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# NEW FIXED POINT THEOREMS FOR SET VALUED MAP ON *G*-METRIC SPACES

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ABSTRACT. By this article, we get a common fixed point for the pair of set-valued maps on a G-complete G-metric spaces in new way. Further, we extend this technique and proved the existence of the coincidence points for a pair of set-valued and single-valued maps on such spaces.

## 1. INTRODUCTION AND PRELIMINARIES

In [1], Banach newly proved the existence of fixed point of self maps satisfying contraction principle on metric space. Afterwards, in [3] Nadler was focused his interest to study and established fixed point on multi-valued mappings and his effort, he proved such in Banach contraction principle version and subsequently many author were contributed their important to develop and extend the concept of Banach contraction principle in many ways.

In 2006, Z.Mustafa and B.Sims [2] introduced the new notion called $G-$metric space which is generalization of metric space. In this direction, several research articles related to fixed point theory on $G-$metric space have appeared.

In this article, we proved the existence of fixed point of a set-valued map defined on $G-$complete $G-$metric space satisfying some simple fractional condition only. Further we extend this concept to prove the existence of common fixed point for a pair of set-valued maps.

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Before proceed further, we need some definitions and notation from [2] in the sequel.

**Definition 1.1.** *Let $S$ be a non-empty set. Suppose that the mapping $G : S \times S \times S \to \mathbb{R}^+$ satisfies:*

(1) $G(l, m, n) = 0$ *if* $l = m = n$,
(2) $0 < G(l, l, m)$ *for all* $l, m \in S$ *with* $l \neq m$,
(3) $G(l, l, m) \leq G(l, m, n)$ *for all* $l, m, n \in S$ *with* $m \neq n$,
(4) $G(l, m, n) = G(l, n, m) = G(n, m, l) = \dots$ *(symmetry in all three variables),*
(5) $G(l, m, n) \leq G(l, c, c) + G(c, m, n)$ *for all* $l, m, n, c \in S$.

*Then $G$ is called a G-metric on $S$ and $(S, G)$ is called a G-metric space.*

To define the fixed point for a set valued map, we need the following notation.

Let $(S, G)$ be a G-metric space and $CB(S)$ denote the collection of non-empty closed bounded subsets of $S$. For $U, V \in CB(S)$

$$D_G(U, V, V) = \inf_{u \in U, v \in V} G(u, v, v)$$

and

$$H(U, V, V) = \max\{\sup_{u \in U} D_G(u, u, V), \sup_{v \in V} D_G(U, v, v)\}.$$

Note that $H(U, V, V) = H(V, U, U)$ and also for any $\epsilon > 0$ and for each $v \in V$, we can find $u \in U$ such that $G(u, v, v) \leq H(U, V, V) + \epsilon$.

Let $S$ be a non-empty set, $F : S \to CB(S)$ be any set valued map and $h : S \to S$ be any self map.

(1) If $l \in F(l)$, then $l$ is called fixed point of $F$.
(2) If $l = h(l) \in F(l)$, then $l$ is called a common fixed point of $F$ and $h$.
(3) If $h(l) \in F(l)$, then $l$ is called a coincidence point of $F$ and $h$.

## 2. Main Results

In this section, we present our main result that the existence of common fixed point for the pair of set valued maps on $G$-complete $G$-metric space which states that

**Theorem 2.1.** *Let $(S, G)$ be a G-complete G-metric space and $F, G : S \to CB(S)$ be two set valued mapping satisfying that*

$$\begin{aligned} H(G(m), F(n), F(n)) &\leq p\,[D_G(F(n), n, n) + D_G(G(m), n, n)] \\ &\quad +\ q\ D_G(F(n), G(m), G(m)), \end{aligned}$$

*for all $m, n \in S$, where $p$ and $q$ are non-negative real numbers with $3p + 2q < 1$, then $F$ and $G$ have a common fixed point.*

