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NORMALITY OF THE EXTENSIONS OF A DOUBLE FUZZY TOPOLOGICAL SPACE

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ABSTRACT. Though the extension of a normal double fuzzy topological space is not necessarily normal, the study identifies certain situations under which the normality is carried over to the extensions.

1. INTRODUCTION

In 2011, A. Ghareeb [2] introduced the concept of normality in double fuzzy topological spaces and studied various types of the same. Recently, Vivek S and S. C. Mathew [7] studied the extensions of a double fuzzy topological space and explored whether the connectedness and compactness of a double fuzzy topological space are carried over to its extensions.

As a continuation of the above study, in this paper we investigate whether the normality of a double fuzzy topological space is carried over to its extensions. Though the extensions need not preserve normality in general, some conditions which ensure the normality of the extended space are obtained.

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2. PRELIMINARIES

Throughout this paper X stands for a non-empty set and some particular sets are identified as, $I = [0,1]$, $I_0 = (0,1]$, $I_1 = [0,1)$ and $I_0 \oplus I_1 = \{(r,s) \in I_0 \times I_1 : r+s \leq 1\}$.

Definition 2.1. *[4] A fuzzy subset g of X is said to be weak if $0 < g(x) < \frac{1}{2}$ for all $x \in X$.*

Definition 2.2. *[6] Let (τ, τ^*) be a pair of functions from I^X to I satisfying*

(i) $\tau(f) + \tau^*(f) \leq 1, \forall f \in I^X$,
(ii) $\tau(\underline{0}) = \tau(\underline{1}) = 1, \tau^*(\underline{0}) = \tau^*(\underline{1}) = 0$,
(iii) $\tau(f_1 \wedge f_2) \geq \tau(f_1) \wedge \tau(f_2)$ *and* $\tau^*(f_1 \wedge f_2) \leq \tau^*(f_1) \vee \tau^*(f_2), f_i \in I^X, i = 1,2$,
(iv) $\tau(\bigvee_{i\in\Delta} f_i) \geq \bigwedge_{i\in\Delta} \tau(f_i)$ *and* $\tau^*(\bigvee_{i\in\Delta} f_i) \leq \bigvee_{i\in\Delta} \tau^*(f_i), f_i \in I^X, i \in \Delta$

Then the pair (τ, τ^*) is called a double fuzzy topology on X and the triplet (X, τ, τ^*) is called a double fuzzy topological space or dfts in short.

Notation: With respect to a given dfts (X, τ, τ^*), we use the following notations: $\tau^+ = \{f \in I^X : \tau(f) > 0\}$, $\tau^+(g) = \{f \in I^X : f = f_1 \vee (f_2 \wedge g), f_1, f_2 \in \tau^+\}$,$(\tau^+)^c = \{f : f^c \in \tau^+\}$ and $R_g f = \{(f_1, f_2) : f_1 \vee (f_2 \wedge g) = f, f_1, f_2 \in I^X\}$.

Note that τ^+ need not be a fuzzy topology. However, if τ^+ is a fuzzy topology then, $\tau^+(g)$ is also a fuzzy topology called the simple extension of τ^+ determined by g.

Definition 2.3. *[3] Let (X, τ, τ^*) be a dfts. For $(r,s) \in I_0 \oplus I_1$, a fuzzy set f is called*

(i) *(r,s)-fuzzy open if $\tau(f) \geq r$ and $\tau^*(f) \leq s$ and*
(ii) *(r,s)-fuzzy closed if f^c is (r,s)-fuzzy open.*

The collections of all (r,s)-fuzzy open sets and (r,s)-fuzzy closed sets are respectively denoted by $\mathcal{O}_{\tau,r,s}$ and $\mathcal{C}_{\tau,r,s}$.

