ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.11, 9019–9029
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.11.6

ON THE UNIQUENESS THEOREMS OF L-FUNCTIONS CONCERNING WEIGHTED SHARING

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ABSTRACT. We mainly study the properties of L-functions using Nevanlinna value distribution theory in the extended selberg class. In this paper, we investigate the relationship between meromorphic functions and L-functions concerning weighted sharing with the help of Nevanlinna value distribution theory. We prove a uniqueness theorem of a meromorphic function and an L-function when they share $(0,0)$ and $(1,1)$. We also get valuable information about the counting of the zeros of L-functions. The results of this paper improve some recent results of W. J. Hao and J. F. Chen [1].

1. INTRODUCTION

L-functions play very important role in the modern number theory. One common thing is that all the L-functions can be described by an Euler product. So all the L-functions can be described as a product taken over prime numbers. Considering unique prime factorization of integers we can represent L-functions as Dirichlet series. We may regard the famous Riemann zeta-function, $\zeta(z) = \sum_{n=1}^{\infty} 1/n^z = \prod_p (1 - 1/p^z)^{-1}$ where $z = \sigma + it$, $\sigma > 1$ and p denotes prime number and the product is taken over all prime numbers, as the prototype. We can get valuable information on the algebraic structure from the value distributions of the L-functions which is not obtainable by the elementary algebraic method. In particular, the distribution of zeros of L-functions is of special

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2020 *Mathematics Subject Classification.* 11M36, 30D35.
Key words and phrases. Meromorphic functions, L-functions, Weighted sharing, Uniqueness.

interest with respect to many problems in multiplicative number theory. One such example is the Riemann hypothesis on the non-vanishing of the Riemann zeta-function in the right half of the critical strip and its impact on the distribution of prime numbers. Riemann hypothesis remains unsolved for more than 150 years though it is among the most famous conjectures of all time.

An L-function is defined by the Dirichlet series $L(z) = \sum_{n=1}^{\infty} a(n)/n^z$ satisfying the assumptions (i) Ramanujan conjecture: For every $\epsilon > 0$, $a(n) \ll n^{\epsilon}$, (ii) Analytic continuation: There exists a nonnegative integer k such that $(z-1)^k L(z)$ is an entire function of finite order, (iii) Every L-function satisfies the functional equation $\lambda_L(z) = \omega\overline{\lambda_L(1-\overline{z})}$, where $\lambda_L(z) = L(z)Q^z \prod_{i=1}^{k} \Gamma(\lambda_i z + \nu_i)$ with positive real numbers Q, λ_i and complex numbers ν_i, ω with $\mathrm{Re}\nu_i \geq 0$ and $\mid \omega \mid = 1$ and (iv) Euler product: $L(z)$ satisfies $L(z) = \prod_p L_p(z)$, where $L_p(z) = \exp(\sum_{k=1}^{\infty} b(p^k)/p^{kz})$ with coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$ and p denotes prime number.

In this paper, we study the uniqueness problems of L-functions and meromorphic functions using Nevanlinna's value distribution theory. Throughout the paper an L-function L means an L-function L with $a(1) = 1$ in the extended Selberg class. Here we use the standard definitions and notations of the value distribution theory [2].

2. Preliminaries

Let ξ and ψ be two nonconstant meromorphic functions in the open complex plane $\mathbb{C}$. We denote by $S(r, \xi)$ any function satisfying $S(r, \xi) = o(T(r, \xi))$ as $r \longrightarrow \infty$, outside a possible exceptional set of finite linear measure. If $\xi - z_0$ and $\psi - z_0$ have the same set of zeros with the same multiplicities, we say that ξ and ψ share z_0 CM (counting multiplicities) and we say that ξ and ψ share z_0 IM (ignoring multiplicities) if we do not consider the multiplicities where $z_0 \in \mathbb{C} \cup \{\infty\}$.

The following gives an account of relevant theorems or definitions for the paper.