*Proof.* Let $l_0 \in S$ be an arbitrary point, let $l_1 \in S$ such that $l_1 \in F(l_0)$ and for any $\epsilon > 0$, then by the definition of $H$, there is a $l_2 \in G(l_0)$ so that

$$\begin{aligned} G(l_2, l_1, l_1) &\leq H(G(l_0), F(l_0), F(l_0)) + \epsilon \\ &\leq p\,[D_G(F(l_0), l_0, l_0) + D_G(G(l_0), l_0, l_0)] \\ &\quad + q\ D_G(F(l_0), G(l_0), G(l_0)) + \epsilon \\ &\leq p\,[G(l_1, l_0, l_0) + G(l_2, l_0, l_0)] +\ q\ G(l_1, l_2, l_2) + \epsilon \\ &\leq p\,[G(l_1, l_0, l_0) + G(l_2, l_1, l_1) + G(l_1, l_0, l_0)] \\ &\quad + 2q\ G(l_2, l_1, l_1) + \epsilon \\ G(l_2, l_1, l_1) &\leq \frac{2p}{1-p-2q} G(l_1, l_0, l_0) + \frac{1}{1-p-2q}\epsilon. \end{aligned}$$

Again we can find a $l_3 \in F(l_2)$ such that

$$\begin{aligned} G(l_3, l_2, l_2) &\leq H(F(l_2), G(l_0), G(l_0)) + \epsilon^2 \\ &= H(G(l_0), F(l_2), F(l_2)) + \epsilon^2 \\ &\leq p\,[D_G(F(l_2), l_2, l_2) + D_G(G(l_0), l_2, l_2)] \\ &\quad + q\ D_G(F(l_2), G(l_0), G(l_0)) + \epsilon^2 \\ &\leq p\,[G(l_3, l_2, l_2) + G(l_2, l_2, l_2)] +\ q\ G(l_3, l_2, l_2) + \epsilon^2 \\ &\leq (p+q)\ G(l_3, l_2, l_2) + \epsilon^2 \\ &\leq (p+2q)\ G(l_3, l_2, l_2) + 2p\ G(l_2, l_1, l_1) + \epsilon^2 \\ G(l_3, l_2, l_2) &\leq \frac{2p}{1-p-2q} G(l_2, l_1, l_1) + \frac{1}{1-p-2q}\epsilon^2. \end{aligned}$$

Take $t = \frac{2p}{1-p-2q}$ and proceeding like this we get a sequence $\{l_i\}_{i\geq 0}$ such that

$$G(l_{i+1}, l_i, l_i) \leq t\ G(l_i, l_{i-1}, l_{i-1}) + \frac{\epsilon^i}{1-p-2q},\ i = 1, 2, 3, \ldots.$$

After some simple calculation we have the inequalities

$$G(l_{i+1}, l_i, l_i) \;\; \leq \;\; t^i \, G(l_1, l_0, l_0) + \; \frac{1}{1-p-2q} \sum_{s=0}^{i-1} t^s \epsilon^{i-s},$$

and since $\epsilon$ was arbitrary so we choose $t \; < \epsilon \; < \; 1$ and for any $j > i$, we get

$$\begin{aligned} G(l_i, l_j, l_j) \;\; &\leq \;\; \sum_{r=0}^{j-i-1} G(l_{i+r}, l_{i+r+1}, l_{i+r+1}) \\ &< \;\; 2\frac{t^i}{1-t} \, G(l_0, l_1, l_1) \; + 2\frac{1}{(1-p-2q)} \frac{\epsilon^{i+1}}{(\epsilon - t)} \frac{1}{(1-\epsilon)}. \end{aligned}$$

Since $t < 1$ and $\epsilon < 1$, we get $G(l_i, l_j, l_j) \to 0$ as $i, j \to \infty$. That is, $\{l_i\}_{i \geq 0}$ is a $G$-Cauchy sequence and hence $l_i \to l \in S$. Next, consider

$$\begin{aligned} D_G(F(l), l, l) \;\; &\leq \;\; D_G(F(l), l_{2i+2}, l_{2i+2}) + \; G(l_{2i+2}, l, l) \\ &\leq \;\; H(F(l), G(l_{2i}), G(l_{2i})) + \; G(l_{2i+2}, l, l) \\ &= \;\; H(G(l_{2i}), F(l), F(l)) + \; G(l_{2i+2}, l, l) \\ &\leq \;\; p \, [D_G(F(l), l, l) + D_G(G(l_{2i}), l, l)] \\ &\qquad + q \; D_G(F(l), G(l_{2i}), G(l_{2i})) \\ &\leq \;\; p \, [D_G(F(l), l, l) + G(l_{2i+2}, l, l)] \\ &\qquad + q \; D_G(F(l), l_{2i+2}, l_{2i+2}). \end{aligned}$$

Making $i \to \infty$ and using the continuity of $G$ we have,

$$D_G(F(l), l, l) \;\; = \;\; 0$$

that is $l \in F(l)$. Similarly $l \in G(l)$. Thus $F$ and $G$ have a common fixed point. □

Next, we present the existence of the coincidence points for a pair of set-valued and single-valued maps on $G$-complete $G$-metric space.

