OBTAINT SUBCLASS OF ANALYTIC FUNCTIONS CONNECTED WITH
CONVOLUTION OF POLYLOGARITHM FUNCTIONS

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ABSTRACT. In this work, we investigate some properties for the subclass
$P_{\beta,\lambda,\gamma,b}(\phi(z))$ of analytic function related with the linear differential operator
$R_{\lambda,\delta}f(z)$ defined by polylogarithm functions. And also, we obtain coefficient
inequalities, extreme points, radii of convexity and starlikeness, growth and
distortion bounds for the subclass $P_{\beta,\lambda,\gamma,b}(\phi(z))$.

1. INTRODUCTION

Let $A$ be the class of functions of the form

\[(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k,\]

which are analytic in the unit disk $U = \{ z : |z| < 1 \}$. For functions $f(z)$ given by
(1.1) and $g(z)$ given by
\[g(z) = z + \sum_{k=2}^{\infty} b_k z^k,\]

the convolution of $f(z)$ and $g(z)$ is defined by
\[(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.\]
If \( f(z) \) and \( g(z) \) are analytic in \( \mathcal{U} \), we state that \( f(z) \) is subordinate to \( g(z) \), i.e. \( f(z) \preceq g(z) \), if a Schwarz function \( w(z) \) exists, with \( w(0) = 0 \) and \( |w| < 1 \) such that \( f(z) = g(w(z)) \). Moreover, if the function \( g(z) \) is univalent in \( \mathcal{U} \), then the above subordination is equivalence holds (see [7, 8]). \( f(z) \prec g(z) \) if and only if \( f(0) = g(0) \), and \( f(\mathcal{U}) \subset g(\mathcal{U}) \).

For \( f \in A \), Al-Oboudi [2] initiated the following differential operator:

\[
D_\delta^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k, \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \delta > 0 : z \in \mathcal{U}).
\]

For \( f \in A \), Ruscheweyh [9] established the following differential operator:

\[
\mathcal{R}_\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} \ast f(z) = z + \sum_{k=2}^{\infty} \frac{(\lambda+k-1)!}{\lambda!(k-1)!} a_k z^k, \quad (\lambda > -1).
\]

Consider \( p \) is the class of functions of the form \( p(z) = 1 + p_1 z + p_2 z^2 + \ldots \) analytic in \( \mathcal{U} \), \( \text{Re} \{p(z)\} > 0 \).

Consider the Polylogarithm function \( E(n, \delta) \) given by

\[
E(n, \delta) = \sum_{k=1}^{\infty} \frac{z^k}{[1 + (k-1)\delta]^n}.
\]

Note that \( E(-1, 1) = \frac{z}{(1-z)^2} \) for \( k = 1, 2, 3, \ldots \) is Koebe function. For further additional information about polylogarithms in theory of univalent functions see Ponnusamy and Sabapathy [7], K. Al Shaqsi and M. Daraus [4] and Ponnusamy [8].

Now we introduce a function \( E^\kappa(n, \delta) \) given by

\[
E(n, \delta) * E^\kappa(n, \delta) = \frac{z}{(1-z)^{\lambda+1}}, \lambda > -1, n \in \mathbb{Z},
\]

thus obtaining the linear operator

\[
(1.2) \quad \mathcal{R}_{\lambda, \delta}^n f(z) = E^\kappa(n, \delta) \ast f(z).
\]

Now we come across the explicit form of the function

\[
E^\kappa(n, \delta) = \sum_{k=1}^{\infty} [1 + (k-1)\delta]^n \frac{(\lambda+k-1)!}{\lambda!(k-1)!} z^k.
\]

From equation (1.2), we define

\[
\mathcal{R}_{\lambda, \delta}^n f(z) = \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n \frac{(\lambda+k-1)!}{\lambda!(k-1)!} a_k z^k.
\]
Note that $R_{0,1}^n = D^n, R_{\lambda,\delta}^0 = D^\lambda$ which give the Salagean differential operator [10] and Ruscheweyh differential operator [9] respectively. It is obvious that the operator includes two well known operators. Also note that $R_{0,\delta}^0 = f(z)$ and $R_{0,\delta}^1 = R_{0,1}^1 = z f'(z)$.

