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ON THE WELL-POSEDNESS OF A TRANSMISSION EIGENVALUE PROBLEM

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ABSTRACT. In the theory of partial differential equations a crucial role plays the well-posedness of a given problem. In this paper we deal with the existence and the unicity of a transmission problem with mixed boundary conditions and the dependence of the solutions from the initial data. First we put the problem in its variational form with the help of Lax-Milgram lemma. Then, using the trace theorem and the fact that our problem is elliptic we prove its existence. Using an important theorem on functional analysis theory [1], we finally show the desired result on well-posedness.

1. INTRODUCTION

We consider the scattering of electromagnetic waves through a penetrable and nonhomogeneous obstacle. It is given the mathematical model of the problem, and we show the existence of the solution and its uniqueness. As tools to achieve these goals we use Green's identities, variational approach and Relich lemma. In this paper we deal with the direct problem, which is a necessary condition for studying the inverse problem. The interior transmission problem, which arises in inverse scattering theory, is a boundary values problem compunded of two partial differential equations of second order defined in a bounded domain that corresponds to the scatterer. The boundary value problem is not elliptic in the sense of Agmon-Doughlas-Nirenberg so the classic thoery of PDE does not

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provide a direct answer for its solvability. Its homogeneous version is referred to as the transmission eigenvalue problem, which is nonlinear and non self-adjoint eigenvalue problem, more specifically

$$\nabla \cdot A\nabla w + k^2 n w = 0 \quad \text{in } \Omega \tag{1.1}$$

$$\Delta u + k^2 u = 0 \quad \text{in } S_R \setminus \overline{\Omega} \tag{1.2}$$

$$w - u^s = -\eta \frac{\partial(u^s + u^i)}{\partial v} + u^i \quad \text{in } \partial\Omega \tag{1.3}$$

$$\frac{\partial w}{\partial v} - \frac{\partial u^s}{\partial v} = \frac{\partial u^i}{\partial v} \quad \text{in } \partial\Omega \tag{1.4}$$

$$\frac{\partial u}{\partial v} = Tu \quad \text{in } \partial S_R, \tag{1.5}$$

where T is the Dirichlet-Neumann operator.

For our purpose, to show that the problem is well-posed we refer to [1] and more specifically to the following theorem

Theorem 1.1. *Let X and Y be two Hilbert spaces and let $A : X \to Y$ be a bijective bounded linear operator with bounded inverse $A^{-1} : Y \to X$, and $B : X \to Y$ a compact linear operator. Then $A + B$ is injective if and only if it is surjective. If $A + B$ is injective (and hence bijective) then the inverse $(A + B)^{-1} : Y \to X$ is bounded.*

2. Variational formulation of the given problem

Let $u_f \in H^1(S_R \backslash \overline{\Omega})$ be the unique solution of the following boundary Dirichlet problem.

$$\begin{cases} \Delta u_\eta + k^2 u_\eta = 0 & in\ S_R \setminus \overline{\Omega} \\ u_\eta = -\eta \frac{\partial(u^s + u^i)}{\partial v} & in\ \partial\Omega \\ u_\eta = 0. \end{cases}$$

The boundary ∂S_R can be chosen such that k^2 is not a Dirichlet eigenvalue for$-\Delta$ in $S_R \setminus \overline{\Omega})$.

We note that the values k^2 ,for which there exist a nonzero solution $u \in H_0^1(\Omega)$ are called Dirichlet eigenvalue of$-\Delta$.

We bring problem (1.1)-(1.5) to its variational form

$$(\nabla\cdot A\nabla w+k^2nw,\varphi)=0$$

$$(\nabla\cdot A\nabla w,\varphi)+(k^2nw,\varphi)=0$$

$$\int_\Omega \nabla\cdot A\nabla w\cdot\overline{\varphi}dx+k^2n\int_\Omega w\overline{\varphi}dx=0$$

$$-\int_\Omega(A\nabla w\nabla\overline{\varphi})dx+\int_{\partial\Omega}(A\nabla w)\cdot v\cdot\overline{\varphi}ds+k^2n\int_\Omega w\overline{\varphi}dx=0$$

$$\int_\Omega(A\nabla w\cdot\nabla\overline{\varphi})dx-k^2n\int_\Omega w\overline{\varphi}dx=\int_{\partial\Omega}\frac{\partial w}{\partial v_A}\overline{\varphi}ds. \tag{2.1}$$

