OBTAIN NEW SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING HADAMARD PRODUCT OF LINEAR OPERATOR WITH POLYLOGARITHM FUNCTIONS

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Abstract. In this article, we explore some standard properties for the new subclass \( P^{n,\lambda}_{b,b}(\phi(\zeta)) \) of analytic function which is associated with the differential operator \( G^{n,\lambda}_{b,b}f(\zeta) \) and also we obtain Briot-Bouquet differential subordination, coefficient inequalities, integral means of inequalities, extreme points and distortion of the class of polylogarithms functions.

1. Introduction

Let \( f(\zeta) \) be the form of analytic functions of class \( A, \)

\[
f(\zeta) = \zeta + \sum_{k=2}^{\infty} a_k \zeta^k,
\]

which are analytic function in \( \mathcal{U} = \zeta : |\zeta| < 1, \) where \( \mathcal{U} \) is the unit disc. Let the function \( f(\zeta) \) is given by (1.1) and \( g(\zeta) \) is given by

\[
g(\zeta) = \zeta + \sum_{k=2}^{\infty} b_k \zeta^k,
\]

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then the Hadamard product of $f(\zeta)$ and $g(\zeta)$ is expressed by

$$(f * g)(\zeta) = \zeta + \sum_{k=2}^{\infty} a_k b_k \zeta^k.$$ 

For $f \in A$, Al-Oboudi [2] introduced the following differential operator:

$$D_n^\delta f(\zeta) = \zeta + \sum_{k=2}^{\infty} \left[ 1 + (k-1)\delta \right]^n a_k \zeta^k, \quad (n \in N_0 = N \cup \{0\}, \delta > 0 : \zeta \in \mathcal{U}).$$

For $f \in A$, Ruscheweyh [9] introduced the following differential operator:

$$R_\lambda f(\zeta) = \zeta (1 - \zeta)^{\lambda+1} * f(\zeta) = \zeta + \sum_{k=2}^{\infty} \frac{(\lambda + k - 1)!}{\lambda! (k-1)!} a_k \zeta^k, \quad (\lambda > -1).$$

Now consider the Polylogarithm function $I(n, \delta)$ given by

$$I(n, \delta) = \sum_{k=1}^{\infty} \frac{\zeta^k}{[1 + (k-1)\delta]^n}.$$ 

Note that $I(-1, 1) = \frac{\zeta}{(1-\zeta)^2}$ for $k = 1, 2, 3, \ldots$ is Koebe function. For more details about polylogarithms in the theory of univalent functions see Ponnusamy and Sabapathy [7], K. Al Shaqsi and M. Daraus [4], and Ponnusamy [8].

Now we introduce a function $I^\kappa(n, \delta)$ given by

$$I^\kappa(n, \delta) = \sum_{k=1}^{\infty} \frac{\zeta^k}{(1 - \zeta)^{\lambda+1}}, \quad (\lambda > -1, n \in Z)$$

and obtain the linear operator

$$(1.2) \quad G_{n,\delta}^{\kappa,\lambda} f(\zeta) = I^\kappa(n, \delta) * f(\zeta).$$

Now we find the explicit form of the function

$$I^\kappa(n, \delta) = \sum_{k=1}^{\infty} \left[ 1 + (k-1)\delta \right]^n \frac{(\lambda + k - 1)!}{\lambda! (k-1)!} \zeta^k.$$ 

From equation (1.2), we define

$$G_{n,\delta}^{\kappa,\lambda} f(\zeta) = \sum_{k=2}^{\infty} \left[ 1 + (k-1)\delta \right]^n \frac{(\lambda + k - 1)!}{\lambda! (k-1)!} a_k \zeta^k.$$ 

Note that $G_{1,0}^{n,0} = D^n, G_{1}^{0,\lambda} = D^\lambda$ which give the Salagean differential operator [10] and Ruscheweyh differential operator [9] respectively. It is obvious that the operator $G_{\delta}^{\kappa,\lambda}$ includes two well known operators. Also note that $G_{1,0}^{0,0} = f(\zeta)$ and $G_{1}^{0,1} = G_{1}^{1,0} = \zeta f'(\zeta)$. 
Let $p$ be the class of functions of the form $p(\zeta) = 1 + p_1 \zeta + p_2 \zeta^2 + \ldots$, analytic in $U$, which satisfy $Re \{p(\zeta)\} > 0$.

