IRREGULAR COLORING OF SOME SPECIAL GRAPHS

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ABSTRACT. For a graph $G$ and a proper coloring $c : V(G) \to \{1, 2, 3, \ldots, k\}$ of the vertices of $G$ for some positive integer $k$, the color code of a vertex $v$ of $G$ (with respect to $c$) is the ordered $(k+1)$-tuple $\text{code}(v) = (a_0, a_1, a_2, \ldots, a_k)$ where $a_0$ is the color assigned to $v$ and $1 \leq i \leq k$, $a_i$ is the number of vertices of $G$ adjacent to $v$ that are colored $i$. The coloring $c$ is irregular if distinct vertices have distinct color codes and the irregular chromatic number $\chi_{ir}(G)$ of $G$ is the minimum positive integer $k$ for which $G$ has an irregular $k$-coloring. In this paper, we obtain the values of irregular coloring for $SF(n,1)$, friendship graph and splitting graph of star graph.

1. INTRODUCTION

Let $G(V, E)$ be simple connected graph. A proper coloring of a graph $G$ is a function $c : V(G) \to N$ having the property that $c(u) \neq c(v)$ for every pair $u, v$ of adjacent vertices of $G$. A $k$-coloring of $G$ uses $k$ colors. The chromatic number $\chi(G)$ of $G$ is the minimum integer $k$ for which $G$ admits a $k$-coloring. In a graph $G$, a proper coloring $c : V(G) \to \{1, 2, 3, \ldots, k\}$ of the vertices of $G$ for some positive integer $k$, the color code of a vertex $v$ of $G$ (with respect to $c$) is the ordered $(k+1)$-tuple $\text{code}(v) = (a_0, a_1, a_2, \ldots, a_k)$, where $a_0$ is the color assigned to $v$ and $1 \leq i \leq k$, $a_i$ is the number of vertices of $G$ adjacent to $v$ that are colored $i$. The coloring $c$ is irregular if distinct vertices have distinct color codes and the
irregular chromatic number $\chi_{ir}(G)$ of $G$ is the minimum positive integer $k$ for which $G$ has an irregular $k$-coloring. Irregular coloring were introduced in [4] and studied further in [5] inspired by the problem in graph theory concerns finding means to distinguish all the vertices of a connected graph. Further some more results of irregular coloring of graphs are discussed in [1, 2, 6]. For graph theoretic terminology we refer to Harary [3]. In this paper, we find that the irregular coloring of $SF(n, 1)$ graph, friendship graph and splitting graph of star graph.

2. MAIN RESULTS

**Definition 2.1.** An $SF(n, m)$ is a graph consisting of a cycle $C_n$, $n \geq 3$ and $n$ set of $m$ independent vertices where each set joins each of the vertices of $C_n$.

**Theorem 2.1.** Let $G = SF(n, 1)$, where $n \geq 3$. Then $2 \binom{k-1}{2} + 1 \leq n \leq 2 \binom{k}{2}$ if and only if $\chi_{ir}(G) = k$.

**Proof.** Let $V(G) = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{u_iu_i; 1 \leq i \leq n\} \cup \{u_iu_{i+1}; 1 \leq i \leq n-1\} \cup u_nu_1$. Assume that $\chi_{ir}(G) = k$. We have to prove that $2 \binom{k-1}{2} + 1 \leq n \leq 2 \binom{k}{2}$. Assume to the contrary that $n \geq 2 \binom{k}{2} + 1$ or $n \leq 2 \binom{k-1}{2}$.

**Case (i):** $n \geq 2 \binom{k}{2} + 1$

Let $A_1, A_2, \ldots, A_{\binom{k}{2}}, A_1', A_2', \ldots, A_{\binom{k}{2}}'$ be the $2 \binom{k}{2}$ distinct 2 element subsets of the set $\{1, 2, \ldots, k\}$, where $A_l = (i, j)$ and $A_l' = (j, i)$, $1 \leq i, j \leq k; 1 \leq l \leq \binom{k}{2}$

and by our assumption $n \geq 2 \binom{k}{2} + 1$, it follows that there exists two vertices $u_i, v_j \in V(G)$ such that $\text{code}(u_i) \neq \text{code}(v_j)$, which is a contradiction. Hence $n \leq 2 \binom{k}{2}$.