Since $u_\eta=-\eta\dfrac{\partial(u^s+u^i)}{\partial v}+u^i$ in $\partial\Omega$ then in $\partial\Omega$ we have $w=0$, and it follows $\dfrac{\partial w}{\partial v_A}=0$ in $\partial\Omega$, so finally

$$\int_\Omega(A\nabla w\cdot\nabla\overline{\varphi})dx-k^2n\int_\Omega w\cdot\overline{\varphi}dx=0.$$

From the Helmholtz equation we obtain

$$\Delta u+k^2u=0$$

$$(\Delta u+k^2u,\varphi)=0$$

$$(\Delta u,\varphi)+k^2(u,\varphi)=0$$

$$\int_{S_R\backslash\overline{\Omega}}\Delta u\cdot\overline{\varphi}dx+k^2\int_{S_R\backslash\overline{\Omega}}u\overline{\varphi}dx=0$$

$$-\int_{S_R\backslash\overline{\Omega}}\nabla u\cdot\nabla\overline{\varphi}dx+\int_{\partial S_R}\overline{\varphi}\frac{\partial u}{\partial v}ds+\int_{\partial\Omega}\overline{\varphi}\frac{\partial u}{\partial v}ds+k^2\int_{S_R\backslash\overline{\Omega}}u\cdot\overline{\varphi}dx=0$$

$$-\int_{S_R\backslash\overline{\Omega}}\nabla u\cdot\nabla\overline{\varphi}dx+\int_{\partial S_R}\overline{\varphi}Tuds+\int_{\partial\Omega}\overline{\varphi}\frac{\partial u}{\partial v}ds+k^2\int_{S_R\backslash\overline{\Omega}}u\cdot\overline{\varphi}dx=0$$

Considering that $u=w-u_\eta$ and (2.1) we have

$$-\int_{S_R\backslash\overline{\Omega}}\nabla(w-u_\eta)\nabla\overline{\varphi}dx+\int_{\partial S_R}\overline{\varphi}T(w-u_\eta)ds+\int_{\partial\Omega}\overline{\varphi}\frac{\partial(w-u_\eta)}{\partial v}ds$$

$$+k^2\int_{S_R\backslash\overline{\Omega}}(w-u_\eta)\overline{\varphi}dx=0$$

$$-\int_{S_R\backslash\overline{\Omega}}\nabla w\cdot\nabla\overline{\varphi}dx+\int_{\partial S_R}\overline{\varphi}Twds+\int_{\partial\Omega}\overline{\varphi}\frac{\partial w}{\partial v}ds+k^2\int_{S_R\backslash\overline{\Omega}}w\cdot\overline{\varphi}dx=0$$

$$\int_{S_R\setminus\overline{\Omega}} \nabla u_\eta \cdot \nabla\overline{\varphi} dx - \int_{\partial S_R} \overline{\varphi} T u_\eta ds + \int_{\partial\Omega} \overline{\varphi}\frac{\partial u^i}{\partial v} ds - k^2 \int_{S_R\setminus\overline{\Omega}} u_\eta \overline{\varphi} dx = 0. \tag{2.2}$$

Now we can have the variational formulation of the problem using identities (2.4) and (2.2).

Find $\omega \in H^1(S_R)$ such that

$$\begin{aligned} &\int_\Omega (\nabla\overline{\varphi} \cdot A\nabla w - k^2 n\overline{\varphi} w)dx + \int_{S_R\setminus\overline{\Omega}} (\nabla\overline{\varphi} \cdot \nabla w - k^2\overline{\varphi} w)dx \\ &- \int_{\partial S_R} \overline{\varphi} T\omega ds = \int_{\partial\Omega} \overline{\varphi}\frac{\partial u^i}{\partial v} ds - \int_{\partial S_R} \overline{\varphi} T u_f ds \\ &+ \int_{S_R\setminus\overline{\Omega}} (\nabla\overline{\varphi} \cdot \nabla u_f - k^2\overline{\varphi} u_f)dx, \end{aligned} \tag{2.3}$$

for every $\varphi \in H^1(S_R)$. Using Green's First Identity we have that $w := \omega|_\Omega$ and $u := \omega|_{S_R\setminus\overline{\Omega}} - u_f$ are solutions of (1.1)-(1.5).