**Definition 1.1.** We define $\mathcal{P}_{\delta,b}^{n,\lambda}(\phi(\zeta))$ be the class of functions $f \in \mathcal{A}$ for which

$$1 + \frac{1}{b} \left( \frac{\zeta(\mathcal{G}_{\delta}^{n,\lambda} f(\zeta))'}{\mathcal{G}_{\delta}^{n,\lambda} f(\zeta)} - 1 \right) \prec \phi(\zeta),$$

where $n, \lambda \in \mathbb{N}, \lambda > 0, b > 0, \phi \in p; \zeta \in U$.

**Definition 1.2.** For $\phi(\zeta) = \frac{1+(1-2\beta)\zeta}{1-\zeta}$, we define $\mathcal{P}_{\delta,b}^{n,\lambda}(\phi(\zeta)) \equiv \mathcal{P}_{\delta,b}^{n,\lambda}(\beta)$ be the class of functions $f \in \mathcal{A}$ for which

$$1 + \frac{1}{b} \left( \frac{\zeta(\mathcal{G}_{\delta}^{n,\lambda} f(\zeta))'}{\mathcal{G}_{\delta}^{n,\lambda} f(\zeta)} - 1 \right) > \beta,$$

where $n, \lambda \in \mathbb{N}, \delta > 0, b > 0, 0 \leq \beta \leq 1$; all $\zeta \in U$.

If $f(\zeta)$ and $g(\zeta)$ are analytic in $U$, we state that $f(\zeta)$ is subordinate to $g(\zeta)$, i.e., $f(\zeta) \prec g(\zeta)$, if there exists a Schwarz function $w(\zeta)$, with $w(0) = 0$ and $|w| < 1$ such that $f(\zeta) = g(w(\zeta))$. Furthermore, if the function $g(\zeta)$ is univalent in $U$, then the above subordination equivalence holds (see [7, 8]). $f(\zeta) \prec g(\zeta)$ if and only if $f(0) = g(0)$, and $f(U) \subset g(U)$.

Note that $\mathcal{P}_{1,1}^{n,\lambda}(\phi(\zeta)) = \mathcal{K}_{\lambda}^{n}(\phi(\zeta)), \mathcal{P}_{1,1}^{n,\lambda}(\beta) = \mathcal{R}_{n}(\beta)$ was studied by K. AlShaqsi and M. Darus [4], $\mathcal{P}_{1,1}^{0,\lambda}(\phi(\zeta)) = S^*\phi(\zeta)$ was studied by Ma and Minda [6], $\mathcal{P}_{1,1}^{0,\lambda}(\beta) = \mathcal{R}_{n}(\beta)$ was introduced and studied by Ahuja [1] and $\mathcal{P}_{1,1}^{0,0}(\beta) = \mathcal{R}_{n}(\beta)$ was introduced and studied by Kadioglu [3].

## 2. Main results

To construct Briot-Bouquet differential subordination theorem, we have to follow the next lemma:

**Lemma 2.1.** Let $\beta, \nu$ be the complex numbers. Let $\phi \in p$ be convex univalent in $U$ with $\phi(0) = 1$ and $Re \{\beta \phi(\zeta) + \nu\} > 0, \zeta \in U$.

**Theorem 2.1.** Let $n, \lambda \in \mathbb{N}, \delta > 0$ and $\phi \in p$, then $\mathcal{P}_{\delta,b}^{n,\lambda+1}(\phi(\zeta)) \subset \mathcal{P}_{\delta,b}^{n,\lambda}(\phi(\zeta))$.

**Proof.** Let $f \in \mathcal{P}_{\delta,b}^{n,\lambda+1}(\phi(\zeta))$ and $p(\zeta) = 1 + \frac{1}{b} \left( \frac{\zeta(\mathcal{G}_{\delta}^{n,\lambda} f(\zeta))'}{\mathcal{G}_{\delta}^{n,\lambda} f(\zeta)} - 1 \right)$, where $p(\zeta)$ is analytic in $U$ with $p(0) = 1$, then

$$\zeta(\mathcal{G}_{\delta}^{n,\lambda} f(\zeta))' = (\lambda + 1)\mathcal{G}_{\delta}^{n,\lambda+1} f(\zeta) - (\lambda)\mathcal{G}_{\delta}^{n,\lambda} f(\zeta).$$
Theorem 2.3. Let
\[
q
\]
Applying Lemma 2.1 in (2.2) we get
\[
1 + \sum_{k=2}^{\infty} (\beta b - b + 1 - k)[1 + (k - 1)\delta]^{n}C(\lambda) |a_k| \leq (1 - \beta)b,
\]
where $C(\lambda) = \frac{(k + \lambda - 1)!}{\lambda!(k-1)!}$, $0 \leq \beta < 1$, $n \in N = N \cup \{0\}$, $\delta > 0$, $b > 0$, $\lambda \geq 0$; $\zeta \in U$.