**Case (ii):** $n \leq 2 \binom{k-1}{2}$

Let $A_1, A_2, \ldots, A_{\binom{k}{2}}, A_1', A_2', \ldots, A_{\binom{k-1}{2}}'$ be the $2 \binom{k-1}{2}$ distinct 2 element subsets of the set $\{1, 2, 3, \ldots, k - 1\}$. We can define a coloring $c$ of $G$ by assigning the 2 distinct colors in $A_l$ and $A_l'$ to the $n$ vertices of $V(G)$, where $1 \leq l \leq \binom{k-1}{2}$. Since $n \leq 2 \binom{k-1}{2}$. Hence $c$ is an irregular coloring with at most $k - 1$ colors. Thus $\chi_{ir}(G) \leq k - 1$, this is a contradiction to our assumption. Hence $n \leq 2 \binom{k-1}{2} + 1$.

From the above two cases, we get $2 \binom{k-1}{2} + 1 \leq n \leq 2 \binom{k}{2}$ and to prove $\chi_{ir}(G) = k$.

Conversely, assume that $2 \binom{k-1}{2} + 1 \leq n \leq 2 \binom{k}{2}$ and to prove $\chi_{ir}(G) = k$. 
Let $A_1, A_2, \ldots, A_{\binom{k}{2}}$, $A'_1, A'_2, \ldots, A'_{\binom{k}{2}}$ be the $2 \binom{k}{2}$ distinct 2 element subsets of the set $\{1, 2, 3, \ldots, k\}$. Since $n \leq 2 \binom{k}{2}$, we can define a coloring $c$ of $G$ by assigning the 2 distinct colors in $A_i$ and $A'_i$ to the $2n$ vertices of $V(G)$. By the argument used in Case (ii), this coloring is irregular and uses at most $k$ colors. Thus $\chi_{ir}(G) \leq k$. On the other hand, since $n \geq 2 \binom{k-1}{2} + 1$ and there are $2 \binom{k-1}{2}$ distinct subsets in $\{1, 2, \ldots, k-1\}$, the argument used in Case (i) shows that there is no irregular coloring of $G$ using $k-1$ or fewer colors. Thus $\chi_{ir}(G) \geq k$ and so $\chi_{ir}(G) = k$. 

**Definition 2.2.** The friendship graph $F_n$ is one-point union of $n$ copies of cycle $C_3$.

**Theorem 2.2.** Let $G = F_n$ be a friendship graph. Then $\binom{k-1}{2} + 1 \leq n \leq \binom{k}{2}$ if and only if $\chi_{ir}(G) = k + 1$.

**Proof.** Let $G = F_n$ be a friendship graph. Assume that $\chi_{ir}(G) = k + 1$. Let

$V(G) = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\} \cup w$

and

$E(G) = \{u_i v_i; 1 \leq i \leq n\} \cup \{w u_i; 1 \leq i \leq n\} \cup \{w v_i; 1 \leq i \leq n\}$

with $\deg(w) = 2n$. Assign $c(w) = k + 1$. We have to prove that $\binom{k-1}{2} + 1 \leq n \leq \binom{k}{2}$. Assume to the contrary that $n \geq \binom{k}{2} + 1$ or $n \geq \binom{k-1}{2}$.

**Case (i):** $n \geq \binom{k}{2} + 1$

Let $A_1, A_2, \ldots, A_{\binom{k}{2}}$ be the $\binom{k}{2}$ distinct 2 element subsets of the set $\{1, 2, 3, \ldots, k\}$, where $A_i = (i, j)$ $1 \leq i, j \leq k; 1 \leq l \leq \binom{k}{2}$ and by our assumption $n \geq \binom{k}{2} + 1$, it follows that there exists two pair of vertices $(u_l, v_l)$ and $(u_m, v_m)$ such that $\text{code}(u_l) = \text{code}(u_m)$ and $\text{code}(v_l) = \text{code}(v_m)$, which is a contradiction. Hence $n \leq \binom{k}{2}$.