Vice Versa, multilying (1.1) and (1.2) with a test φ function then using the boundary codition (1.4), (1.5) we have that $\omega = w$ in Ω and $\omega = u + u_f$ in $S_R \setminus \overline{\Omega}$ where $\omega \in H^1(S_R)$ and satisfies (2.7), where the pair (w, u) is a solution of (1.1)-(1.5). Now using the Lax-Milgram lemma we obtain

$$\int_\Omega (\nabla\overline{\varphi} \cdot A\nabla\omega + w\overline{\varphi} - w\overline{\varphi} - k^2 n\overline{\varphi}\omega) + \int_{S_R\setminus\overline{\Omega}} (\nabla\overline{\varphi}\nabla\omega + w\overline{\varphi} - w\overline{\varphi} - k^2\overline{\varphi}\omega)dx -$$

$$- \int_{\partial S_R} \overline{\varphi}(T - T_0 + T_0)\omega ds = \int_{\partial\Omega} \overline{\varphi}\frac{\partial u^i}{\partial v} ds - \int_{\partial S_R} \overline{\varphi} T u_f ds + \int_{S_R\setminus\overline{\Omega}} (\nabla\overline{\varphi} \cdot \nabla u_f - k^2\overline{\varphi} u_f)dx$$

In $H^1(S_R) \times H^1(S_R)$ we define the sesquilinear continuous form

$$\begin{aligned} a_1(\psi, \varphi) := &\int_\Omega (\nabla\overline{\varphi} \cdot A\nabla\psi + \overline{\varphi}\psi)dx + \int_{S_R\setminus\overline{\Omega}} (\nabla\overline{\varphi} \cdot \nabla\psi + \overline{\varphi}\psi)dx \\ &- \int_{\partial S_R} \overline{\varphi} T_0 \psi ds, \end{aligned}$$

$\varphi, \psi \in H^1(S_R)$ and

$$a_2(\psi, \varphi) := -\int_\Omega (nk^2 + 1)\overline{\varphi}\psi dx - \int_{S_R\setminus\overline{\Omega}} (k^2 + 1)\overline{\varphi}\psi dx - \int_{\partial S_R} \overline{\varphi}(T - T_0)\psi ds$$

$\varphi, \psi \in H^1(S_R)$, where T_0 is defined from [2]. We define also the bounded linear conjugate functional

$$F(\varphi) = \int_{\partial\Omega} \overline{\varphi}\frac{\partial u^i}{\partial v} ds - \int_{\partial S_R} \overline{\varphi} T u_f ds + \int_{S_R\setminus\overline{\Omega}} (\nabla\overline{\varphi} \cdot \nabla u_f - k^2\overline{\varphi} u_f)dx.$$

The problem is written as

$$a_1(\omega,\varphi)+a_2(\omega,\varphi)=f(\varphi), \qquad \forall\varphi\in H^1(S_R).$$

From assumption $\overline{\xi}Re(A)\xi \geq \gamma|\xi|^2$, $\xi\in C^3$, $x\in\overline{\Omega}$ the trace theorem and the inequality of T_0 operator the following inequalities hold