Definition 2.1. *([8], Definition 1.3, 1.4) Let ξ be a meromorphic function defined in the complex plane. Let n be a positive integer and $\alpha \in \mathbb{C} \cup \{\infty\}$. By $N(r, \alpha; \xi \mid\leq n)$ we denote the counting function of the α points of ξ with multiplicity $\leq n$ and by*

$\overline{N}(r, \alpha; \xi \mid\leq n)$ *the corresponding one for which we do not count the multiplicity. Also by* $N(r, \alpha; \xi \mid\geq n)$ *we denote the counting function of the* α *points of* ξ *with multiplicity* $\geq n$ *and by* $\overline{N}(r, \alpha; \xi \mid\geq n)$ *the corresponding one for which we do not count the multiplicity. We define*

$$N_n(r, \alpha; \xi) = \overline{N}(r, \alpha; \xi) + \overline{N}(r, \alpha; \xi \mid\geq 2) + \cdots + \overline{N}(r, \alpha; \xi \mid\geq n).$$

Considering CM sharing in 2007 Steuding [12] proved the following uniqueness theorem of L-functions.

Theorem 2.1. *([12], Theorem 7.11) Let* L *and* G *be two L-functions with* $a(1) = 1$ *and* $\alpha \neq \infty$ *be a complex number. If* L *and* G *share* α *CM, then* $L \equiv G$.

Remark 2.1. *[3] In 2016 Hu and Li taking* $L = 1 + 2/4^s$ *and* $G = 1 + 3/9^s$ *proved that Theorem 2.1 is not true for* $\alpha = 1$.

In 2010 Li [9] study the uniqueness problems of meromorphic functions and L-functions and proved the following theorem.

Theorem 2.2. *([9], Theorem 1) Let* F *be a nonconstant meromorphic function having finitely many poles and* L *be a nonconstant L-function. If* F *and* L *share* α *CM and* β *IM then* $L \equiv F$, *where* α *and* β *are two distinct finite values.*

Definition 2.2. *([4], Definition 6, 7, [5], Definition 5) Let* ξ *and* ψ *be two meromorphic functions defined in the complex plane and* n *be an integer* (≥ 0) *or infinity. For* $\alpha \in \mathbb{C} \cup \{\infty\}$ *we denote by* $E_n(\alpha; \xi)$ *the set of all zeros of* $\xi - \alpha$ *where a zero of multiplicity* k *is counted* k *times if* $k \leq n$ *and* $n + 1$ *times if* $k > n$. *If* $E_n(\alpha; \xi) = E_n(\alpha; \psi)$, *we say that* ξ, ψ *share the value* α *with weight* n.

We write ξ, ψ share (α, n) to mean that ξ, ψ share the value α with weight n. Clearly if ξ , ψ share (α, n) then ξ, ψ share (α, m) for all integers $m, 0 \leq m < n$. Also we note that ξ, ψ share a value α IM or CM if and only if ξ, ψ share $(\alpha, 0)$ or (α, ∞) respectively.

In 2015, Wu and Hu [13] cosidering weighted sharing proved the following uniqueness theorem of L-functions.

Theorem 2.3. *([13], Theorem 1.5) Let* L *and* G *be two L-functions, and let* α, $\beta \in \mathbb{C}$ *be two distinct values. Take two positive integers* n_1, n_2 *with* $n_1 n_2 > 1$. *If* $E_{n_1}(\alpha, L) = E_{n_1}(\alpha, G)$, *and* $E_{n_2}(\beta, L) = E_{n_2}(\beta, G)$, *then* $L \equiv G$.

In 2017, Liu, Li and Yi [10] proved the following uniqueness theorem of L-functions.

Theorem 2.4. *([10], Theorem 1.1) Let $j \geq 1$ and $k \geq 1$ be integers such that $j > 3k + 6$. Also let L be an L-function and F be a nonconstant meromorphic function. If $\{F^j\}^{(k)}$ and $\{L^j\}^{(k)}$ share 1 CM then $F \equiv dL$ for some constant d satisfying $d^j = 1$.*

Considering weighted sharing in 2018 Hao and Chen [1] proved the following theorem.