\[\zeta(\mathcal{G}^{n,\lambda}_{\delta} f(\zeta))' = (\lambda + 1)\mathcal{G}^{n,\lambda}_{\delta} f - (\lambda)\mathcal{G}^{n,\lambda}_{\delta}.
\]

Hence
\[
1 + \sum_{k=2}^{\infty} (\beta b - b + 1 - k)[1 + (k - 1)\delta]^{n}C(\lambda) |a_k| \leq (1 - \beta)b,
\]
and let $c$ be real number such that $c > -1$, then $\mathcal{F}$ defined by $\mathcal{F} = \frac{c+1}{\zeta} \int_{0}^{1} \mathcal{F}(t) dt$ belongs to the class $f \in \mathcal{P}_{\delta,b}^{n,\lambda}(\phi(\zeta))$.

\[
q
\]
Proof. Let $f \in \mathcal{P}_{\delta,b}^{n,\lambda}(\phi(\zeta))$, then $q(\zeta) = 1 + \frac{1}{b} \zeta(\mathcal{G}^{n,\lambda}_{\delta} f(\zeta))' = \phi(\zeta)$, where $q(\zeta)$ is analytic in $\mathcal{U}$ with $q(0) = 1$, then
\[
\zeta(\mathcal{G}^{n,\lambda}_{\delta} f(\zeta))' = (\lambda + 1)\mathcal{G}^{n,\lambda}_{\delta} f - (\lambda)\mathcal{G}^{n,\lambda}_{\delta}.
\]

Applying Lemma 2.1 in (2.2) we get $q < \phi$. Hence the theorem is proved.

Remark 2.1. If we put $b = 1$ in the above theorem, we obtain the result of K. Al-Shaqsi and M. Darus [4].

Theorem 2.2. Let $f \in \mathcal{P}_{\delta,b}^{n,\lambda}(\phi(\zeta))$ and let $c$ be real number such that $c > -1$, then $\mathcal{F}$ defined by $\mathcal{F} = \frac{c+1}{\zeta} \int_{0}^{1} \mathcal{F}(t) dt$ belongs to the class $f \in \mathcal{P}_{\delta,b}^{n,\lambda}(\phi(\zeta))$.

Proof. Let $f \in \mathcal{P}_{\delta,b}^{n,\lambda}(\phi(\zeta))$, then $q(\zeta) = 1 + \frac{1}{b} \zeta(\mathcal{G}^{n,\lambda}_{\delta} f(\zeta))' = \phi(\zeta)$, where $q(\zeta)$ is analytic in $\mathcal{U}$ with $q(0) = 1$, then
\[
\zeta(\mathcal{G}^{n,\lambda}_{\delta} f(\zeta))' = (\lambda + 1)\mathcal{G}^{n,\lambda}_{\delta} f - (\lambda)\mathcal{G}^{n,\lambda}_{\delta}.
\]

Let $q(\zeta) = 1 + \frac{1}{b} \zeta(\mathcal{G}^{n,\lambda}_{\delta} f(\zeta))' = \phi(\zeta)$, then we get
\[
1 + \sum_{k=2}^{\infty} (\beta b - b + 1 - k)[1 + (k - 1)\delta]^{n}C(\lambda) |a_k| \leq (1 - \beta)b.
\]
Applying Lemma 2.1 in (2.2) we get $q < \phi$. Hence the theorem is proved.