$$\begin{aligned}
a_1(\varphi,\varphi) &= \int_\Omega (\nabla\overline{\varphi}\cdot A\nabla\varphi+\overline{\varphi}\varphi)dx+\int_{S_R\setminus\overline{\Omega}}(\nabla\overline{\varphi}\cdot\nabla\varphi+\overline{\varphi}\varphi)dx\\
&\quad -\int_{\partial S_R}\overline{\varphi}T_0\psi ds \geq \int_\Omega(\nabla\overline{\varphi}\cdot A\nabla\varphi)dx+\int_\Omega\overline{\varphi}\varphi dx\\
&\quad +\int_{S_R\setminus\overline{\Omega}}\nabla\overline{\varphi}\cdot\nabla\varphi dx+\int_{S_R\setminus\overline{\Omega}}(\overline{\varphi}\varphi)dx+\tilde{C}||\varphi||^2_{H^{1/2}(\partial S_R)}\\
&\geq \gamma|\xi|^2+||\varphi||^2_{L^2(\Omega)}+||\nabla\varphi||^2_{L^2(\Omega)}+|||\varphi||^2_{L^2(S_R\setminus\overline{\Omega})}+[\tilde{\tilde{C}}]||\varphi||^2_{H^1(S_R)}\\
&\geq [\tilde{\tilde{C}}]||\varphi||^2_{H^1(S_R)},
\end{aligned}$$

which show that $a_1(.,.)$ is stricly coercive. From Lax-Milgram lemma the operator $A:H^1(S_R)\to H^1(S_R)$, $a_1(\omega,\varphi)=(A\omega,\varphi)_{H^1(S_R)}$ has bounded inverse. From the compactness of $I_m: H^1(S_R)\to L^2(S_R)$ and $T-T_0: H^{1/2}(\partial S_R)\to H^{-1/2}(\partial S_R)$, the operator $B: H^1(S_R)\to H^1(S_R)$ defined as $a_2(\omega,\varphi)=(B\omega,\varphi)_{H^1(S_R)}$ is compact. Since our problem satisfies the condition of theorem [1], to prove the existence of the solution we need to show its uniqueness.

3. Uniqueness of the problem

Theorem 3.1. *Problem* $(1.1)-(1.5)$ *has at most one solution.*

Proof. Let $w\in H^1(\Omega)$ and $u^s\in H^1_{loc}(R^2\setminus\overline{\Omega})$ be the solution of (1.1)-(1.5) with incident wave $u^i=0$. Let S_R be a disk centered at the origin with radius R that contain the closure $\overline{\Omega}$. We apply Green's first identity in Ω and $(R^2\setminus\overline{\Omega})\cap S_R$, obtaining the following

$$\begin{aligned}
&(\nabla\cdot A\nabla w+k^2nw,w)=0\\
&(\nabla\cdot A\nabla w,w)+(k^2nw,w)=0\\
&\int_\Omega\nabla\cdot A\nabla w\cdot\overline{w}dy+k^2n\int_\Omega w\overline{w}dy=0\\
&-\int_\Omega(A\nabla w\cdot\nabla\overline{w})dy+\int_{\partial\Omega}(A\nabla w)\cdot v\cdot\overline{w}ds+k^2n\int_\Omega w\cdot\overline{w}dy=0
\end{aligned}$$

$$\int_{\Omega}(A\nabla w\cdot\nabla\overline{w})dy-k^2n\int_{\Omega}w\cdot\overline{w}dy=\int_{\partial\Omega}\frac{\partial w}{\partial v_A}\overline{w}ds. \tag{3.1}$$

For the Helmholtz equation we have

$$\Delta u^s+k^2u^s=0$$

$$(\Delta u^s+k^2u^s,u^s)=0$$

$$(\Delta u^s,u^s)+k^2(u^s,u^s)=0$$

$$\int_{S_R\setminus\overline{\Omega}}\Delta u^s\cdot\overline{u}^sdy+k^2\int_{S_R\setminus\Omega}u^s\overline{u}^sdy=0$$

$$-\int_{S_R\setminus\overline{\Omega}}\nabla u^s\cdot\nabla\overline{u}^sdy+\int_{\partial S_R}\overline{u}^s\frac{\partial u^s}{\partial v}ds+\int_{\partial\Omega}\overline{u}^s\frac{\partial u^s}{\partial v}ds+k^2\int_{S_R\setminus\overline{\Omega}}u^s\overline{u}^sdy=0$$

$$\int_{S_R\setminus\overline{\Omega}}(|\nabla u^s|^2-k^2|u^s|^2)dy=\int_{\partial S_R}\overline{u}^s\frac{\partial u^s}{\partial v}ds-\int_{\partial\Omega}\overline{u}^s\frac{\partial u^s}{\partial v}ds. \tag{3.2}$$