Theorem 2.5. *([1], Theorem 1.7) Let L be an L-function and F be a meromorphic function defined in the complex plane $\mathbb{C}$ with finitely many poles. Let α_1, $\alpha_2 \in \mathbb{C}$ be distinct and m_1,m_2 be positive integers such that $m_1 m_2 > 1$. If $E_{m_j}(\alpha_j, F) = E_{m_j}(\alpha_j, L)$, $j = 1, 2$, then $L \equiv F$.*

Now the following question comes naturally.

Question 2.1. *Can we reduce the weight of the sharing of values in the Theorem 2.5 ?*

Definition 2.3. *([4], Definition 4). Let two nonconstant meromorphic functions ξ and ψ share a value α IM . We denote by $\overline{N}_*(r, \alpha; \xi, \psi)$ the counting function of those α-points of ξ whose multiplicities are not equal to the multiplicities of the corresponding α-points of ψ, where each α-point is counted only once.*

Definition 2.4. *([7], Definition 1.4) We denote by $N_0(r, 0; \xi^{(k)})$ $(\overline{N}_0(r, 0; \xi^{(k)}))$ the counting function (reduced counting function) of those zeros of $\xi^{(k)}$ which are not the zeros of the nonconstant meromorphic function ξ.*

Definition 2.5. *([7], Definition 1.5) We denote by $N_\otimes(r, 0; \xi^{(k)})$ $(\overline{N}_\otimes(r, 0; \xi^{(k)}))$ the counting function (reduced counting function) of those zeros of $\xi^{(k)}$ which are not the zeros of $\xi(\xi - 1)$.*

Definition 2.6. *([7], Definition 1.6) We denote by $N_\oplus(r, 0; \xi^{(k)})$ $(\overline{N}_\oplus(r, 0; \xi^{(k)}))$ the counting function (reduced counting function) of those zeros of $\xi^{(k)}$ which are not the zeros of $\xi - 1$.*

Throughout the paper we mean by ξ, ψ two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$.

3. Main Results

Using weighted sharing we try to solve Question 2.1 and prove the following theorem.

Theorem 3.1. *Let f be a nonconstant meromorphic function and L be a non-constant L-function. If $E_0(0, f) = E_0(0, L)$, $E_1(1, f) = E_1(1, L)$ and $\overline{N}(r, 0; f) + \overline{N}(r, \infty; f) = S(r, f)$ then either $L \equiv f$ or $T(r, L) = N(r, 0; L \mid\leq 2) + S(r, L)$ and $T(r, f) = N(r, 0; L' \mid\leq 1) + S(r, L)$.*

Remark 3.1. *In Theorem 2.5 the weights of the sharing are m_1, m_2 such that $m_1 m_2 > 1$. So in Theorem 3.1 the weights of the sharing reduced to 0 and 1.*

3.1. Lemmas.

In this subsection we present some necessary lemmas.

Henceforth we denote by Ψ the function defined by

$$\Psi = \Psi_{\xi,\psi} = \left(\frac{\xi''}{\xi'} - \frac{2\xi'}{\xi - 1}\right) - \left(\frac{\psi''}{\psi'} - \frac{2\psi'}{\psi - 1}\right).$$

Lemma 3.1. *([5], Lemma 1) If ξ and ψ share $(1, 1)$ and $\Psi \not\equiv 0$ then*

(i) $N(r, 1; \xi \mid\leq 1) \leq N(r, \infty; \Psi) + S(r, \xi) + S(r, \psi)$,

(ii) $N(r, 1; \psi \mid\leq 1) \leq N(r, \infty; \Psi) + S(r, \xi) + S(r, \psi)$.