### Definition 1.1
We define $P_{\beta,\lambda,\delta,b}(\phi(z))$ be the class of the functions $f \in A$ for which

$$1 + \frac{1}{b} \left( \frac{z (R_{\lambda,\delta}^n f(z))'}{R_{\lambda,\delta}^n f(z)} - 1 \right) - \beta \left| \frac{z (R_{\lambda,\delta}^n f(z))'}{R_{\lambda,\delta}^n f(z)} - 1 \right| < \phi(z),$$

where $n, \lambda \in N_0, \beta > 0, \delta > 0, b > 0, \phi \in p; z \in U$.

### Definition 1.2
For $\phi(z) = \frac{1+(-2)z}{(1-z)}$, we define $P_{\beta,\lambda,\delta,b}^n (\phi(z)) = P_{\beta,\lambda,\delta,b}^n (\alpha), f \in A$ for which

$$1 + \frac{1}{b} \left( \frac{z (R_{\lambda,\delta}^n f(z))'}{R_{\lambda,\delta}^n f(z)} - 1 \right) - \beta \left| \frac{z (R_{\lambda,\delta}^n f(z))'}{R_{\lambda,\delta}^n f(z)} - 1 \right| > \alpha,$$

where $n, \lambda \in N_0, \beta > 0, \delta > 0, b > 0, \phi \in p, 0 \leq \alpha \leq 1; z \in U$.

Note that $P_{0,0,1,1}^n (\phi(z)) = K_0^n (\phi(z)), P_{0,0,1,1}^n (\alpha) = R_{\lambda}^n (\alpha)$ studied by K. AlShaqsi and M. Darus [4], $P_{0,0,1,1}^0 (\phi(z)) = S^* (\phi(z))$ studied by Ma and Minda [6], $P_{0,0,1,1}^0 (\alpha) = R_{\lambda} (\alpha)$ introduced and studied by Ahuja [1] and $P_{0,0,1,1}^0 (\alpha) = R_{\alpha} (\alpha)$ introduced and studied by Kadioglu [3].

### 2. Main results

#### Theorem 2.1
Let $f(z)$ be defined by (1.1). Then $f \in A$ if and only if

$$\sum_{k=2}^{n} (kb\beta - b\beta - k + 1 - b + b\alpha) [1 + (k - 1)\delta]^n C(\lambda) \left| a_k \right| \leq (1 - \alpha)b,$$

where $C(\lambda) = \frac{(k+\lambda-1)!}{\lambda!(k-1)!}, 0 \leq \alpha < 1, n \in N_0 = N \cup \{0\}, \delta > 0, b > 0, \lambda \geq 0; z \in U$.

**Proof.** Suppose that the inequality (2.1) is true and $|z| < 1$. Then it is proved that the values of (1.3) lies in a circle centered at $w = 1$ whose radius is $(1 - \alpha)b$. It is sufficient to show that $\left| 1 + \frac{1}{b} \left( \frac{z (R_{\lambda,\delta}^n f(z))'}{R_{\lambda,\delta}^n f(z)} - 1 \right) - \beta \left| \frac{z (R_{\lambda,\delta}^n f(z))'}{R_{\lambda,\delta}^n f(z)} - 1 \right| - \alpha + 1 \right| < 1$, which gives $\sum_{k=2}^{n} (kb\beta - b\beta - k + 1 - b + b\alpha) [1 + (k - 1)\delta]^n C(\lambda) \left| a_k \right| \leq (1 - \alpha)b$. Hence the condition (2.1) holds.
Conversely, let us assume that the function \( f \) defined by (1.1) is in the class \( P_{\beta,\lambda, \delta, b}^n(\alpha) \), then \( \Re (1 + \frac{b}{\lambda} \left( \frac{z(\mathcal{R}_{\lambda, \delta}^n(z)^\prime - 1)}{\mathcal{R}_{\lambda, \delta}^n(z)} - \beta \right)) > \alpha \), by the value of \( z \) on the real axis, let \( z \to 1^- \) through real values, we obtain the result

\[
\sum_{k=2}^{\infty} (kb\beta - b\beta - k + 1 - b + b\alpha)[1 + (k - 1)\delta]nC(\lambda) |a_k| \leq (1 - \alpha)b.
\]

Hence the result is sharp for the function \( f(z) = z + (1 - \alpha)b \).