Theorem 2.3. Let $f(\zeta)$ be defined by (1.1). Then $f \in \mathcal{A}$ if and only if
\[
\sum_{k=2}^{\infty} (\beta b - b + 1 - k)[1 + (k - 1)\delta]^{n}C(\lambda) |a_k| \leq (1 - \beta)b,
\]
where $C(\lambda) = \frac{(k + \lambda - 1)!}{\lambda!(k-1)!}$, $0 \leq \beta < 1$, $n \in N = N \cup \{0\}$, $\delta > 0$, $b > 0$, $\lambda \geq 0$; $\zeta \in U$.

\[
\sum_{k=2}^{\infty} (\beta b - b + 1 - k)[1 + (k - 1)\delta]^{n}C(\lambda) |a_k| \leq (1 - \beta)b.
\]

Proof. Suppose that the inequality (2.3) is true and $|\zeta| < 1$. Then it is sufficient to show that $\left| 1 + \frac{1}{b} \zeta(\mathcal{G}^{n,\lambda}_{\delta} f(\zeta))' - 1 \right| < 1$, which gives $\sum_{k=2}^{\infty} (\beta b - b + 1 - k)[1 + (k - 1)\delta]^{n}C(\lambda) |a_k| \leq (1 - \beta)b$. It is clear that the values of (2.3) lies in a circle centered at $w = 1$ whose radius is $(1 - \beta)b$. Hence the condition (2.3) holds.

Conversely, let us consider that the function $f$ defined by (1.1) is in the class $\mathcal{P}_{\delta,b}^{n,\lambda}(\beta)$, then $Re \left( 1 + \frac{1}{b} \zeta(\mathcal{G}^{n,\lambda}_{\delta} f(\zeta))' - 1 \right) > \beta$, by the value of $\zeta$ on the real axis. Let $\zeta \to 1^-$ through real values, we obtain the result.
Hence the result is sharp for the function \( f(\zeta) = \zeta + \frac{(1-\beta)b}{(\beta b-b+1-k)[1+(k-1)\delta]}C(\lambda) \).

**Corollary 2.1.** Let \( f \in \mathcal{P}^{n,\lambda}_{\delta,b}(\beta) \), then we have \( a_k \leq \frac{(1-\beta)b}{(\beta b-b+1-k)[1+(k-1)\delta]}C(\lambda) \).

Put \( b = 1 \), then the class \( \mathcal{P}^{n,\lambda}_{\delta,b}(\phi) \) analogous to the class \( \mathcal{K}^{n,\lambda}_{\delta,b}(\beta) \) introduced by M. Thirucheran, A. Anand and T. Stalin [5].

**Corollary 2.2.** Let \( f \in \mathcal{P}^{n,\lambda}_{\delta,b}(\beta) \), then we have \( a_k \leq \frac{(1-\beta)b}{(\beta b-b+1-k)[1+(k-1)\delta]}C(\lambda) \).

Put \( b = 1, \delta = 1 \) we obtain the corollary which analogous to the result of K. AlShaqui and M. Darus [4].

**Corollary 2.3.** Let \( f \in \mathcal{P}^{n,\lambda}_{\delta,b}(\beta) \), then we have \( a_k \leq \frac{(1-\beta)b}{(\beta b-b+1-k)[1+(k-1)\delta]}C(\lambda) \).

**Theorem 2.4.** Let \( f \in \mathcal{P}^{n,\lambda}_{\delta,b}(\beta) \) and suppose that \( f(\zeta) = \zeta + \frac{(1-\beta)b}{\phi(\lambda,\delta)}\zeta^k \), \( k = 2,3,...,|\epsilon_k| = 1 \), where \( \phi(\lambda,\delta) = (\beta b-b+1-k)[1+(k-1)\delta]C(\lambda) \). If there exists \( w(\zeta) \) given by \( w(\zeta)^{k-1} = \frac{\phi(\lambda,\delta)}{(1-\beta)b\epsilon_k} \sum_{k=2}^{\infty} a_k \zeta^{k-1} \), then for \( \zeta = r e^{i\theta}, 0 < r < 1, \int_0^{2\pi} |f(\zeta)|^\mu d\theta \leq \int_0^{2\pi} |g(\zeta)|^\mu d\theta, \mu > 0 \).

**Proof.** To complete the theorem we have to prove that

\[
\int_0^{2\pi} \left|1 + \sum_{k=2}^{\infty} a_k \zeta^{k-1}\right|^\mu d\theta \leq \int_0^{2\pi} \left|1 + \frac{(1-\beta)b\epsilon_k}{\phi(\lambda,\delta)} \zeta^{k-1}\right|^\mu d\theta.
\]

Using Littlewood subordination theorem, it is sufficient to prove that

\[
1 + \sum_{k=2}^{\infty} a_k \zeta^{k-1} < 1 + \frac{(1-\beta)b\epsilon_k}{\phi(\lambda,\delta)} \zeta^{k-1}.
\]