Lemma 3.2. *([5], Lemma 3) Let ξ and ψ share $(1, 0)$ and $\Psi \not\equiv 0$. Then*

$$\begin{aligned} N(r, \Psi) \leq\ & \overline{N}(r, \infty; \xi \mid\geq 2) + \overline{N}(r.0; \xi \mid\geq 2) + \overline{N}(r, \infty; \psi \mid\geq 2) + \overline{N}(r, 0; \psi \mid\geq 2) \\ +\ & \overline{N}_*(r, 1; \xi, \psi) + \overline{N}_\otimes(r, 0; \xi') + \overline{N}_\otimes(r, 0; \psi'). \end{aligned}$$

Lemma 3.3. *([6], Lemma). If k is a positive integer then*

$$N_0(r, 0; \xi^{(k)}) \leq k\overline{N}(r, \infty; \xi) + N(r, 0; \xi \mid< k) + k\overline{N}(r, 0; \xi \mid\geq k) + S(r, \xi).$$

Lemma 3.4. *([7], Lemma 2.4) If ξ and ψ share $(1, 1)$ then*

$$\begin{aligned} &\overline{N}_0(r, 0; \psi') + \overline{N}(r, 1; \psi \mid\geq 2) + \overline{N}_*(r, 1; \xi, \psi) \\ \leq& 3\overline{N}(r, 0; \psi) + 3\overline{N}(r, \infty; \psi) + S(r, \psi). \end{aligned}$$

Lemma 3.5. *([12], Theorem 7.9) Let L be an L-function with degree d. Then*

$$T(r, L) = \frac{d}{\pi} r \log r + O(r).$$

Lemma 3.6. *([11], Lemma 4.6) Let L be an L-function. Then $N(r, \infty; L) = S(r, L) = O(\log r)$.*

Lemma 3.7. *Let f be a nonconstant meromorphic function and L be an L-function. If f and L share $(0,0)$ and $(1,1)$ such that $\overline{N}(r,0;f)+\overline{N}(r,\infty;f)=S(r,f)$, then $S(r,f)=S(r,L)=O(\log r)$.*

Proof. Since f and L share $(1,1)$ therefore we have by the second fundamental theorem

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,\infty;f)+S(r,f) \\ &= \overline{N}(r,1;L)+S(r,f) \\ &\leq T(r,L)+S(r,f). \end{aligned}$$

This shows that every $S(r,f)$ is replaceable by $S(r,L)$. Since f and L share $(0,0)$ and $(1,1)$ and every $S(r,f)$ is replaceable by $S(r,L)$, therefore we have by the second fundamental theorem

$$\begin{aligned} T(r,L) &\leq \overline{N}(r,0;L)+\overline{N}(r,1;L)+\overline{N}(r,\infty;L)+S(r,L) \\ &= \overline{N}(r,0;f)+\overline{N}(r,1;f)+S(r,L) \\ &= T(r,f)+S(r,L). \end{aligned}$$

This shows that every $S(r,L)$ is replaceable by $S(r,f)$. Hence by Lemma 3.6 we have $S(r,f)=S(r,L)=O(\log r)$. This completes the proof. □

3.2. **Proof of the main result.** Here we give the proof of the Theorem 3.1.

Proof. Let $\Phi=\Psi_{L,f}$. We have to consider the following two cases.

Case I. Let $\Phi\equiv 0$. Integrating we have

$$L-1\equiv\frac{f-1}{P-Q(f-1)}, \tag{3.1}$$

where $P(\neq 0)$ and Q are constants.

If $Q=0$ then from (3.1) we get

$$L-1\equiv d(f-1), \tag{3.2}$$

where $d=\frac{1}{P}$ is a nonzero constant.

Since L and f share $(0,0)$, therefore from (3.2) we have $d=1$. Hence from (3.2) we have $L\equiv f$.

