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A NOTE ON COMPUTATION OF NUMBER OF FUZZY BITOPOLOGICAL SPACE

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ABSTRACT. In this paper, the number of fuzzy bitopological space with having 2-open sets, 3-open sets, 4-open sets and 5-open sets are computed. Moreover some results for finding number of fuzzy bitopological space are given.

1. INTRODUCTION

A fuzzy set is defined by Zadeh [1], is a function from a set X to $[0,1]$. Extending this definition to a more general set than the unit interval; as an example we have a complete lattice, many researchers developed various results. General topology was one of the first branches of pure mathematics to which fuzzy sets have been applied systemically. Starting from single topology it is extended to bitopology and tritopology etc. Krishnamurthy [2] computed sharper bound for number of distinct topology, namely, 2^{2^n}. Sharp [3] have shown a result which lead us to know that only discrete topology has cardinal $> 3/4\ 2^n$ and on the cardinality of connected, non-connected, non-T_0, connected and non-T_0 topologies some other bounds are derived. Stanley in [4], found all non-homeomorphic topologies with n-points and $> 7/16\ 2^n$ open set. Further, he determined which of these are T_0 and also computed $\tau(n,k)$, the number of topology on a finite set having n elements and k open set, for large values of k, namely $3.2^{n-3} < k < 2^n$

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and also T_0 topologies on a set, having $n+1 \leq k \leq n+3$ open sets. For computing the number of chain topological spaces, Kamel [5] formulated special case, and maximal elements with the natural generalization. Ragnarsson and Tenner in [6], studied the smaller possible number of points with having k-open sets in a topological space. In 2006, Benoumhani [7], also computed $\tau(n,k)$ for $2 \leq k \leq 12$ and number of T_0 topologies on X with having $n+4, n+5$ and $n+6$ open sets.

In 1968, Chang [8], introduced the concept of fuzzy topological spaces as an application of fuzzy sets to general topological spaces. Many researchers has developed various properties of topology and they extended general topology to fuzzy topology. Benoumhani and Kolli [9] also studied about fuzzy topologies and partitions. Jaballah and Saidi [10] studied about the Length of maximal chains and number of ideals in commutative rings. Recently, Benoumhani and Jaballah [11] also studied about fuzzy topological spaces and computed some results for finding number of fuzzy topology.

In 1963, Kelly [12], introduced the concept of bitopological space in his journal of London Mathematical Society. He initiated his study about bitopological space by the use of quasimetric and its conjugate. Reilly [13], Patty [14], and other researchers also worked and given their contribution to bitopological spaces. The concept of fuzzy bitopological spaces was introduced by, Kandil [15]. Recently Muchahary and Basumatary [16] has discussed on neutrosophic bitopological space.

Throughout this paper, X will denote a non-empty set of cardinality n and $\mathcal{M}$, a totally ordered set consisting of $m \geq 2$ elements. Also by $\mathcal{F}$, we will denote collection of all fuzzy subsets of X, with membership values in $\mathcal{M}$.

Here $\mathcal{F}$ is partially ordered by $\mu \leq \nu \iff \mu(x) \leq \nu(x)$for every $x \in X$. With the same partial order this set is a complete lattice. We also have $\mu < \nu \iff \mu \leq \nu$ and $\mu(x) < \nu(x)$, for some $x \in X$.

The fuzzy subsets $0_{\mathcal{F}}(x) = 0$ for every $x \in X$, and $1_{\mathcal{F}}(x) = 1$ for every $x \in X$. For every fuzzy subset μ different from $0_{\mathcal{F}}$ and $1_{\mathcal{F}}$, we have $0_{\mathcal{F}} < \mu < 1_{\mathcal{F}}$.

We will use symbols $\tau_{\mathcal{F}}(n,m,k), (\tau_i,\tau_j)_{\mathcal{F}}(n,m,k)$ to denote number of fuzzy topologies and number of fuzzy bitopological spaces respectively on X, with membership values in $\mathcal{M}$, having k open set .