Let \( 1 + \sum_{k=2}^{\infty} a_k \zeta^{k-1} < 1 + \frac{(1-\beta)b\epsilon_k}{\phi(\lambda,\delta)} (w(\zeta))^{k-1} \), therefore

\[
(w(\zeta))^{k-1} = \frac{\phi(\lambda,\delta)}{(1-\beta)b\epsilon_k} \sum_{k=2}^{\infty} a_k \zeta^{k-1}.
\]

Hence \( w(0) = 0 \). Furthermore, if \( f \in \mathcal{A} \) it satisfies \( \phi(\lambda,\delta) \leq (1-\beta)b \),

\[
|w(\zeta)|^{k-1} = \left|\frac{\phi(\lambda,\delta)}{(1-\beta)b\epsilon_k} \sum_{k=2}^{\infty} a_k |\zeta^{k-1}| \right| \leq |\zeta| < 1.
\]

Hence the theorem is completed. □
Let us define new subclass $P_{n,\lambda}^{\delta,b}(\beta) \subset P_{\delta,b}^{n,\lambda}(\beta)$, which contains the function $f \in A$. Now we determine the extreme points of the subclass $P_{\delta,b}^{n,\lambda}(\beta)$.

**Theorem 2.5.** Let

$$f_1(\zeta) = \zeta, f_k(\zeta) = \zeta + \sum_{k=2}^{\infty} \eta_k \frac{(1 - \beta)b}{\phi(\lambda, \delta)} \zeta^k, k = 2, 3, \ldots$$

Then $f \in P_{\delta,b}^{n,\lambda}(\beta)$ if and only if $f(\zeta) = \sum_{k=1}^{\infty} \eta_k f_k(\zeta)$ where $\eta_k > 0$ and $\sum_{k=1}^{\infty} \eta_k = 1$.

**Proof.** Let

$$f(\zeta) = \sum_{k=1}^{\infty} \eta_k f_k(\zeta)$$

$$= \zeta + \sum_{k=2}^{\infty} \eta_k \frac{(1 - \beta)b}{\phi(\lambda, \delta)} \zeta^k$$

$$= \sum_{k=2}^{\infty} \eta_k \frac{(1 - \beta)b}{\phi(\lambda, \delta)} (\phi(\lambda, \delta))$$

$$= (1 - \beta)b \sum_{k=1}^{\infty} \eta_k$$

$$= (1 - \beta)b(1 - \eta_1) < (1 - \beta)b,$$

which shows that $f \in P_{\delta,b}^{n,\lambda}(\beta)$.

Conversely, suppose that $f \in P_{\delta,b}^{n,\lambda}(\beta)$. Since $|a_k| \leq \frac{(1 - \beta)b}{\phi(\lambda, \delta)}$, $k = 2, 3, \ldots$. Let $\eta_k \leq \frac{\phi(\lambda, \delta)}{(1 - \beta)b}, \eta_1 = 1 - \sum_{k=2}^{\infty} \eta_k$. Then we obtain $f(\zeta) = \sum_{k=1}^{\infty} \eta_k f_k(\zeta)$. □

**Theorem 2.6.** The class $P_{\delta,b}^{n,\lambda}(\beta)$ is convex.

**Proof.** Let the function $f_j(\zeta) = \zeta + \sum_{k=2}^{\infty} a_{k,j} \zeta^k, a_{k,j} \geq 0, j = 1, 2$ lies in the class $f \in P_{\delta,b}^{n,\lambda}(\beta)$. It is sufficient to prove that $h(\zeta) = (\mu + 1)f_1(\zeta) - \mu f_2(\zeta), 0 \leq \zeta \leq 1$. 


Since \( h(\zeta) = \zeta + \sum_{k=2}^{\infty} [(1 + \mu)a_{k,1} - \mu a_{k,2}] \zeta^k \), this implies that
\[
\sum_{k=2}^{\infty} (\beta b - b + 1 - k)[1 + (k - 1)\delta]nC(\lambda)(1 + \mu)a_{k,1} \\
+ (\beta b - b + 1 - k)[1 + (k - 1)\delta]nC(\lambda)(\mu)a_{k,2} \\
\leq (1 + \mu)(1 - \beta)b + \mu(1 - \beta)b \\
\leq (1 - \beta)b.
\]
Therefore \( h \in \mathcal{P}_{\delta,b}^{n,\lambda}(\beta) \). Hence \( \mathcal{P}_{\delta,b}^{n,\lambda}(\beta) \) is convex.