Theorem 2.1. *[8] Let (X, τ, τ^*) be a dfts. Then for any $(r,s) \in I_0 \oplus I_1$, $\mathcal{C}_{\tau,r,s}$ is a meet-semilattice.*

Definition 2.4. *[1] Let $g \in I^X$. Then $f_1, f_2 \leq g$ are said to be quasi coincident with respect to g if $\exists x \in X$ such that $f_1(x) + f_2(x) > g(x)$ and is denoted by $f_1 q f_2[g]$.*

The non-quasi coincidence of f_1 and f_2 with respect to g is denoted by $f_1\bar{q}f_2[g]$. By $\bar{q}[g]$ we denote the set $\{(f_1, f_2) : f_1\bar{q}f_2[g]\}$.

In particular, if $g = \underline{1}$ then we simply say f_1 and f_2 are non-quasi coincident and is denoted by $f_1\bar{q}f_2$.

Definition 2.5. *[2] A dfts (X, τ, τ^*) is said to be normal if for each $(r, s) \in I_0 \oplus I_1$ and $f_1, f_2 \in \mathcal{C}_{\tau,r,s}$ with $f_1\bar{q}f_2$, there exists $g_1, g_2 \in \mathcal{O}_{\tau,r,s}$ such that $f_1 \leq g_1, f_2 \leq g_2$ and $g_1\bar{q}g_2$.*

Definition 2.6. *[5] Let P be a poset and $A \subset P$. Define*

$$\uparrow A = \{b \in P : \exists a \in A, b \geq a\} \text{ and } \downarrow A = \{b \in P : \exists a \in A, b \leq a\}.$$

For a singleton set $\{a\}$ denote $\uparrow a = \uparrow \{a\}$ and $\downarrow a = \downarrow \{a\}$.

Definition 2.7. *[7] Let (X, τ, τ^*) be a dfts and $g \in I^X$. For $\alpha \in I_0$ and $\beta \in I_1$ with $\alpha \geq \tau(g)$, $\beta \leq \tau^*(g)$ and $\alpha + \beta \leq 1$, define $\tau_0, \tau_0^* : I^X \to I$ by*

(i) $\tau_0(g) = \alpha$, $\tau_0^*(g) = \beta$ *and*

(ii) *for all $f \in I^X \setminus \{g\}$,*

$$\tau_0(f) = \max\left\{\tau(f), \bigvee \{\tau(f_1) \wedge \tau(f_2) \wedge \alpha : (f_1, f_2) \in R_g f\ \}\right\}$$

$$\tau_0^*(f) = \min\left\{\tau^*(f), \bigwedge \{\tau^*(f_1) \vee \tau^*(f_2) \vee \beta : (f_1, f_2) \in R_g f\ \}\right\}$$

Then the triplet (X, τ_0, τ_0^*) is a dfts called the (g, α, β)-extension of (X, τ, τ^*).

3. Normality of the Extensions of a DFTS

In general, the normality of a dfts is not carried over to its extensions as shown by the following example.

Example 3.1. *Let $X = I$ and $\mathcal{B} = \{\chi_{[0,a]} : a \in (0, 1)\}$. Now define $\tau, \tau^* : I^X \to I$ by*

$$\tau(f) = \begin{cases} 1, & \text{if } f = \underline{1} \text{ or } \underline{0} \\ a, & \text{if } f = \chi_{[0,a]} \in \mathcal{B} \\ 0, & \text{elsewhere} \end{cases} \quad \text{and } \tau^*(f) = \begin{cases} 0, & \text{if } f = \underline{1} \text{ or } \underline{0} \\ 1-a, & \text{if } f = \chi_{[0,a]} \in \mathcal{B} \\ 1, & \text{elsewhere} \end{cases}$$

Then, (X, τ, τ^*) is a normal dfts.

