] = \int_{t_0}^{t} \frac{ds}{a^{1/\lambda}(s)}, \quad R[t_0, t] = \infty \text{ as } t \to \infty, \tag{1.2}$$

and we define $Z(t) = y(t) + A(t)y(t - \eta(t))$, $Z^{[1]} = a(Z'')^\lambda$, $Z^{[2]} = (Z^{[1]})'$. If $y(t)$, $Z(t)$, $Z'(t)$, $Z^{[1]}(t) \in C^1\big([t_*, +\infty)\big)$ for all $t \in [t_*, +\infty)$ and $y(t)$ satisfies equation (1.1) for some $t_* \geq t_0$, then the function y is called a solution of equation (1.1). Such a solution (which is non-trivial) of equation(1.1) is called oscillatory if it has a sequence of large zeros lending to ∞; it is non-oscillatory otherwise.

After more than a quarter of a century, there has been an increasing interest in studying the oscillation and asymptotic behavior of differential equations and their applications. We refer the monographs Agarwal et al. [11], Erbe et al. [7], Gyori and Ladas [5]. In particular, a wide attention was made over last few years on the oscillation and non-oscillation of first/second order differential equations. Comparatively, there are not many related results for third-order which received less attention in the literature. Many authors have worked on various aspects of oscillation and asymptotic behavior of third order see in [1, 2, 4, 6, 8–10, 12–16] and the references cited therein.

B. Baculíková et al. [1], J. Džurina et al. [6], E.Thandapani and T. Li [4] studied a oscillatory results of third-order neutral differential equations

$$\Big[a(t)[x(t) + p(t)x(\delta(t))'']^\gamma\Big]' + q(t)x^\gamma(\tau(t)) = 0, \quad t \geq t_0.$$

B. Baculíková [2], Y. Jiang et al. [15] studied several oscillation results for third-order neutral differential equation using the Riccati / comparison method

$$\Big[a(t)[x(t) + p(t)x(\delta(t))'']^\alpha\Big]' + q(t)f(x(\tau(t))) = 0, \quad t \geq t_0.$$

If $\lambda = 1$, Jiang and Li [16] established asymptotic behavior of equation (1.1) using generalized Riccati and integral averaging technique.

Till necessarily, there is no paper provide oscillation criteria of equation (1.1) under the condition of $\lambda \geq 1$ and canonical case of $a(t)$. Therefore, we present some new results for all solutions of equation (1.1) to be either oscillates or tends to 0 asymptotically by employing generalized Riccati substitutions and integral averaging technique under (1.2) and also present related examples.

2. MAIN RESULTS

In this section, we prove our main results followed by lemmas.

Lemma 2.1 ([16], Lemma 3). *Assume that* $u(t) > 0$, $u'(t) > 0$ *and* $u''(t) \leq 0$ *for* $t \geq t_0$. *If* $\sigma \in C([t_0,\infty),[0,\infty))$, $\sigma(t) \leq t$ *and* $\lim_{t\to\infty} \sigma(t) = \infty$, *then for every* $\alpha \in (0,1)$, *there exists a* $T_\alpha \geq t_0$ *such that* $\frac{u(\sigma(t))}{\sigma(t)} \geq \frac{\alpha u(t)}{t}$ *for* $t \geq T_\alpha$.

Lemma 2.2 ([16], Lemma 4). *Assume that* $u(t) > 0$, $u'(t) > 0$ *and* $u''(t) \leq 0$ *for* $t \geq t_0$. *Then for each* $\beta \in (0,1)$, *there exists a* $T \geq t_0$ *such that* $u(t) \geq \beta \frac{tu'(t)}{2}$ *for* $t \geq T_\beta$.

