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ON STRUCTURES OF RIGHT DUO SEMINEARRING

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ABSTRACT. A right ideal of a seminearring S need not be two-sided ideal in general. We concentrate on those seminearrings which exhibit this property. It is quite natural for us to investigate the distinct properties of such seminearrings. In this paper we introduce the concept of right duo seminearring and prove some of the salient features of a right duo seminearrings when S is a regular seminearring. We also characterise such a seminearring.

1. INTRODUCTION

In 1967, V.G. Van Hoorn and B. Van Rootselaar introduced the algebraic structure from monoids is seminearring [7]. $(S,+)$ and $(S,.)$ are semigroups with right distributive law where addition and multiplication as a binary operations is known as a right seminearring $(S,+,.)$ [3]. If for all $l \in S$, $l+0=0+l=l$ and $l.0=0.l=0$ then S is an absorbing zero 0. Semigroup mapping, linear sequential machines, etc,. are the seminearring applications. $(\Gamma,+)$ is a semigroup mapping sets with absorbing zero, $\mathfrak{M}(\Gamma)$ is the form of pointwise addition and composition of mapping is the natural example of seminearring. If every right ideal of S is two-sided then it is called a right duo seminearring. It is clear that a right ideal of S need not be two-sided ideal. For instance in the seminearring S of 2×2 matrices with absorbing zero over $\mathbb{Z}^+$ the subset,

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$$I = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{Z}^+ \right\}$$

is a right ideal. Since

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ e & f \end{pmatrix} = \begin{bmatrix} ac+be & ad+bf \\ 0 & 0 \end{bmatrix} \in I.$$

It is not a left ideal, since

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in I$$

but

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \notin I.$$

In this example, I is a one sided ideal which is not two sided. It is very natural for us to study and investigate the different properties of a seminearring S which actually bless it with the right duo structure. Motivated by this, we make an attempt in this paper to unify and consolidate the salient features of certain special structures possessing the right duo structure with a view to obtain full charecterisation of right duo seminearring.

2. PRELIMINARIES

Inside this section we compile all the terminologies that are used in our paper relating to the theory of seminearring. A seminearring S is known as right(left) normal seminearring if $a \in aS(Sa)$ for every $a \in S$. S is normal if it is both a left as well as a right normal [2]. S is idempotent whenever $a^2 = a$ for every $a \in S$. $(S, +, .)$ is a seminearring and $\mathcal{I}$ is a non-empty subset of S is a left (respectively right) ideal of S if $(i) m+y \in \mathcal{I}\ \forall\ m, y \in \mathcal{I}$, $(ii) r.u \in \mathcal{I}(u.r \in \mathcal{I})\ \forall\ u \in \mathcal{I}\ and\ r \in S$ conditions hold [6]. $\mathcal{I}$ is called an ideal of S if it is a both left as well as a right ideal of S. An element a of a seminearring S is called distributive if for all $x, y \in S$, $a(x+y) = ax + ay$; S will be called distributive if each of its element is distributive. A seminearring S is called distributively generated, if S contains a multiplicative subsemigroup D of distributive elements which generates $(S, +)$. If A, B are the non-empty subsets of a seminearring S, then AB will denote the set of all finite sums of the form $\sum a_k b_k$ with $a_k \in A$ and $b_k \in B$. In particular, for each $a \in S$, $aS(Sa)$ will denote the set of all finite sums of the form $\sum as_k(\sum s_k a)$ with $s_k \in S$. Since S is right distributive, $Sa = \{sa : s \in S\}$. Clearly $aS(Sa)$ is a right (left) ideal of S [5]. The ideal $aS(Sa)$ will be called the principal right(left) ideal generated by a. For any subset S of S, $< S >$ will denote the ideal generated by S [4]. A seminearring S is regular if for each $a \in S$, there exists $b \in S$ such that $a = aba$ [6]. We notice that if $\mathcal{I}$ is an ideal of a seminearring S, then $\mathcal{I}$ is a subseminearring of S. For a subset $\mathcal{I}$ of S, we notice that $(i) S^1\mathcal{I} = \mathcal{I} \cup S\mathcal{I}$ is the left ideal($S\mathcal{I} \subseteq \mathcal{I}$) generated by $\mathcal{I}$ $(ii)\mathcal{I}S^1 = \mathcal{I} \cup \mathcal{I}S$ is the right ideal($\mathcal{I}S \subseteq \mathcal{I}$) generated by $\mathcal{I}$ and $(iii)S^1\mathcal{I}S^1 = \mathcal{I} \cup S\mathcal{I} \cup \mathcal{I}S \cup S\mathcal{I}S$ generated by $\mathcal{I}$. A seminearring S is right(left) regular seminearring if $a \in a^2S(Sa^2)$ for each $a \in S$. A seminearring S is intra regular if $a \in Sa^2S$ for $a \in S$. A seminearring S is band if each $a \in S$ is idempotent. A seminearring S is said to be left duo if every left ideal of S is two sided and duo if every one sided ideal of S is two sided [1]. Throughout this paper S stands for a right seminearring with an absorbing zero.