Let $Q \neq 0$. If $P+Q \neq 0$ then from (3.1) and lemma 3.7 we get by the second fundamental theorem

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,\frac{P+Q}{Q};f)+S(r,f) \\ &= \overline{N}(r,\infty;L)+S(r,f) \\ &\leq S(r,L)+S(r,f) \\ &= S(r,f), \end{aligned}$$

which is a contradiction. Therefore $P+Q=0$ and so from (3.1) we get

$$(L-\frac{Q-1}{Q})f \equiv \frac{1}{Q}.$$

If we put $c=\frac{Q-1}{Q}$ then $c \neq 1$ and from above we get $(L-c)f \equiv 1-c$, which contradicts that f and L share $(0,0)$.

Case II. Let $\Phi \not\equiv 0$.

Let $\Omega = \frac{L-1}{f-1}$. Since f, g share $(1,1)$ we get by Lemma 3.4, Lemma 3.6 and Lemma 3.7

$$\begin{aligned} \overline{N}(r,0;\Omega) &\leq \overline{N}_*(r,1;L,f)+\overline{N}(r,\infty;f) \\ &\leq 3\overline{N}(r,0;f)+4\overline{N}(r,\infty;f)+S(r,f) \\ &= S(r,f) \\ &= S(r,L). \end{aligned}$$

and

$$\begin{aligned} \overline{N}(r,\infty;\Omega) &\leq \overline{N}_*(r,1;L,f)+\overline{N}(r,\infty;L) \\ &\leq 3\overline{N}(r,0;f)+3\overline{N}(r,\infty;f)+\overline{N}(r,\infty;L)+S(r,f) \\ &= \overline{N}(r,\infty;L)+S(r,f) \\ &= \overline{N}(r,\infty;L)+S(r,L) \\ &= S(r,L). \end{aligned}$$

Since $L' = \Omega(f-1)(\frac{\Omega'}{\Omega}+\frac{f'}{f-1})$, we see that possible zeros of L' occur from the following sources:

(i) zeros of Ω, (ii) zeros of $f-1$, and (iii) zeros of $\frac{\Omega'}{\Omega}+\frac{f'}{f-1}$.

Let z_1 be a simple zero of $f-1$. Since L and f share $(1,1)$, z_1 is neither a zero nor a pole of Ω. On the other hand z_1 is a simple pole of $\frac{\Omega'}{\Omega}+\frac{f'}{f-1}$. Hence z_1 is not a zero of L'.

Therefore by Lemma 3.4, Lemma 3.6 and Lemma 3.7 we get

$$
\begin{aligned}
\overline{N}(r,0;L') &\leq \overline{N}(r,0;\Omega)+\overline{N}(r,1;f\mid\geq 2)+T(r,\frac{\Omega'}{\Omega}+\frac{f'}{f-1})\\
&\leq 3\overline{N}(r,0;f)+3\overline{N}(r,\infty;f)+N(r,\frac{\Omega'}{\Omega})+N(r,\frac{f'}{f-1})+S(r,L)\\
&\leq \overline{N}(r,0;\Omega)+\overline{N}(r,\infty;\Omega)+\overline{N}(r,1;f)+\overline{N}(r,\infty;f)+S(r,L)\\
&\leq N(r,1;f\mid\leq 1)+\overline{N}(r,1;f\mid\geq 2)+S(r,L)\\
&\leq N(r,1;f\mid\leq 1)+3\overline{N}(r,0;f)+3\overline{N}(r,\infty;f)+S(r,L)\\
&= N(r,1;f\mid\leq 1)+S(r,L). \qquad (3.3)
\end{aligned}
$$

Again since L and f share $(1,1)$, by Lemma 3.1, Lemma 3.2, Lemma 3.3 and lemma 3.6 we get

$$
\begin{aligned}
N(r,1;f\mid\leq 1) &\leq \overline{N}(r,0;L\mid\geq 2)+\overline{N}(r,0;f\mid\geq 2)+\overline{N}(r,1;f\mid\geq 2)\\
&+ \overline{N}_{\otimes}(r,0;L')+\overline{N}_{\otimes}(r,0;f')+\overline{N}(r,\infty;L\mid\geq 2)\\
&+ \overline{N}(r,\infty;f\mid\geq 2)\\
&\leq \overline{N}(r,0;L\mid\geq 2)+\overline{N}(r,1;L\mid\geq 2)+\overline{N}_{\otimes}(r,0;L')\\
&+ N_0(r,0;f')+S(r,f)\\
&\leq \overline{N}(r,0;L')+S(r,L). \qquad (3.4)
\end{aligned}
$$