Theorem 2.1. *Let* $(H_1) - (H_3)$ *and* (1.2) *holds. If there exists* $\zeta \in C^1([t_0,\infty),\mathbb{R})$, *such that for all sufficiently large* $t_k > t_1 \geq t_0$ $(k = 2,3)$ *and for some* $b, c \in (0,1)$, *we have*

$$\int_{t_2}^{\infty} \left[\Phi(s) - \frac{\zeta(s)a(s)}{(\lambda+1)^{\lambda+1}} \left(\frac{\zeta'(s)}{\zeta(s)} + (\lambda+1)\omega^{1/\lambda}(s) \right) \right] ds = \infty \tag{2.1}$$

and

$$\int_{t_3}^{\infty} \int_{v}^{\infty} \frac{1}{a^{1/\lambda}(u)} \left(\sum_{j=1}^{m} \xi_j \int_{u}^{\infty} B_j(s)\,ds \right)^{1/\lambda} du\,dv = \infty, \tag{2.2}$$

where

$$\Phi(t) = \zeta(t)\Big\{ (a(t)\omega(t))' + a(t)\omega^{\frac{\lambda+1}{\lambda}}(t) + \frac{(1-A_0)^\lambda b^\lambda c^\lambda}{2^\lambda t^\lambda} \sum_{j=1}^{m} \xi_j B_j(t)(t-\sigma_j(t))^{2\lambda} \Big\}. \tag{2.3}$$

Then every solution $y(t)$ *of* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} y(t) = 0$.

Proof. Conversely, assume $y(t)$ be a non-oscillatory solution of equation (1.1). Without loss of generality, we may suppose that there exists a $t_1 \geq t_0$ such that $y(t) > 0, y(t-\eta(t)) > 0$ and $y(t+\sigma_j(t)) > 0$ for all $t \geq t_1$ and $j = 1,2,...,m$. Then we have $Z(t) > 0$ for all $t \geq t_1$, in view of (1.1) and (H_3), we have

$$Z^{[2]}(t) + \sum_{j=1}^{m} B_j(t)\xi_j y^\lambda(t-\sigma_j(t)) \leq 0. \tag{2.4}$$

Assumption of (1.2), there exists following two cases:

$$(C_1): \quad Z(t) > 0, \quad Z'(t) > 0, \quad Z''(t) > 0, \quad \text{and} \quad Z^{[2]}(t) \leq 0,$$

or

$$(C_2): \quad Z(t) > 0, \quad Z'(t) < 0, \quad Z''(t) > 0, \quad \text{and} \quad Z^{[2]}(t) \leq 0.$$

Assume (C_1) holds. By virtue $Z(t) > 0$ and $Z'(t) > 0$ that

$$y(t) = Z(t) - A(t)y(t-\eta(t)) \geq Z(t) - A_0 Z(t-\eta(t)) \geq (1-A_0)Z(t),$$

that is

$$y(t) \geq (1-A_0)Z(t). \tag{2.5}$$

Using (2.5) in (2.4), we get

$$Z^{[2]}(t) \leq -(1-A_0)^{\lambda} \sum_{j=1}^{m} \xi_j B_j(t) Z^{\lambda}(t-\sigma_j(t)). \tag{2.6}$$

Define

$$\delta(t) = \zeta(t)\left[\frac{Z^{[1]}(t)}{(Z'(t))^{\lambda}} + a(t)\omega(t)\right]. \tag{2.7}$$

Then $\delta(t) > 0$ and

$$\begin{aligned} \delta'(t) &= \zeta'(t)\left[\frac{Z^{[1]}(t)}{(Z'(t))^{\lambda}} + a(t)\omega(t)\right] + \zeta(t)\left[\frac{Z^{[1]}(t)}{(Z'(t))^{\lambda}} + a(t)\omega(t)\right]' \\ &= \frac{\zeta'(t)}{\zeta(t)}\delta(t) + \zeta(t)\left\{(a(t)\omega(t))' + \frac{Z^{[2]}(t)}{(Z'(t))^{\lambda}} - \lambda a(t)\left(\frac{Z''(t)}{Z'(t)}\right)^{\lambda+1}\right\}. \end{aligned} \tag{2.8}$$

From (2.7) we have

$$\left(\frac{Z''(t)}{Z'(t)}\right)^{\lambda+1} = \left[\frac{\delta(t)}{\zeta(t)a(t)} - \omega(t)\right]^{\frac{\lambda+1}{\lambda}}. \tag{2.9}$$

Substituting (2.6) and (2.9) in (2.8), we have

$$\begin{aligned} \delta'(t) \leq\; & \frac{\zeta'(t)}{\zeta(t)}\delta(t) + \zeta(t)(a(t)\omega(t))' - \zeta(t)(1-A_0)^{\lambda}\sum_{j=1}^{m}\xi_j B_j(t)\frac{Z^{\lambda}(t-\sigma_j(t))}{(Z'(t))^{\lambda}} \\ & -\lambda\,\zeta(t)\,a(t)\left[\frac{\delta(t)}{\zeta(t)a(t)} - \omega(t)\right]^{\frac{\lambda+1}{\lambda}}. \end{aligned} \tag{2.10}$$