Adding both sides of (3.1), (3.2) we have

$$\begin{aligned}&\int_{\Omega}(A\nabla w\cdot\nabla\overline{w}-k^2n|w|^2)dy+\int_{S_R\setminus\overline{\Omega}}(|\Delta u^s|^2-k^2|u^s|^2)dy=\\&=\int_{\partial\Omega}\overline{w}\frac{\partial w}{\partial v_A}ds-\int_{\partial\Omega}\overline{u}^s\frac{\partial u^s}{\partial v}ds+\int_{\partial S_R}\overline{u}^s\frac{\partial u^s}{\partial v}ds.\end{aligned} \tag{3.3}$$

Using the transmission condition

$$\begin{aligned}&\int_{\partial\Omega}\overline{w}\frac{\partial w}{\partial v_A}ds-\int_{\partial\Omega}\overline{u}^s\frac{\partial u^s}{\partial v}ds=\int_{\partial\Omega}\overline{w}\frac{\partial u^s}{\partial v}ds-\int_{\partial\Omega}\overline{u}^s\frac{\partial u^s}{\partial v}ds=\\&=\int_{\partial\Omega}(\overline{w}-\overline{u}^s)\frac{\partial u^s}{\partial v}ds=\int_{\partial\Omega}(\overline{w}-\overline{u}^s)(\frac{u^s-w}{\eta})ds=\\&=\frac{1}{\eta}\int_{\partial\Omega}|w-u^s|^2ds.\end{aligned}$$

So finally relation (3.3) can be given as

$$\begin{aligned}&\int_{\Omega}(A\nabla w\cdot\nabla\overline{w}-k^2n|w|^2)dy+\int_{S_R\setminus\overline{\Omega}}(|\Delta u^s|^2-k^2|u^s|^2)dy\\&=\frac{1}{\eta}\int_{\partial\Omega}|w-u^s|^2ds+\int_{\partial S_R}\overline{u}^s\frac{\partial u^s}{\partial v}ds\end{aligned}$$

From the assumptions, $\operatorname{Im}(A)\leq 0,\operatorname{Im}(n)\leq 0$ and $\eta\geq\eta_0\geq 0$, the

$$\operatorname{Im}(\int_{\partial S_R}\overline{u}^s\frac{\partial u^s}{\partial v}ds)\leq 0$$

which Implies that $\text{Im}(\int_{\partial S_R} u^s \frac{\partial \overline{u}^s}{\partial v} ds) \geq 0$. From Relich lemma $u^s = 0$ in $R^2 \backslash \overline{\partial S_R}$, so $u^s = 0$ in $R^2 \backslash \overline{\Omega}$. From the transmission condition we have $w = 0$ and $\frac{\partial w}{\partial v_A} = 0$ in $\partial\Omega$. This result can be extended in Ω as well, not only in its boundary due to the unique principle of continuation. To this purpose, first we extend $Re(A)$ as a differentiable continuous symetric function with positive values in $\overline{S_R}$ and $\text{Im}(A)$ as a differentiable continuous symmetric function with positive values with compact support in S_R. We define $w = 0$ in $S_R \setminus \overline{\Omega}$. Since $w = 0$ and $\frac{\partial w}{\partial v_A} = 0$ in $\partial\Omega$ then $w \in H^1(S_R)$ and $\nabla \cdot A\nabla w + k^2 n w = 0$ in S_R. So the condition of theorem 1.1 are satisfied for function w, we can use the principle of continuity where $q = 0$ and since $w = 0$ in $S_R \setminus \overline{\Omega}$, then $w = 0$ in S_R. $\square$

So we proved the existence and uniqueness of the solution of problem (1.1)-(1.5). Using Lax-Milgram lemma we have an estimate of the solution that is continuously dependent of the initial data, so we can state that the problem is well-posed.

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