**Theorem 2.2.** *Let $(S, G)$ be a G-complete G-metric space and $F, G : S \to CB(S)$ be two set valued mapping and $h, k : S \to S$ satisfying that*

**(A):** *$F(S) \subseteq k(S)$ and $G(S) \subseteq h(S)$ with $h(S)$ and $k(S)$ are both closed.*

**(B):** $H(G(m), F(n), F(n)) \leq p \, G(h(n), h(n), k(m))$
$$+q\frac{D_G(k(m),G(m),G(m))D_G(h(n),F(n),F(n))}{2[1+G(h(n),h(n),k(m))]}$$

*for all $m, n \in S$, where $p$ and $q$ are non-negative real numbers with $p + q < \frac{1}{2}$, then there are $a, b \in S$ such that $h(a) \in F(a), k(b) \in G(b), h(a) = k(b)$ and $F(a) = G(b)$.*

*Proof.* Let $l_0 \in S$ be an arbitrary point and let $\epsilon > 0$, since $F(l_0)$ is non-empty and by $(A)$, there is some $l_1 \in S$ such that $k(l_1) \in F(l_0)$ and again from $(A)$, we can find $l_2 \in S$ with $h(l_2) \in G(l_1)$ so that

$$G(h(l_2), k(l_1), k(l_1)) \leq H(G(l_1), F(l_0), F(l_0)) + \epsilon$$

in the same argument we have $l_3 \in S$ with $k(l_3) \in F(l_2)$ such that

$$G(k(l_3), h(l_2), h(l_2)) \leq H(F(l_2), G(l_1), G(l_1)) + \epsilon^2.$$

Continue like this, we generally get,

$$m_{2i-1} = k(l_{2i-1}) \in F(l_{2i-2}), m_{2i} = h(l_{2i}) \in G(l_{2i-1}), i = 1, 2, \ldots$$

such that

$$\begin{aligned} G(m_{2i}, m_{2i-1}, m_{2i-1}) &\leq H(G(l_{2i-1}), F(l_{2i-2}), F(l_{2i-2})) + \epsilon^{2i-1} \\ G(m_{2i+1}, m_{2i}, m_{2i}) &\leq H(F(l_{2i}), G(l_{2i-1}), G(l_{2i-1})) + \epsilon^{2i}. \end{aligned}$$

Now from $(B)$, we have

$$\begin{aligned} &G(m_{2i}, m_{2i-1}, m_{2i-1}) \\ &\qquad \leq H(G(l_{2i-1}), F(l_{2i-2}), F(l_{2i-2})) + \epsilon^{2i-1} \\ &\qquad \leq \epsilon^{2i-1} + p\, G(h(l_{2i-2}), h(l_{2i-2}), k(l_{2i-1})) \\ &\qquad\quad + \left\{\frac{q\, D_G(k(l_{2i-1}), G(l_{2i-1}), G(l_{2i-1}))}{2[1 + G(h(l_{2i-2}), h(l_{2i-2}), k(l_{2i-1}))]} \right. \\ &\qquad\qquad \left. \times\ D_G(h(l_{2i-2}), F(l_{2i-2}), F(l_{2i-2}))\right\} \\ &\qquad \leq \epsilon^{2i-1} + \ p\, G(m_{2i-2}, m_{2i-2}, m_{2i-1}) \\ &\qquad\quad + q\ \frac{G(m_{2i-1}, m_{2i}, m_{2i})G(m_{2i-2}, m_{2i-1}, m_{2i-1})}{2[1 + G(m_{2i-2}, m_{2i-2}, m_{2i-1})]} \\ &\qquad \leq p\, G(m_{2i-1}, m_{2i-2}, m_{2i-2}) \\ &\qquad\quad + 2qG(m_{2i}, m_{2i-1}, m_{2i-1}) + \epsilon^{2i-1} \\ &G(m_{2i}, m_{2i-1}, m_{2i-1}) \\ &\qquad \leq \frac{p}{1-2q}\ G(m_{2i-1}, m_{2i-2}, m_{2i-2}) + \frac{\epsilon^{2i-1}}{1-2q} \end{aligned}$$