2. PRELIMINARY

Definition 2.1. *Let X be a universal set. Then a function $A : X \to [0,1]$ define a fuzzy set on X , where A is called the membership function and $\mu_A(x)$ is called membership grade of x.*

We also write fuzzy set

$A = \{(x, \mu_A(x)) : x \in X\}$, Where each pair $(x, \mu_A(x))$ is called singleton.

Definition 2.2. *A fuzzy topology τ on a set X consists of a collection of fuzzy subsets of X called open sets, satisfying the following three axioms:*

(1) *The fuzzy subsets $0_{\mathcal{F}}$ and $1_{\mathcal{F}}$ are in τ.*
(2) *The union $\vee_{i\in I} u_i$ of any collection $\{u_i : i \in I\}$ of elements of τ is also in τ.*
(3) *The intersection $u_1 \cap u_2$ of any two elements u_1 and u_2 of τ is also in τ.*

The members of τ is called open set and the pair (X, τ) is called fuzzy topological space. The existence of a fuzzy topology τ in $\mathcal{F}$ with membership values in $\mathcal{M}$ implies necessarily that $0_{\mathcal{F}}$ and $1_{\mathcal{F}}$ are open sets in τ.

Example 1. *If $X = \{a, b\}$ then $0_{\mathcal{F}} = \{(a, 0), (b, 0)\}$, $1_{\mathcal{F}} = \{(a, 1), (b, 1)\}$, $F_1 = \{(a, 0.3), (b, 0.4)\}$, $F_2 = \{(a, 1), (b, 0)\}$, $F_3 = \{(a, 0.2), (b, 1)\}$, $F_4 = \{(a, 0.1), (b, 0.4)\}$, $F_5 = \{(a, 0.3), (b, 0)\}$, etc. are fuzzy subsets of X. Here $\tau_1 = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_1\}$, $\tau_2 = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_2\}, \tau_3 = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_1, F_4\}, ...$ are fuzzy topologies on X. We have trivially $\tau_{\mathcal{F}}(n, m, 2) = 1$ and $\tau_{\mathcal{F}}(n, m, 3) = m^{n-2}$.*

Proposition 2.1. *[11] The number of fuzzy topologies on X, whose membership values lies in $\mathcal{M}$ is finite if X and $\mathcal{M}$ are both finite and conversely.*

Theorem 2.1. *[11] The number $\tau_{\mathcal{F}}(n, m, 4)$ of fuzzy topologies in $\mathcal{F}$ having exactly 4 open sets is given by: $\tau_{\mathcal{F}}(n, m, 4) = (\frac{m(m+1}{2})^n - 3m^n + 2^{n-1} + 2$.*

Theorem 2.2. *[11] The number $\tau_{\mathcal{F}}(n, m, 5)$ of fuzzy topologies in $\mathcal{F}$ having exactly 5 open sets is given by: $\tau_{\mathcal{F}}(n, m, 5) = \binom{m+2}{3}^n - 4\binom{m+1}{2}^n + 5m^n - (m-1)^n + (2m-1)^n - 2^{n+1}$.*

Theorem 2.3. *[11] For $n \geq m \geq 2$, the number of fuzzy topology in $\mathcal{F}$ having k open set where $m^n - m^{n-2} < k < m^n$ and $k = m^n - m^{n-2}$ are*

(i) *$\tau_{\mathcal{F}}(n, m, k) = 0$ for $m^n - m^{n-2} < k < m^n$, amd*
(ii) *$\tau_{\mathcal{F}}(n, m, m^n - m^{n-2}) = n(n-1)$.*

Definition 2.3. *[15] A fuzzy bitopological space is a triple (X, τ_1, τ_2), where τ_1 and τ_2 are arbitrary fuzzy topologies on X.*