Now, let $g = \chi_{[\frac{1}{2},1]}$ and (X, τ_0, τ_0^*) be the $(g, 1, 0)$-extension of (X, τ, τ^*). Then, consider $f = \chi_{(\frac{1}{2},1]}$. Clearly, $f\bar{q}g^c$ and for any $(r, s) \in I_0 \oplus I_1$ with $r \leq \frac{1}{2}, s \geq \frac{1}{2}$,

$f, g^c \in \mathcal{C}_{\tau_0,r,s}$. But, there does not exist two fuzzy sets $h_1, h_2 \in \mathcal{O}_{\tau_0,r,s}$ such that $f \leq h_1, g^c \leq h_2$ and $h_1 \bar{q} h_2$. Hence, (X, τ_0, τ_0^*) is not a normal dfts.

The concept of fuzzy topology on a fuzzy set was introduced by Chakraborty and Ahsanullah in [1]. Inspired from that we define double fuzzy topology on a fuzzy set as follows:

Definition 3.1. *Let $g \in I^X$. Define a pair of maps $\tau, \tau^* :\downarrow g \to I$ satisfying*

(i) $\tau(f) + \tau^*(f) \leq 1, \forall f \in\downarrow g$,
(ii) $\tau(\underline{0}) = \tau(g) = 1, \tau^*(\underline{0}) = \tau^*(g) = 0$,
(iii) $\tau(f_1 \wedge f_2) \geq \tau(f_1) \wedge \tau(f_2)$ *and* $\tau^*(f_1 \wedge f_2) \leq \tau^*(f_1) \vee \tau^*(f_2), f_i \in\downarrow g, i = 1, 2$,
(iv) $\tau(\bigvee_{i\in\Delta} f_i) \geq \bigwedge_{i\in\Delta} \tau(f_i)$ *and* $\tau^*(\bigvee_{i\in\Delta} f_i) \leq \bigvee_{i\in\Delta} \tau^*(f_i), f_i \in\downarrow g, i \in \Delta$

Then the pair (τ, τ^*) is called a double fuzzy topology on g. When $g = \underline{1}$, it becomes a double fuzzy topology on X.

Given a double fuzzy topology (τ, τ^*) on X and $g \in I^X$, there exists a double fuzzy topology on g induced by (τ, τ^*) as shown below:

Theorem 3.1. *Let (X, τ, τ^*) be a dfts and $g \in I^X$. Now, for $f \in\downarrow g$ define*

$$\tau|_g(f) = \vee\{\tau(h) : h \wedge g = f\} \text{ and } \tau^*|_g(f) = \wedge\{\tau^*(h) : h \wedge g = f\}.$$

Then, $\left(\tau|_g, \tau^*|_g\right)$ is a double fuzzy topology on g, called the double fuzzy subspace topology on g.

Proof. Straightforward. □

Lemma 3.1. *Let (X, τ, τ^*) be a dfts and (X, τ_0, τ_0^*) be the (g, α, β) extension of (X, τ, τ^*). Then, $\left(g, \tau|_g, \tau^*|_g\right) = \left(g, \tau_0|_g, \tau_0^*|_g\right)$*

Proof. Clearly, by definition of τ_0, $\tau_0|_g(f) \geq \tau|_g(f)$, $\forall f \leq g$. Suppose $\tau_0|_g(f) > \tau|_g(f)$ for some $f \leq g$.

$\implies \exists h \in I^X$ with $h \wedge g = f$ such that $\tau_0(h) > \tau|_g(f) \geq \tau(h)$

$\implies \exists h \in I^X$ with $h \wedge g = f$ such that $\bigvee \{\tau(f_1) \wedge \tau(f_2) \wedge \alpha : (f_1, f_2) \in R_g h\} > \tau|_g(f)$

$\implies \exists (f_1, f_2) \in R_g h$ such that $\tau(f_1) \wedge \tau(f_2) \wedge \alpha > \tau|_g(f)$

$\implies \tau(f_1) \wedge \tau(f_2) > \tau|_g(f)$

$$\implies \tau(f_1 \vee f_2) \geq \tau(f_1) \wedge \tau(f_2) > \tau|_g(f). \tag{3.1}$$

Again,

$$h \wedge g = f \Longrightarrow \big(f_1 \vee (f_2 \wedge g)\big) \wedge g = f$$
$$\Longrightarrow (f_1 \vee f_2) \wedge g = f.$$

Therefore, $\tau|_g(f) \geq \tau(f_1 \vee f_2)$, which is a contradiction to (3.1).