**Case (ii):** $n \geq \binom{k-1}{2}$

Let $A_1, A_2, \ldots, A_{\binom{k-1}{2}}$ be the $\binom{k-1}{2}$ distinct 2 element subsets of the set $\{1, 2, 3, \ldots, k-1\}$. We can define a coloring $c$ of $G$ by assigning the 2 distinct colors in $A_i$ to the $n$ vertices of $V(G)$, where $1 \leq l \leq \binom{k-1}{2}$. Since $n \geq \binom{k-1}{2}$. Then $c$ is an irregular coloring with at most $k-1$ colors and $c(w) = 1$. Thus $\chi_{ir}(G) \leq k$, which is a contradiction to our assumption. Hence $n \geq \binom{k-1}{2} + 1$. From the above two cases we get $\binom{k-1}{2} + 1 \leq n \leq \binom{k}{2}$.

Conversely, assume that $\binom{k-1}{2} + 1 \leq n \leq \binom{k}{2}$ and to prove $\chi_{ir}(G) = k + 1$. Let $A_1, A_2, \ldots, A_{\binom{k}{2}}$ be the $\binom{k}{2}$ distinct 2 element subsets of the set $\{1, 2, 3, \ldots, k\}$. 


Since \( n \leq \binom{k}{2} \), we can define a coloring of \( G \) by assigning the 2 distinct colors in \( A_i \) to the \( n \) vertices of \( V(G) \). By the argument used in Case (ii), this coloring is irregular and uses at most \( k \) colors. Assign \( c(w) = k + 1 \). Thus \( \chi_{ir}(G) \leq k + 1 \). On the other hand, Since \( n \geq \binom{k-1}{2} + 1 \) and there are \( \binom{k-1}{2} + 1 \) distinct subsets in \( \{1, 2, \ldots, k-1\} \), the argument used in case (i) shows that there is no irregular coloring of \( G \) using \( k - 1 \) or fewer colors. Assign \( c(w) = k + 1 \). Thus \( \chi_{ir}(G) \geq k + 1 \) and hence \( \chi_{ir}(G) = k + 1 \). □

**Definition 2.3.** A tree containing exactly one vertex which is not a pendent vertex is called a star graph \( K_{1,n} \). For a graph \( G \), the splitting graph \( Spl(G) \) of a graph \( G \) is obtained by adding a new vertex \( v' \) corresponding to each vertex \( v \) of \( G \) such that \( N(v) = N(v') \).

**Theorem 2.3.** If \( G \) is a splitting graph of \( K_{1,n} \) then \( \chi_{ir}(S(K_{1,n})) = n + 1 \).

**Proof.** Let \( G \) be a splitting graph of \( K_{1,n} \) with vertices \( V(G) = \{v, v_1, v_2, \ldots, v_n, v', v'_1, v'_2, \ldots, v'_n\} \) and \( E(G) = \{vv_i; 1 \leq i \leq n\} \cup \{v_i v'_i; 1 \leq i \leq n\} \cup \{v'_i v'_j; 1 \leq i, j \leq n\} \). First to prove that \( \chi_{ir}(S(K_{1,n})) \geq n + 1 \). In \( G \), \( N(v'_i) = N(v_j) \) for all \( 1 \leq i, j \leq n \). Therefore, we need \( n \) distinct colors for the vertices set \( \{v_i\} \) and \( \{v'_i\} \), where \( 1 \leq i \leq n \), since \( N(v_i) \neq N(v'_i) \). But \( v' \) is adjacent to all the vertices of \( v'_i, 1 \leq i \leq n \). Hence assign the color \( n + 1 \) to \( v' \). Thus \( \chi_{ir}(S(K_{1,n})) \geq n + 1 \). Next to prove that \( \chi_{ir}(S(K_{1,n})) \leq n + 1 \). The following \( n + 1 \) coloring for \( S(K_{1,n}) \) is irregular. For \( 1 \leq i \leq n \), assign the color \( i \) for \( v_i \) and \( v'_i \), \( i + 1 \) for \( v \) and \( v' \). Since \( \deg(v_i) \neq \deg(v'_i) \) and \( \deg(v) \neq \deg(v') \), it follows that \( \text{code}(v_i) \neq \text{code}(v'_i) \) and \( \text{code}(v) \neq \text{code}(v') \). Hence \( \chi_{ir}(S(K_{1,n})) \neq n + 1 \). Thus, we get \( \chi_{ir}(S(K_{1,n})) = n + 1 \). □

**References**


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