**Theorem 2.2.** Let

\[
f_1(z) = z, f_k(z) = z + \sum_{k=2}^{\infty} \eta_k \frac{(1 - \alpha)b}{\psi(\lambda)} z^k, k = 2, 3, ...
\]

where \( \psi(\lambda) = \sum_{k=2}^{\infty} (kb\beta - b\beta - k + 1 - b + b\alpha)[1 + (k - 1)\delta]nC(\lambda) \). Then \( f \in P_{\beta,\lambda, \delta, b}^n(\alpha) \) if and only if it can be expressed in the form \( f(z) = \sum_{k=1}^{\infty} \eta_k f_k(z) \) where \( \eta_k > 0 \) and \( \sum_{k=1}^{\infty} \eta_k = 1 \).

**Proof.** Let

\[
f(z) = \sum_{k=1}^{\infty} \eta_k f_k(z)
\]

\[
= z + \sum_{k=2}^{\infty} \eta_k \frac{(1 - \alpha)b}{\psi(\lambda)} z^k
\]

\[
= \sum_{k=2}^{\infty} \eta_k \frac{(1 - \alpha)b}{\psi(\lambda)} \psi(\lambda)
\]

\[
= (1 - \alpha)b \sum_{k=1}^{\infty} \eta_k
\]

\[
= (1 - \alpha)b(1 - \eta_1) < (1 - \alpha)b
\]

which shows that \( f \in P_{\beta,\lambda, \delta, b}^n(\alpha) \).

Conversely, suppose that \( f \in P_{\beta,\lambda, \delta, b}^n(\alpha) \). Since \( |a_k| \leq \frac{(1 - \alpha)b}{\psi(\lambda)}, k = 2, 3, ... \) Let \( \eta_k \leq \frac{\psi(\lambda)}{(1 - \alpha)b}, \eta_1 = 1 - \sum_{k=2}^{\infty} \eta_k \). Then we obtain \( f(z) = \sum_{k=1}^{\infty} \eta_k f_k(z) \). \( \square \)
Theorem 2.3. Let \( f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k \), \( f \in \mathcal{P}_{\beta,\lambda,\delta,b}^n(\alpha) \), then for \( |z| = r \), we have

\[
\begin{align*}
    r - \frac{(1 - \alpha)b}{(\beta b + \alpha b - b - 1)(1 + \delta)^n(\lambda + 1)} r^2 &\leq |f(z)| \\
    \leq r + \frac{(1 - \alpha)b}{(\beta b + \alpha b - b - 1)(1 + \delta)^n(\lambda + 1)} r^2
\end{align*}
\]

and

\[
\begin{align*}
    1 - \frac{2(1 - \alpha)b}{(\beta b + \alpha b - b - 1)(1 + \delta)^n(\lambda + 1)} r &\leq |f'(z)| \\
    \leq 1 + \frac{2(1 - \alpha)b}{(\beta b + \alpha b - b - 1)(1 + \delta)^n(\lambda + 1)} r
\end{align*}
\]

Putting \( \beta = 0, \delta = 1 \) in the above theorem the result obtained is analogue the results of M. Thirucheran, M. Vinothkumar and T. Stalin [5].

Corollary 2.1. Let \( f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k \), \( f \in \mathcal{P}_{\beta,\lambda,\delta,b}^n(\alpha) \), then for \( |z| = r \) we have

\[
\begin{align*}
    r - \frac{(1 - \alpha)b}{(\alpha b - b - 1)(2)^n(\lambda + 1)} r^2 &\leq |f(z)| \\
    \leq r + \frac{(1 - \alpha)b}{(\alpha b - b - 1)(2)^n(\lambda + 1)} r^2
\end{align*}
\]

and

\[
\begin{align*}
    1 - \frac{2(1 - \alpha)b}{(\alpha b - b - 1)(2)^n(\lambda + 1)} r &\leq |f'(z)| \\
    \leq 1 + \frac{2(1 - \alpha)b}{(\alpha b - b - 1)(2)^n(\lambda + 1)} r
\end{align*}
\]

Putting \( \beta = 0, \delta = 1, b = 1 \) in the above theorem the result obtained is analogue to the results of K. AlShaqqi and M. Darus [4].