3. Main Results

We begin this section with the charecterisation of right duo seminearring.

Proposition 3.1. *A semirinearring S is right duo iff $R_1R_2 \subseteq R_1 \cap R_2$ for every pair of right ideals R_1, R_2 of S.*

Proof. Let S satisfy the condition. $R_1R_2 \subseteq R_1 \cap R_2$. Then for any right ideal R of S. $SR \subseteq S \cap R = R$ and so S is right duo.

Conversely, if S is right duo, then for any two right ideals R_1, R_2 of S, we have $R_1R_2 \subseteq R_1S \subseteq R_1$ and $R_1R_2 \subseteq SR_2 \subseteq R_2$. Hence $R_1R_2 \subseteq R_1 \cap R_2$. ☐

Proposition 3.2. *A right duo seminearring is regular iff every principal left ideal is idempotent. Further if the right duo seminearring has identity, then it is regular iff every proper principal left ideal is idempotent.*

Proof. Let S be a regular right duo seminearring and I is a left ideal of S. If $a \in I$, then $a = axa$, for some $x \in S$ and so $a \in I^2$, showing that I is idempotent. Hence in particular, every principal left ideal of S is idempotent.

Conversely, let S be a right duo seminearring satisfying the condition. For any $a \in S$, the principal left ideal generated by a is $a \cup Sa$ and being idempotent, we have $a \cup Sa = (a \cup Sa)(a \cup Sa) = a^2 \cup aSa \cup Sa^2 \cup SaSa \subseteq a^2 \cup aSa \cup a(a \cup$

$aS) \cup Sa(a \cup aS) \subseteq a^2 \cup aSa$, as $Sa \subseteq a \cup aS$, S being right duo. Hence $a = a^2$ or $a \in aSa$, so that a is regular. Thus S is regular.

Suppose S has identity. If for Sa, the principal left ideal generated by a, we have $Sa = S$, than $a \in Sa = (Sa)a \subseteq (aS)a$ and so a is regular. If Sa is proper then as in the earlier proof, a is seen to be regular and so S is regular.

A similar proof holds for left duo seminearrings. □

Corollary 3.1. *A right normal seminearring is regular iff every principal left ideal is idempotent.*

Remark 3.1. *The following example shows that the existence of identity is neccessary for the second half of Proposistion 3.2 to hold.*
We consider the seminearring $(S, +, .)$ where $S = \{a, b, c\}$ and the semigroup operations $'+'$ and $'.'$ in S as follows:

+	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

.	a	b	c
a	a	a	a
b	a	b	b
c	a	b	b

Let $S = \{a, b, c\}$ where $a.x = x.a = a$ for $x = a, b, c$, $b^2 = c^2 = b$ and $b.c = c.b = b$.
The seminearring S has no identity and being commutative, is right duo. Here $R = \{a\}$ and $R = \{a, b\}$ are the only proper principal right ideals and are idempotent. However, S is not regular, as c is not regular.