and

$$\begin{aligned}
G(m_{2i+1}, m_{2i}, m_{2i}) &\leq H(F(l_{2i}), G(l_{2i-1}), G(l_{2i-1})) + \epsilon^{2i} \\
&= H(G(l_{2i-1}), F(l_{2i}), F(l_{2i})) + \epsilon^{2i} \\
&\leq \epsilon^{2i} + p\, G(h(l_{2i}), h(l_{2i}), k(l_{2i-1})) \\
&\quad + \{\frac{q\, D_G(k(l_{2i-1}), G(l_{2i-1}), G(l_{2i-1}))}{2[1 + G(h(l_{2i}), h(l_{2i}), k(l_{2i-1}))]} \\
&\qquad \times D_G(h(l_{2i}), F(l_{2i}), F(l_{2i}))\} \\
&\leq \epsilon^{2i} + \ p\, G(m_{2i}, m_{2i}, m_{2i-1}) \\
&\quad + q \frac{G(m_{2i-1}, m_{2i}, m_{2i}) G(m_{2i}, m_{2i+1}, m_{2i+1})}{2[1 + G(m_{2i}, m_{2i}, m_{2i-1})]} \\
&\leq \ 2p\, G(m_{2i}, m_{2i-1}, m_{2i-1}) \\
&\quad + 2q G(m_{2i+1}, m_{2i}, m_{2i}) + \epsilon^{2i}.
\end{aligned}$$

Hence

$$G(m_{2i+1}, m_{2i}, m_{2i}) \leq \frac{2p}{1-2q}\, G(m_{2i}, m_{2i-1}, m_{2i-1}) + \frac{\epsilon^{2i}}{1-2q}$$

in both cases we have

$$G(m_{i+1}, m_i, m_i) \leq \frac{2p}{1-2q}\, G(m_i, m_{i-1}, m_{i-1}) + \frac{\epsilon^{i}}{1-2q}, i = 2, 3, \dots.$$

By the usual argument as in the proof of above theorem, we may show that $\{m_i\}$ is a $G$-Cauchy sequence and hence, it converges to $m \in S$. Also from $(A)$ and $(2.1)$, there are $a, b \in S$ such that $h(a) = m = k(b)$. Then

$$\begin{aligned}
D_G(F(a), h(a), h(a)) &\leq D_G(F(a), m_{2i}, m_{2i}) + G(m_{2i}, h(a), h(a)) \\
&\leq H(F(a), G(l_{2i-1}), G(l_{2i-1})) + G(m_{2i}, h(a), h(a)) \\
&= H(G(l_{2i-1}), F(a), F(a)) + G(m_{2i}, h(a), h(a))
\end{aligned}$$

$$\begin{aligned}
&\leq G(m_{2i}, h(a), h(a)) + p\, G(h(a), h(a), k(l_{2i-1})) \\
&\quad + \{\frac{q\, D_G(h(a), h(a), F(a))}{[1 + G(h(a), h(a), k(l_{2i-1}))]} \\
&\qquad \times D_G(k(l_{2i-1}), G(l_{2i-1}), G(l_{2i-1}))\} \\
&\leq G(m_{2i}, h(a), h(a)) + p\, G(h(a), h(a), m_{2i-1}) \\
&\quad + q\, \frac{G(m_{2i-1}, m_{2i}, m_{2i}) D_G(h(a), h(a), F(a))}{[1 + G(h(a), h(a), m_{2i-1})]}
\end{aligned}$$

allowing $i \to \infty$ and using the continuity of $G$ we get,

$$D_G(F(a), h(a), h(a)) = 0,$$

that is $h(a) \in F(a)$. Similarly $k(b) \in G(b)$. The equality $F(a) = G(b)$ follows directly from $(B), h(a) \in F(a), k(b) \in G(b)$ and $h(a) = k(b)$. $\square$

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