Corollary 2.2. Let \( f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k \), \( f \in \mathcal{P}_{\beta,\lambda,\delta,b}^n(\alpha) \), then for \( |z| = r \) we have

\[
\begin{align*}
    r - \frac{(1 - \alpha)}{(\alpha - 2)(2)^n(\lambda + 1)} r^2 &\leq |f(z)| \\
    \leq r + \frac{(1 - \alpha)}{(\alpha - 2)(2)^n(\lambda + 1)} r^2
\end{align*}
\]

and

\[
\begin{align*}
    1 - \frac{2(1 - \alpha)}{(\alpha - 2)(2)^n(\lambda + 1)} r &\leq |f'(z)| \\
    \leq 1 + \frac{2(1 - \alpha)}{(\alpha - 2)(2)^n(\lambda + 1)} r
\end{align*}
\]

Theorem 2.4. The class \( \mathcal{P}_{\beta,\lambda,\delta,b}^n(\alpha) \) is convex.

Proof. Let the function \( f_1(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \), \( a_{k,j} \geq 0, j = 1, 2 \) lies in the class \( f \in \mathcal{P}_{\beta,\lambda,\delta,b}^n(\alpha) \). It is sufficient to prove that \( h(z) = (\gamma + 1)f_1(z) - \gamma f_2(z), 0 \leq z \leq 1, \)
the class \( f \in \mathcal{P}_{\beta, \lambda, \delta, b}^n(\alpha) \). Since \( h(z) = z + \sum_{k=2}^{\infty} [(1 + \gamma)a_{k,1} - \gamma a_{k,2}] z^k \), which implies that
\[
\sum_{k=2}^{\infty} (kb\beta - b\beta - k + 1 - b + b\alpha)[1 + (k - 1)\delta]^nC(\lambda)(1 + \gamma)a_{k,1} \\
+ (kb\beta - b\beta - k + 1 - b + b\alpha)[1 + (k - 1)\delta]^nC(\lambda)(\gamma)a_{k,2} \\
\leq (1 + \gamma)(1 - \alpha)b + \gamma(1 - \alpha)b \\
\leq (1 - \alpha)b
\]
therefore \( h \in \mathcal{P}_{\beta, \lambda, \delta, b}^n(\alpha) \). Hence \( \mathcal{P}_{\beta, \lambda, \delta, b}^n(\alpha) \) is convex.

Theorem 2.5. Let \( f \in \mathcal{P}_{\beta, \lambda, \delta, b}^n(\alpha) \), then \( f \) is close-to-convex of order \( \sigma (0 \leq \sigma < 1) \) in the disc \( |z| < r_1 \), where \( r_1 := \left( (1 - \sigma)[(kb\beta - b\beta - k + 1 - b + b\alpha][1 + (k - 1)\delta]^nC(\lambda)] \right)^{\frac{1}{k-1}}, (k \geq 2) \).

Theorem 2.6. Let \( f \in \mathcal{P}_{\beta, \lambda, \delta, b}^n(\alpha) \), then \( f \) is starlike of order \( \sigma (0 \leq \sigma < 1) \) in the disc \( |z| < r_2 \), where \( r_2 := \inf \left( (1 - \sigma)[(kb\beta - b\beta - k + 1 - b + b\alpha][1 + (k - 1)\delta]^nC(\lambda)] \right)^{\frac{1}{k-1}}, (k \geq 2) \).

Theorem 2.7. Let \( f \in \mathcal{P}_{\beta, \lambda, \delta, b}^n(\alpha) \), then \( f \) is convex of order \( \sigma (0 \leq \sigma < 1) \) in the disc \( |z| < r_3 \), where \( r_3 := \inf \left( (1 - \sigma)[(kb\beta - b\beta - k + 1 - b + b\alpha][1 + (k - 1)\delta]^nC(\lambda)] \right)^{\frac{1}{k-1}}, (k \geq 2) \).

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