Proposition 3.3. *A right duo seminearring is regular iff it is left regular.*

Proof. Let S be a right duo seminearring. If S is left regular, then for any $a \in S$, we see that $a \in Sa^2 = (Sa)a \subseteq (aS \cup a)a = aSa \cup a^2$ and so $a = a^2$ or $a \in aSa$. Hence a is regular and so S is regular.

Conversely, if S is regular, then for any $a \in S$ we have $a \in aSa = a(Sa) \subseteq a(aS \cup a) = a^2S \cup a^2$.

Hence $a = a^2$ or $a \in a^2S$ and so in either case $a \in a^2S$. Thus S is right regular. Hence the result. □

The following proposition gives a necessary and sufficient condition for a right duo seminearring to be intra-regular.

Proposition 3.4. *A right duo sminearring S is intra-regular iff every principal ideal is idempotent.*

Proof. Clearly in any intra-regular seminearring, principal ideals are idempotent.

Conversely, let S be such that every principal ideal is idempotent. Then if $a \in S$, we have $a \in SaSaS$. Since S is right duo, $SaS \subseteq aS$ and so $a \in (Sa)(aS) = Sa^2S$, which shows that a is intra regular.

Hence S is intra regular. □

Proposition 3.5. *In a right duo seminearring S, for any two idempotents $e, f \in S$, we have $efe = fe$. In particular, the idempotents form a band.*

Proof. If $x \in S$, then the right ideal xS being two sided, we have $SxS \subseteq xS$. It follws that $ef = eef \in SeS \subseteq eS$ and so $ef = ea$, for some $a \in S$. Hence $efe = ae = fe$.

Consequently $(fe)^2 = fe$ and so idempotents form a band. □

We now give here a necessary and sufficient condition for any seminearring to be duo and obtaied a charecterisation of regular duo seminearring.

Proposition 3.6. *A seminearring S is duo iff $LR \subseteq L \cap R$ for every left ideal L and every right ideal R of S.*

Proof. Let S satisfy the condition $LR \subseteq L \cap R$ Then we have $SR \subseteq S \cap R = R$ and $LS \subseteq L \cap S = L$. Hence L and R are two-sided and so S is a duo seminearring. Conversely, if S is duo, then for any left ideal L and right ideal R, we have $LR \subseteq LS \subseteq L$ and $LR \subseteq SR \subseteq R$ and so $LR \subseteq L \cap R$. □

Theorem 3.1. *A seminearring S is a regular duo seminearring iff for each $a \in S$, $J^2(a) = L(a) = R(a)$, where $L(a)$, $R(a)$ and $J(a)$ denote respectively the principal left, right and two-sided ideals generated by a.*

Proof. Let for any $a \in S$, $J^2(a) = L(a) = R(a)$. If L is any left ideal of S, then for any $b \in L$, we have $bS \subseteq L(b)S = R(b)S \subseteq R(b) = L(b) \subseteq L$ and so L is two sided. Similarly, we see that every right ideal R is also two sided. Hence S is a duo seminearring.

Now $a \in L(a) = J^2(a) = (S^1aS^1)(S^1aS^1) =(S^1a)(S^1S^1)(aS^1) \subseteq (aS^1)S(S^1a)$, as $L(a) = R(a) \subseteq aSa$ and so a is regular. Hence S is regular.

Conversely let S be a regular duo seminearring. Then by regularity of S, it follows that $J(a)$ is idempotent for any $a \in S$. Also S being duo, we also have $L(a) = J(a)$ and $R(a) = J(a)$.

Hence $J^2(a) = J(a) = L(a) = R(a)$ proving the theorem. $\square$

4. Conclusion

It is worth notify that, a right ideal is not a left ideal even in near ring theory. This makes the interest to study of such a seminearring. In this work observes some fruitful results of right duo seminearrings which is not adhere the above rule.

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