Using the inequality

$$X^{\frac{\lambda+1}{\lambda}}-(X-Y)^{\frac{\lambda+1}{\lambda}}\leq Y^{\frac{1}{\lambda}}\left[\Big(\frac{\lambda+1}{\lambda}\Big)X-\Big(\frac{1}{\lambda}\Big)Y\right].$$

Take $X=\frac{\delta(t)}{\zeta(t)a(t)}$ and $Y=\omega(t)$. Now

$$\left[\frac{\delta(t)}{\zeta(t)a(t)}\right]^{\frac{\lambda+1}{\lambda}}-\left[\frac{\delta(t)}{\zeta(t)a(t)}-\omega(t)\right]^{\frac{\lambda+1}{\lambda}}\leq(\omega(t))^{\frac{1}{\lambda}}\left[\Big(\frac{\lambda+1}{\lambda}\Big)\frac{\delta(t)}{\zeta(t)a(t)}-\Big(\frac{1}{\lambda}\Big)\omega(t)\right],$$

which implies

$$\left[\frac{\delta(t)}{\zeta(t)a(t)}-\omega(t)\right]^{\frac{\lambda+1}{\lambda}}\geq\frac{\delta^{\frac{\lambda+1}{\lambda}}(t)}{(\zeta(t)a(t))^{\frac{\lambda+1}{\lambda}}}-\Big(\frac{\lambda+1}{\lambda}\Big)\frac{\omega^{\frac{1}{\lambda}}(t)}{\zeta(t)a(t)}\delta(t)+\frac{\omega^{\frac{\lambda+1}{\lambda}}(t)}{\lambda},$$

which with (2.10) gives

$$\begin{aligned}\delta'(t)\ \leq\ &\frac{\zeta'(t)}{\zeta(t)}\delta(t)+\zeta(t)(a(t)\omega(t))'-\zeta(t)(1-A_0)^{\lambda}\sum_{j=1}^{m}\xi_jB_j(t)\frac{Z^{\lambda}(t-\sigma_j(t))}{(Z'(t))^{\lambda}}\\&-\frac{\lambda}{(\zeta(t)a(t))^{\frac{1}{\lambda}}}\delta^{\frac{\lambda+1}{\lambda}}(t)+(\lambda+1)\omega^{\frac{1}{\lambda}}(t)\delta(t)-\zeta(t)a(t)\omega^{\frac{\lambda+1}{\lambda}}(t).\end{aligned}$$

If $b\in(0,1)$ with the help of Lemma 2.1, then we have

$$\begin{aligned}\delta'(t)\ \leq\ &\frac{\zeta'(t)}{\zeta(t)}\delta(t)+\zeta(t)(a(t)\omega(t))'\\&-\zeta(t)(1-A_0)^{\lambda}\sum_{j=1}^{m}\xi_jB_j(t)b^{\lambda}\left(\frac{(t-\sigma_j(t))}{t}\right)^{\lambda}\frac{Z^{\lambda}(t-\sigma_j(t))}{(Z'(t-\sigma_j))^{\lambda}}\\&-\frac{\lambda}{(\zeta(t)a(t))^{\frac{1}{\lambda}}}\delta^{\frac{\lambda+1}{\lambda}}(t)+(\lambda+1)\omega^{\frac{1}{\lambda}}(t)\delta(t)-\zeta(t)a(t)\omega^{\frac{\lambda+1}{\lambda}}(t).\end{aligned}$$