Hence, $\tau_0|_g(f) = \tau|_g(f),\ \forall f \in \downarrow g$. Similarly, $\tau_0^*|_g(f) = \tau^*|_g(f),\ \forall f \in \downarrow g$. □

Theorem 3.2. *Let (X, τ, τ^*) be a normal dfts and $g \in 2^X$ such that $g \in \mathcal{C}_{\tau,1,0}$. Then, the (g, α, β) extension (X, τ_0, τ_0^*) of (X, τ, τ^*) is normal if $\Big(g^c, \tau|_{g^c}, \tau^*|_{g^c}\Big)$ is normal.*

Proof. For $(r, s) \in I_0 \oplus I_1$ consider $f_1, f_2 \in \mathcal{C}_{\tau_0,r,s}$ such that $f_1 \bar{q} f_2$.

Case I: $r > \alpha$ and $s < \beta$.

We have $\bigvee\{\tau(h_1) \wedge \tau(h_2) \wedge \alpha : (h_1, h_2) \in R_g f\} \leq \alpha$ for any $f \in I^X$. Therefore, $\tau_0(f) \geq r \Rightarrow \tau(f) \geq r$. Similarly, $\tau_0^*(f) \leq s \Rightarrow \tau^*(f) \leq s$. Thus, if $r > \alpha$ and $s < \beta$ then, $\mathcal{O}_{\tau_0,r,s} = \mathcal{O}_{\tau,r,s}$ and $\mathcal{C}_{\tau_0,r,s} = \mathcal{C}_{\tau,r,s}$.

Then by the normality of (X, τ, τ^*), there exist $g_1, g_2 \in \mathcal{O}_{\tau,r,s}$ such that $f_1 \leq g_1, f_2 \leq g_2$ and $g_1 \bar{q} g_2$.

Case II: $r \leq \alpha$ or $s \geq \beta$.

We have,

$$f_1 \wedge g = [h_1 \vee (h_2 \wedge g)]^c \wedge g, \text{ for some } (h_1, h_2) \in R_g f_1 \text{ and } h_1, h_2 \in \mathcal{O}_{\tau,r,s}$$
$$= h_1^c \wedge (h_2^c \wedge g) \text{ since } g \in 2^X.$$

Therefore $f_1 \wedge g \in \mathcal{C}_{\tau,r,s}$ since $\mathcal{C}_{\tau,r,s}$ is a meet-semilattice.

Similarly $f_2 \wedge g \in \mathcal{C}_{\tau,r,s}$ and $(f_1 \wedge g)\bar{q}(f_2 \wedge g)$. Hence, there exist $m_1, n_1 \in \mathcal{O}_{\tau,r,s}$ such that $f_1 \wedge g \leq m_1$, $f_2 \wedge g \leq n_1$ and $m_1 \bar{q} n_1$.

Further, $f_1 \wedge g^c, f_2 \wedge g^c \in \mathcal{C}_{\tau_0|_{g^c},r,s} = \mathcal{C}_{\tau|_{g^c},r,s}$ and $(f_1 \wedge g^c)\bar{q}(f_2 \wedge g^c)$. Therefore there exist $m_2, n_2 \in \mathcal{O}_{\tau|_{g^c},r,s}$ such that $(f_1 \wedge g^c) \leq m_2$, $(f_2 \wedge g^c) \leq n_2$ and $m_2 \bar{q} n_2$ since $\Big(g^c, \tau|_{g^c}, \tau^*|_{g^c}\Big)$ is normal.