By the second fundamental theorem and Lemma 3.3 we get

$$
\begin{aligned}
T(r,f) &\leq \overline{N}(r,1;f)+\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f)\\
&\leq N(r,1;f\mid\leq 1)+N_0(r,0;f')+S(r,f)\\
&\leq N(r,1;f\mid\leq 1)+\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f)\\
&= N(r,1;f\mid\leq 1)+S(r,f).
\end{aligned}
$$

so that

$$N(r,1;f\mid\leq 1)=T(r,f)+S(r,f)=T(r,f)+S(r,L). \qquad (3.5)$$

Since L and f share $(1,1)$ by Lemma 3.3 and lemma 3.7 we get

$$\begin{aligned}\overline{N}(r,1;L\mid\geq 2) &= \overline{N}(r,1;f\mid\geq 2)\\ &\leq N_0(r,0;f')\\ &\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f)\\ &= S(r,f)\\ &= S(r,L).\end{aligned}$$

Now by lemma 3.6, lemma 3.7, (3.4) and the second fundamental theorem we get

$$\begin{aligned}T(r,L) &\leq \overline{N}(r,\infty;L)+N(r,0;L)+\overline{N}(r,1;L)-N_{\oplus}(r,0;L')+S(r,L)\\ &= N(r,0;L)+N(r,1;f\mid\leq 1)-N_{\oplus}(r,0;L')+S(r,L)\\ &\leq N(r,0;L)+\overline{N}(r,0;L')-N_{\oplus}(r,0;L')+S(r,L)\\ &= N(r,0;L)+\overline{N}_{\oplus}(r,0;L')-N_{\oplus}(r,0;L')+\overline{N}(r,1;L\mid\geq 2)+S(r,L)\\ &= N(r,0;L)+\overline{N}_{\oplus}(r,0;L')-N_{\oplus}(r,0;L')+S(r,L)\\ &\leq N(r,0;L)+S(r,L)\\ &\leq T(r,L)+S(r,L).\end{aligned}$$

Hence

$$T(r,L)=N(r,0;L)+S(r,L) \tag{3.6}$$

and

$$N_{\oplus}(r,0;L')-\overline{N}_{\oplus}(r,0;L')=S(r,L). \tag{3.7}$$

From (3.7) we get

$$N(r,0;L\mid\geq 3)\leq 3\{N_{\oplus}(r,0;L')-\overline{N}_{\oplus}(r,0;L')\}=S(r,L).$$

Hence from (3.6) we get

$$T(r,L)=N(r,0;L\mid\leq 2)+S(r,L).$$

Again from (3.7) we get by Lemma 3.4

$$\begin{aligned}\overline{N}(r,0;L'\mid\geq 2) &\leq \overline{N}(r,1;L\mid\geq 3)+2\{N_{\oplus}(r,0;L')-\overline{N}_{\oplus}(r,0;L')\}\\ &\leq \overline{N}(r,1;L\mid\geq 2)+S(r,L)\\ &= S(r,L).\end{aligned}$$

So, from (3.3), (3.4) and (3.5) we obtain

$$N(r,0;L' \mid\leq 1) \leq T(r,f) + S(r,L). \tag{3.8}$$

and

$$T(r,f) \leq N(r,0;L' \mid\leq 1) + S(r,L). \tag{3.9}$$

From (3.8) and (3.9) we have

$$T(r,f) = N(r,0;L' \mid\leq 1) + S(r,L).$$

This proves the theorem. □

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