If $c\in(0,1)$ and take $u(t)=Z'(t)$ by the help of Lemma 2.2, we have

$$\begin{aligned}\delta'(t) &\le \frac{\zeta'(t)}{\zeta(t)}\delta(t)+\zeta(t)(a(t)\omega(t))' \\ &\quad -\zeta(t)\frac{(1-A_0)^\lambda b^\lambda c^\lambda}{2^\lambda t^\lambda}\sum_{j=1}^{m}\xi_j B_j(t)(t-\sigma_j(t))^{2\lambda} \\ &\quad -\frac{\lambda}{(\zeta(t)a(t))^{\frac{1}{\lambda}}}\delta^{\frac{\lambda+1}{\lambda}}(t)+(\lambda+1)\omega^{\frac{1}{\lambda}}(t)\delta(t)-\zeta(t)a(t)\omega^{\frac{\lambda+1}{\lambda}}(t) \\ &\le \zeta(t)(a(t)\omega(t))'-\zeta(t)a(t)\omega^{\frac{\lambda+1}{\lambda}}(t) \\ &\quad -\zeta(t)\frac{(1-A_0)^\lambda b^\lambda c^\lambda}{2^\lambda t^\lambda}\sum_{j=1}^{m}\xi_j B_j(t)(t-\sigma_j(t))^{2\lambda} \\ &\quad +\left[\frac{\zeta'(t)}{\zeta(t)}+(\lambda+1)\omega^{\frac{1}{\lambda}}(t)\right]\delta(t)-\left[\frac{\lambda}{(\zeta(t)a(t))^{\frac{1}{\lambda}}}\right]\delta^{\frac{\lambda+1}{\lambda}}(t) \\ &\le -\Phi(t)+\left[\frac{\zeta'(t)}{\zeta(t)}+(\lambda+1)\omega^{\frac{1}{\lambda}}(t)\right]\delta(t)-\left[\frac{\lambda}{(\zeta(t)a(t))^{\frac{1}{\lambda}}}\right]\delta^{\frac{\lambda+1}{\lambda}}(t),\end{aligned} \tag{2.11}$$

where $\Phi(t)$ is defined in (2.3). Now using the following inequality, for all $\lambda > 0$, then for all $U, V > 0$, one has

$$Uv - Vv^{\frac{\lambda+1}{\lambda}} \le \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}}\frac{U^{\lambda+1}}{V^\lambda}. \tag{2.12}$$

Set $U = \frac{\zeta'(t)}{\zeta(t)} + (\lambda+1)\omega^{\frac{1}{\lambda}}(t)$, $V = \frac{\lambda}{(\zeta(t)a(t))^{\frac{1}{\lambda}}}$, one can obtain that

$$\delta'(t) \le -\Phi(t) + \frac{\zeta(t)a(t)}{(\lambda+1)^{\lambda+1}}\left(\frac{\zeta'(t)}{\zeta(t)} + (\lambda+1)\omega^{1/\lambda}(t)\right). \tag{2.13}$$

Integrating (2.13) from t_2 to t, gives

$$\int_{t_2}^{t}\left(\Phi(t) - \frac{\zeta(t)a(t)}{(\lambda+1)^{\lambda+1}}\left(\frac{\zeta'(t)}{\zeta(t)} + (\lambda+1)\omega^{1/\lambda}(t)\right)\right)ds \le -\delta(t)+\delta(t_2),$$

letting $t \to \infty$, which contradicts to (2.1).

Assume (C_2) holds. Let $Z(t) > 0$, $Z'(t) < 0$, $t \ge t_3$, then there exists a constant $c_1 \ge 0$, such that $\lim_{t\to\infty} Z(t) = c_1$. We claim that $c_1 = 0$. Assume on the contrary that $c_1 > 0$, we have $c_1 + \epsilon > Z(t) > c_1$ for any $\epsilon > 0$, $t \ge t_3$. Choose

$0 < \epsilon < \frac{c_1(1-A_0)}{A_0}$, we obtain from (H_1), we have

$$\begin{aligned} y(t) &= Z(t) - A(t)y(t-\eta(t)) > c_1 - A(t)y(t-\eta(t)) \\ &> c_1 - A_0 Z(t-\eta(t)) \\ &> c_1 - A_0(L+\epsilon) := N(c_1+\epsilon) > NZ(t), \end{aligned} \tag{2.14}$$

where $N = \frac{c_1 - A_0(c_1+\epsilon)}{c_1+\epsilon} > 0$. Using (2.4) and (2.14), we obtain

$$\begin{aligned} Z^{[2]}(t) &\leq -\sum_{j=1}^{m} \xi_j\, B_j(t)\, y^\lambda(t-\sigma_j(t)) \\ &\leq -N^\lambda \sum_{j=1}^{m} \xi_j\, B_j(t)\, Z^\lambda(t-\sigma_j(t)) \\ &\leq -N^\lambda \sum_{j=1}^{m} \xi_j\, B_j(t)\, Z^\lambda(t). \end{aligned} \tag{2.15}$$