Also, $(f_1 \wedge g) \vee (f_1 \wedge g^c) = f_1$ and $(f_2 \wedge g) \vee (f_2 \wedge g^c) = f_2$. Hence, $f_1 \leq (m_1 \wedge g) \vee m_2 = g_1$, say and $f_2 \leq (n_1 \wedge g) \vee n_2 = g_2$, say with $g_1, g_2 \in \mathcal{O}_{\tau_0,r,s}$. Moreover,

$$g_1(x) + g_2(x) = \begin{cases} m_1(x) + n_1(x), & \text{if } g(x) = 1 \\ m_2(x) + n_2(x), & \text{if } g(x) \neq 1 \end{cases}$$

and therefore $g_1 \bar{q} g_2$. □

Theorem 3.3. *Let (X, τ, τ^*) be a normal dfts and g be a weak fuzzy subset of X satisfying (i) $g \in \mathcal{C}_{\tau,1,0}$ (ii) $\tau^+ \backslash \{\underline{0}\} \subseteq \uparrow g$ and (iii) $(f_1, f_2) \in \bar{q}[\underline{1}] \Longrightarrow (f_1, f_2) \in \bar{q}[g^c]$ for $f_1, f_2 \in [\tau^+(g)]^c \setminus \{\underline{1}\}$. Then the (g, α, β) extension (X, τ_0, τ_0^*) of (X, τ, τ^*) is normal if $(g^c, \tau|_{g^c}, \tau^*|_{g^c})$ is normal.*

Proof. Consider $f_1, f_2 \in \mathcal{C}_{\tau_0,r,s}$ such that $f_1 \bar{q} f_2$ for some for $(r, s) \in I_0 \oplus I_1$.

Case I: $r > \alpha$ and $s < \beta$.
Similar to Case I of Theorem 3.2.

Case II: $r \leq \alpha$ or $s \geq \beta$.

Claim 1: $f_1 \leq g^c, f_2 \leq g^c$.
If $f_1, f_2 \in \mathcal{C}_{\tau,r,s}$, then $\tau^+ \setminus \{\underline{0}\} \subseteq \uparrow g \Rightarrow f_1 \leq g^c, f_2 \leq g^c$.
Suppose $f_1, f_2 \in \mathcal{C}_{\tau_0,r,s} \setminus \mathcal{C}_{\tau,r,s}$. Then, we have $f_1 = [h_1 \vee (h_2 \wedge g)]^c$ for some $(h_1, h_2) \in R_g f_1^c$ and $h_1, h_2 \in \tau^+$. For, $f_1 \notin \mathcal{C}_{\tau,r,s} \Rightarrow \tau(f_1^c) < r$ or $\tau^*(f_1^c) > s$.
If $0 < \tau(f_1^c) < r$ then, $f_1^c, \underline{0} \in \tau^+$ and $(f_1^c, \underline{0}) \in R_g f_1^c$.
If $\tau(f_1^c) = 0$, then $\tau_0(f_1^c) \geq r \neq 0 \Rightarrow \bigvee\{\tau(h_1) \wedge \tau(h_2) \wedge \alpha : (h_1, h_2) \in R_g f_1^c\} \geq r$. Therefore, there exists $(h_1, h_2) \in R_g f_1^c$ such that $h_1, h_2 \in \tau^+$.
If $\tau(f_1^c) \geq r$ and $\tau^*(f_1^c) > s$ then, $f_1^c, \underline{0} \in \tau^+$ and $(f_1^c, \underline{0}) \in R_g f_1^c$.
Thus, $f_1 = [h_1 \vee (h_2 \wedge g)]^c = h_1^c \wedge (h_2^c \vee g^c) \leq h_1^c \leq g^c$ since $h_1 \in \tau^+$.
i.e., $f_1 \leq g^c$. Similarly, $f_2 \leq g^c$.
Again,
$$\begin{aligned}\tau_0|_{g^c}(f_1^c \wedge g^c) &= \vee\{\tau_0(h) : h \wedge g^c = f_1^c \wedge g^c, h \in I^X\} \\ &\geq \tau_0(f_1^c \wedge g^c) \text{ since } (f_1^c \wedge g^c) \wedge g^c = (f_1^c \wedge g^c) \\ &= \tau_0(f_1^c) \text{ since } \tau_0(g^c) = 1 \\ &\geq r\end{aligned}$$
and $\tau_0^*|_{g^c}(f_1^c \wedge g^c) \leq \tau_0^*(f_1^c) \leq s$. Therefore $f_1 \vee g \in \mathcal{C}_{\tau_0|_{g^c},r,s} = \mathcal{C}_{\tau|_{g^c},r,s}$, by Lemma 3.1. Similarly, $f_2 \vee g \in \mathcal{C}_{\tau|_{g^c},r,s}$.