Example 2. *If $X = \{a, b, c\}$ then $0_{\mathcal{F}} = \{(a, 0), (b, 0), (c, 0)\}$, $1_{\mathcal{F}} = \{(a, 1), (b, 1), (c, 1)\}$, $F_1 = \{(a, 0), (b, 0.4), (c, 0)\}$, $F_2 = \{(a, 0), (b, 1), (c, 0)\}$, $F_3 = \{(a, 1), (b, 0.5), (c, 0.5)\}$, $F_4 = \{(a, 0.2), (b, 0.6), (c, 0)\}$, $F_5 = \{(a, 0.1), (b, 0.2), (c, 0.2)\}, \ldots$ etc. are fuzzy subsets of X. Here $\tau_1 = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_1\}, \tau_2 = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_2\}, \tau_3 = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_3\}$, $\tau_4 = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_1, F_3\}, \ldots$ are fuzzy topologies on X. Then (X, τ_1, τ_2), (X, τ_1, τ_3), (X, τ_2, τ_3) etc. are fuzzy bitopological spaces.*

3. Main Result

For any finite $n \geq 1$ and finite $m \geq 2$

Result 3.1. *The number of fuzzy bitopological space in $\mathcal{F}$ consisting of exactly two open set in both the fuzzy topology is $(\tau_i, \tau_j)_{\mathcal{F}}(n, m, 2) = 1$ as $\tau_{\mathcal{F}}(n, m, 2) = 1$.*

Result 3.2. *The number of fuzzy bitopological space $(\tau_i, \tau_j)_{\mathcal{F}}(n, m, 3)$ in $\mathcal{F}$ consisting of exactly three open set in both the fuzzy topology is $(\tau_i, \tau_j)_{\mathcal{F}}(n, m, 3) = \frac{(\tau_{\mathcal{F}}(n,m,3)\tau_{\mathcal{F}}(n,m,3)+1)}{2} = \frac{(m^{2n}-3m^n+2)}{2}$, where $\tau_{\mathcal{F}}(n, m, 3) = m^n - 2$.*

Result 3.3. *The number of fuzzy bitopological space $(\tau_i, \tau_j)_{\mathcal{F}}(n, m, 4)$ in $\mathcal{F}$ consisting of exactly four open set in both the fuzzy topology is $(\tau_i, \tau_j)_{\mathcal{F}}(n, m, 4) = \frac{(\tau_{\mathcal{F}}(n,m,4)(\tau_{\mathcal{F}}(n,m,4)+1))}{2}$, where $\tau_{\mathcal{F}}(n, m, 4) = (\frac{m(m+1)}{2})^n - 3m^n + 2^{n-1} + 2$.*

Example 3. *If $X = \{a, b\}$ and $\mathcal{M} = \{0, 0.3, 1\}$ then fuzzy subsets of X with membership values in $\mathcal{M}$ are $0_{\mathcal{F}} = \{(a, 0), (b, 0)\}$, $1_{\mathcal{F}} = \{(a, 1), (b, 1)\}$, $F_1 = \{(a, 0), (b, 0.3)\}$, $F_2 = \{(a, 0), (b, 1)\}$, $F_3 = \{(a, 1), (b, 0)\}$, $F_4 = \{(a, 1), (b, 0.3)\}$, $F_5 = \{(a, 0.3), (b, 0)\}$, $F_6 = \{(a, 0.3), (b, 0.3)\}$, $F_7 = \{(a, 0.3), (b, 1)\}$. Here $n = 2, m = 3$, so*

$$\begin{aligned}
\tau_{\mathcal{F}}(2, 3, 4) &= \left(\frac{3(3+1)}{2}\right)^2 - 3 \times 3^2 + 2^{2-1} + 2 \\
&= \left(\frac{3 \times 4}{2}\right)^2 - 27 + 2 + 2 \\
&= 6^2 - 23 \\
&= 36 - 23 \\
&= 13
\end{aligned}$$

These fuzzy topologies are $\tau_1 = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_1, F_2\}$, $\tau_2 = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_1, F_4\}$, $\tau_3 = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_1, F_6\}$, $\tau_4 = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_1, F_7\}$, $\tau_5 = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_2, F_3\}$, $\tau_6 = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_2, F_7\}$, $\tau_7 = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_3, F_4\}$, $\tau_8 = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_3, F_5\}$, $\tau_9 = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_4, F_5\}$, $\tau_{10} = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_4, F_6\}$, $\tau_{11} = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_5, F_6\}$, $\tau_{12} = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_5, F_7\}$, $\tau_{13} = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, F_6, F_7\}$. *So, the number of fuzzy bitopological space is*

$$\begin{aligned}(\tau_i, \tau_j)_{\mathcal{F}}(2,3,4) &= \frac{\tau_{\mathcal{F}}(2,3,4)(\tau_{\mathcal{F}}(2,3,4)+1)}{2} \\ &= \frac{13(13+1)}{2} = \frac{13 \times 14}{2} \\ &= 13 \times 7 = 91.\end{aligned}$$