Integrating (2.15) from t to ∞ yields

$$Z^{[1]}(t) \geq N^\lambda \sum_{j=1}^{m} \xi_j \int_t^\infty B_j(s)\, Z^\lambda(s)\, ds, \tag{2.16}$$

and using the fact that $Z(t) > c_1$ in (2.16), we find

$$Z''(t) \geq \frac{c_1 N}{a^{1/\lambda}(t)} \left(\sum_{j=1}^{m} \xi_j \int_t^\infty B_j(s)\, ds \right)^{1/\lambda}. \tag{2.17}$$

Again, integrating (2.17) from t to ∞ yields

$$-Z'(t) \geq c_1 N \int_t^\infty \frac{1}{a^{1/\lambda}(u)} \left(\sum_{j=1}^{m} \xi_j \int_u^\infty B_j(s)\, ds \right)^{1/\lambda} du. \tag{2.18}$$

Finally, integrating (2.18) from t_4 to ∞, we deduce that

$$Z(t_4) \geq c_1 N \int_{t_4}^\infty \int_v^\infty \frac{1}{a^{1/\lambda}(u)} \left(\sum_{j=1}^{m} \xi_j \int_u^\infty B_j(s)\, ds \right)^{1/\lambda} du\, dv.$$

The last inequality contradict (2.2), we have $c_1 = 0$. By making use of $0 \leq y(t) \leq Z(t)$, that concludes $\lim_{t\to\infty} y(t) = 0$. □

Next, we examine the oscillation results of solutions of (1.1) by Philos-type [3]. Let $\mathbb{S}_0 = \{(t,s) : a \leq s < t < +\infty\}$, $\mathbb{S} = \{(t,s) : a \leq s \leq t < +\infty\}$ the continuous function $E(t,s)$, $E : \mathbb{S} \to \mathbb{R}$ belongs to the class function $\Re$

(i) $E(t,t)=0$ for $t\geq t_0$ and $E(t,s)>0$ for $(t,s)\in\mathbb{S}_0$,

(ii) $\frac{\partial E(t,s)}{\partial s}\leq 0$, $(t,s)\in\mathbb{S}_0$ and some locally integrable function $e(t,s)$ such that

$$-E(t,s)\left[\frac{\zeta'(t)}{\zeta(t)}+(\lambda+1)\omega^{\frac{1}{\lambda}}(t)\right]=\frac{\partial E(t,s)}{\partial s}+e(t,s),$$

for all $(t,s)\in\mathbb{S}_0$.

Theorem 2.2. *Let* $(H_1)-(H_3)$, (1.2) *and* (2.2) *holds. If there exists* $\zeta\in C^1([t_0,\infty),\mathbb{R})$, *such that for all sufficiently large* $t_5>t_1\geq t_0$ *and for some* $E\in\Re$, *we have*

(2.19)
$$\limsup_{t\to\infty}\frac{1}{E(t,t_5)}\int_{t_4}^{t}\left(E(t,s)\Phi(s)-\frac{1}{(\lambda+1)^{\lambda+1}}\frac{\zeta(s)a(s)|e(t,s)|^{\lambda+1}}{E^{\lambda}(t,s)}\right)ds=\infty,$$

where $\Phi(t)$ *defined in* (2.3). *Then every solution* $y(t)$ *of* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty}y(t)=0$.