Claim 2: $(f_1 \vee g)\bar{q}(f_2 \vee g)$.
Suppose $\exists x \in X$ such that $(f_1 \vee g)(x) + (f_2 \vee g)(x) > 1$.

<u>Case 1.</u> $(f_1 \vee g)(x) = g(x)$ and $(f_2 \vee g)(x) = f_2(x)$.
Then, $(f_1 \vee g)(x) + (f_2 \vee g)(x) > 1 \Rightarrow f_2(x) > g^c(x)$, which is a contradiction since $f_2 \leq g^c$.

<u>Case 2.</u> $(f_1 \vee g)(x) = f_1(x)$ and $(f_2 \vee g)(x) = g(x)$.
Similar to Case 1.

Case 3. $(f_1 \vee g)(x) = g(x)$ and $(f_2 \vee g)(x) = g(x)$.

Then, $(f_1 \vee g)(x) + (f_2 \vee g)(x) > 1 \Rightarrow g(x) > \frac{1}{2}$, which is a contradiction since g is a weak fuzzy subset.

Case 4. $(f_1 \vee g)(x) = f_1(x)$ and $(f_2 \vee g)(x) = f_2(x)$.

This contradicts the assumption that $f_1 \bar{q} f_2$.

Hence the claim.

Thus, $(f_1 \vee g)\bar{q}(f_2 \vee g)$ and by (iii), $(f_1 \vee g)\bar{q}(f_2 \vee g)[g^c]$.

Then, there exist $g_1, g_2 \in \mathcal{O}_{\tau|_{g^c},r,s}$ such that $(f_1 \vee g) \leq g_1, (f_2 \vee g) \leq g_2$ and $g_1 \bar{q} g_2[g^c]$. Also,

$$\tau|_{g^c}(g_1) \geq r \Rightarrow \vee\{\tau(h) : h \wedge g^c = g_1, h \in I^X\} \geq r. \tag{3.1}$$

Now since $\tau(g^c) = 1$, $\tau(g_1) \geq \tau(h) \wedge \tau(g^c) = \tau(h)$ for all $h \in I^X$ such that $g_1 = h \wedge g^c$. Hence, $\tau(g_1) \geq \vee\{\tau(h) : h \wedge g^c = g_1, h \in I^X\} \geq r$ by (3.1).

Therefore, $\tau_0(g_1) \geq \tau(g_1) \geq r$. Similarly, $\tau_0^*(g_1) \leq s$ which implies $g_1 \in \mathcal{O}_{\tau_0,r,s}$. Equivalently $g_2 \in \mathcal{O}_{\tau_0,r,s}$ and $g_1 \bar{q} g_2$. Hence, (X, τ_0, τ_0^*) is normal. □

4. Conclusion

It is shown that the extension of a normal dfts need not be normal. However, certain situations under which the normality of the space is carried over to its extensions are identified. But, the investigation in the converse direction remains unexplored.

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