Result 3.4. *The number of fuzzy bitopological space* $(\tau_i, \tau_j)_{\mathcal{F}}(n, m, 5)$ *in* $\mathcal{F}$ *consisting of exactly five open set in both the fuzzy topology is* $(\tau_i, \tau_j)_{\mathcal{F}}(n, m, 5) = \frac{\tau_{\mathcal{F}}(n,m,5)(\tau_{\mathcal{F}}(n,m,5)+1)}{2}$, *where* $\tau_{\mathcal{F}}(n, m, 5) = \binom{m+2}{3}^n - 4\binom{m+1}{2}^n + 5m^n - (m-1)^n + (2m-1)^n - 2^{n+1}$.

Example 4. *Let* $X = \{a, b\}$ *and* $\mathcal{M} = \{0, 0.6, 1\}$ *then fuzzy subsets of* X *with membership values in* $\mathcal{M}$ *are* $0_{\mathcal{F}} = \{(a,0),(b,0)\}$, $1_{\mathcal{F}} = \{(a,1),(b,1)\}, F_1 = \{(a,0),(b,1)\}$, $F_2 = \{(a,0),(b,0.6)\}$, $F_3 = \{(a,0.6),(b,1)\}$, $F_4 = \{(a,0.6),(b,0)\}$, $F_5 = \{(a,0.6),(b,0.6)\}, F_6 = \{(a,1),(b,0.6)\}$, $F_7 = \{(a,1),(b,0)\}$. *Here* $n = 2$, $m = 3$. *Therefore*

$$\begin{aligned}\tau_{\mathcal{F}}(2,3,5) &= \binom{3+2}{3}^2 - 4\binom{3+1}{2}^2 + 5 \times 3^2 - (3-1)^2 + (2 \times 3 - 1)^2 - 2^{2+1}. \\ &= \binom{5}{3}^2 - 4\binom{4}{2}^2 + 45 - 4 + 25 - 8 \\ &= \left(\frac{5!}{2!3!}\right)^2 - 4 \times \left(\frac{4!}{2!2!}\right)^2 + 70 - 12 \\ &= 100 - 144 + 58 = 14.\end{aligned}$$

So, the number of fuzzy bitopological space is

$$\begin{aligned}(\tau_i, \tau_j)_{\mathcal{F}}(2,3,5) &= \frac{(\tau_{\mathcal{F}}(2,3,5)(\tau_{\mathcal{F}}(2,3,5)+1))}{2} \\ &= \frac{14(14+1)}{2} = \frac{14 \times 15}{2} \\ &= 7 \times 15 = 105.\end{aligned}$$

Theorem 3.1. *The number of fuzzy bitopological space in $\mathcal{F}$ having k open set in both fuzzy topology*

(a) $(\tau_i, \tau_j)_{\mathcal{F}}(n, m, k) = 0$ *for* $m^n - m^{n-2} < k < m^n$ *and*
(b) $(\tau_i, \tau_j)_{\mathcal{F}}(n, m, k) = \frac{(\tau_F(n,m,k)(\tau_{\mathcal{F}}(n,m,k)+1)}{2} = \frac{n^4 - 2.n^3 + 2.n^2 - n}{2}$ *where* $k = m^n - m^{n-2}$ *and* $\tau_{\mathcal{F}}(n, m, k) = n(n-1)$.

4. CONCLUSION

If we can determine number of fuzzy topologies having k open set $\tau_{\mathcal{F}}(n, m, k)$ on a finite set then we can also determine number of fuzzy bitopological space $(\tau_i, \tau_j)_{\mathcal{F}}(n, m, k)$.

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