Proof. Proceeding by the similar argument as in proof of Theorem 2.1, we obtain the inequality (2.11),

$$\delta'(t)\;\leq\;-\Phi(t)+P(t)\delta(t)-Q(t)\delta^{\frac{\lambda+1}{\lambda}}(t), \tag{2.20}$$

where $\Phi(t)$ is defined in (2.3) and set

$$P(t)=\frac{\zeta'(t)}{\zeta(t)}+(\lambda+1)\omega^{\frac{1}{\lambda}}(t),\quad Q(t)=\frac{\lambda}{(\zeta(t)a(t))^{\frac{1}{\lambda}}}.$$

Multiplying $E(t,s)$ integrating (2.20) from t_5 to t, one can get that

$$\begin{aligned}
\int_{t_5}^{t}E(t,s)\Phi(s)ds &\leq \int_{t_5}^{t}E(t,s)\Big\{-\delta'(s)+P(s)\delta(s)-Q(s)\delta^{\frac{\lambda+1}{\lambda}}(s)\Big\}ds\\
&= E(t,t_5)\delta(t_5)+\int_{t_5}^{t}\left\{\frac{\partial}{\partial s}E(t,s)+E(t,s)\Phi(s)\right\}\delta(s)\,ds\\
&\quad -\int_{t_5}^{t}E(t,s)Q(s)\delta^{\frac{\lambda+1}{\lambda}}(s)\,ds\\
&\leq E(t,t_5)\delta(t_5)+\int_{t_5}^{t}|e(t,s)|\delta(s)\,ds-\int_{t_5}^{t}E(t,s)Q(s)\delta^{\frac{\lambda+1}{\lambda}}(s)\,ds\\
&\leq E(t,t_5)\delta(t_5)+\int_{t_5}^{t}\left\{|e(t,s)|\delta(s)-E(t,s)Q(s)\delta^{\frac{\lambda+1}{\lambda}}(s)\right\}ds.
\end{aligned}$$

Now using the inequality (2.12), set $U = |e(t,s)|$ and $V = E(t,s)Q(t)$, we get that

$$\int_{t_5}^{t} E(t,s)\Phi(s)ds \le E(t,t_5)\delta(t_5) + \int_{t_5}^{t} \frac{1}{(\lambda+1)^{\lambda+1}} \frac{\zeta(s)a(s)|e(t,s)|^{\lambda+1}}{E^{\lambda}(t,s)}\, ds.$$

Hence

$$\frac{1}{E(t,t_5)} \int_{t_5}^{t} \left(E(t,s)\Phi(s) - \frac{1}{(\lambda+1)^{\lambda+1}} \frac{\zeta(s)a(s)|e(t,s)|^{\lambda+1}}{E^{\lambda}(t,s)} \right) ds \le \delta(t_5)$$

which contradicts to (2.19). □

Example 1. *Consider the equation*

$$\left(\left(\left[y(t) + (1/5)y(t/5)\right]''\right)^3\right)' + \frac{6}{t^7} y^3(t/4) + \frac{24}{t^7} y^3(t/2) = 0, \quad t \ge 1. \tag{2.21}$$

Compared with (1.2)*, wee see that* $\lambda = 3$*,* $a(t) = 1$*,* $A(t) = A_0 = 1/5(\le 1)$*,* $\eta = t/5$*,* $B_1(t) = 6/t^7$*,* $B_2(t) = 24/t^7$*,* $\sigma_1(t) = t/4$ *and* $\sigma_2(t) = t/2$*. By taking* $\zeta(t) = t$*,* $\omega(t) = 0$*. The conditions* $(H_1) - (H_3)$ *and* $R[t_0, t] = \infty$*, are satisfied. Then,*

$$\int_{t_2}^{\infty} \left[\frac{b^3 c^3}{32750 s^3} (3\xi_1 + 786\xi_2) - \frac{1}{256} \right] ds = \infty$$

and

$$\int_{t_3}^{\infty} \int_{v}^{\infty} \Big(\int_{u}^{\infty} \frac{(6\xi_1 + 24\xi_2)}{s^7}\, ds \Big)^{1/3} du\, dv = \infty,$$

where $b, c \in (0,1)$ *and* $\xi_1, \xi_2 > 0$*, we see that* (2.1) *and* (2.2) *are clearly satisfied. So we get the equation* (2.21) *is oscillatory and* $x(t) = t^{-1}$ *is a one such solution of equation* (2.21).

3. Conclusion

We present some new oscillatory and asymptotic properties are obtained by means of inequality technique and generalized Riccati substitution if $\lambda \ge 1$. Our obtained results are improve and extend some of the results of Y. Jiang, T. Li [16]. In addition, we can consider the oscillation of equation (1.1) under $R[t_0, t] < \infty$ and we can try to get some oscillation criteria of equation (1.1) if $A(t) < 0$ and arbitrary choose of λ in the future work.

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