**Theorem 2.7.** Let \( f \in \mathcal{P}_{\delta,b}^{n,\lambda}(\beta) \), then \( f \) is close-to-convex of order \( \sigma (0 \leq \sigma < 1) \) in the disc \( |\zeta| < r_1 \), where \( r_1 := \left( \frac{(1-\sigma)[(\beta b-b+1-k)[1+(k-1)\delta]nC(\lambda)]}{(k-\sigma)(1-\beta)b} \right)^{\frac{1}{1+r}} \).

**Theorem 2.8.** Let \( f \in \mathcal{P}_{\delta,b}^{n,\lambda}(\beta) \), then \( f \) is starlike of order \( \sigma (0 \leq \sigma < 1) \) in the disc \( |\zeta| < r_2 \), where \( r_2 := \inf \left( \frac{(1-\sigma)[(\beta b-b+1-k)[1+(k-1)\delta]nC(\lambda)]}{(k-\sigma)(1-\beta)b} \right)^{\frac{1}{1+r}} \), \( k \geq 2 \).

**Theorem 2.9.** Let \( f \in \mathcal{P}_{\delta,b}^{n,\lambda}(\beta) \), then \( f \) is convex of order \( \sigma (0 \leq \sigma < 1) \) in the disc \( |\zeta| < r_3 \), where \( r_3 := \inf \left( \frac{(1-\sigma)[(\beta b-b+1-k)[1+(k-1)\delta]nC(\lambda)]}{(k-\sigma)(1-\beta)b} \right)^{\frac{1}{1+r}} \), \( k \geq 2 \).

**Theorem 2.10.** Let \( f(\zeta) = \zeta + \sum_{k=2}^{\infty} |a_k| \zeta^k \) be in the class \( f \in \mathcal{P}_{\delta,b}^{n,\lambda}(\beta) \), then for \( |\zeta| = r \), we have
\[
r - \frac{(1 - \beta)b}{(\beta b - b - 1)(1 + \delta)n(\lambda + 1)} r^2 \leq |f(\zeta)| \leq r + \frac{(1 - \beta)b}{(\beta b - b - 1)(1 + \delta)n(\lambda + 1)} r^2
\]
and
\[
1 - \frac{2(1 - \beta)b}{(\beta b - b - 1)(1 + \delta)n(\lambda + 1)} r \leq |f'(\zeta)| \leq 1 + \frac{2(1 - \beta)b}{(\beta b - b - 1)(1 + \delta)n(\lambda + 1)} r.
\]

Putting \( \delta = 1 \) in the above theorem we obtain the result which analogue the results of M. Thirucheran, M. Vinothkumar and T. Stalin. [5]

**Corollary 2.4.** Let \( f(\zeta) = \zeta + \sum_{k=2}^{\infty} |a_k| \zeta^k \) be in the class \( f \in \mathcal{P}_{\delta,b}^{n,\lambda}(\beta) \), then for \( |\zeta| = r \) we have
\[
r - \frac{(1 - \beta)b}{(\beta b - b - 1)(2^n(\lambda + 1))} r^2 \leq |f(\zeta)| \leq r + \frac{(1 - \beta)b}{(\beta b - b - 1)(2^n(\lambda + 1))} r^2
\]
and
\[
1 - \frac{2(1 - \beta)b}{(\beta b - b - 1)(2^n(\lambda + 1))} r \leq |f'(\zeta)| \leq 1 + \frac{2(1 - \beta)b}{(\beta b - b - 1)(2^n(\lambda + 1))} r.
\]
Putting $\delta = 1, b = 1$ in the above theorem we obtain result which is analogue the results of K. AlShaqsi and M. Darus. [4]

**Corollary 2.5.** Let $f \in \mathcal{P}_{\delta, b}^{n, \lambda}(\beta)$ be in the class $f \in \mathcal{P}_{\beta, \delta, b}^{n, \lambda}(\beta)$, then for $|\zeta| = r$ we have

$$r - \frac{(1 - \beta)}{(\beta - 2)(2)^n(\lambda + 1)}r^2 \leq |f(\zeta)| \leq r + \frac{(1 - \beta)}{(\beta - 2)(2)^n(\lambda + 1)}r^2$$

and

$$1 - \frac{2(1 - \beta)}{(\beta - 2)(2)^n(\lambda + 1)}r \leq |f'(\zeta)| \leq 1 + \frac{2(1 - \beta)}{(\beta - 2)(2)^n(\lambda + 1)}r.$$

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