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OBTAIN SUBCLASS OF ANALYTIC FUNCTIONS CONNECTED WITH CONVOLUTION OF POLYLOGARITHM FUNCTIONS

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ABSTRACT. In this work, we investigate some properties for the subclass $\mathcal{P}^n_{\beta,\lambda,\delta,b}(\phi(z))$ of analytic function related with the linear differential operator $\mathcal{R}^n_{\lambda,\delta}f(z)$ defined by polylogarithm functions. And also, we obtain coefficient inequalities, extreme points, radii of convexity and starlikeness, growth and distortion bounds for the subclass $\mathcal{P}^n_{\beta,\lambda,\delta,b}(\phi(z))$.

1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions of the form

$$\mathsf{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U} = z : \ |z| < 1$. For functions $\mathsf{f}(z)$ given by (1.1) and $\mathsf{g}(z)$ given by

$$\mathsf{g}(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

the convolution of $\mathsf{f}(z)$ and $\mathsf{g}(z)$ is defined by

$$(\mathsf{f} * \mathsf{g})(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

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If $\mathsf{f}(z)$ and $\mathsf{g}(z)$ are analytic in $\mathcal{U}$, we state that $\mathsf{f}(z)$ is subordinate to $\mathsf{g}(z)$, i.e. $\mathsf{f}(z) \prec \mathsf{g}(z)$, if a Schwarz function $w(z)$ exists, with $w(0) = 0$ and $|w| < 1$ such that $\mathsf{f}(z) = \mathsf{g}(w(z))$. Moreover, if the function $\mathsf{g}(z)$ is univalent in $\mathcal{U}$, then the above subordination is equivalence holds (see [7,8]). $\mathsf{f}(z) \prec \mathsf{g}(z)$ if and only if $\mathsf{f}(0) = \mathsf{g}(0)$, and $\mathsf{f}(\mathcal{U}) \subset \mathsf{g}(\mathcal{U})$.

For $\mathsf{f} \in \mathcal{A}$, Al-Oboudi [2] initiated the following differential operator:

$$\mathcal{D}_\delta^n \mathsf{f}(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k, \quad (n \in N_0 = N \cup \{0\}, \delta > 0 : z \in \mathcal{U}).$$

For $\mathsf{f} \in \mathcal{A}$, Ruscheweyh [9] established the following differential operator:

$$\mathcal{R}^\lambda \mathsf{f}(z) = \frac{z}{(1-z)^{\lambda+1}} * \mathsf{f}(z) = z + \sum_{k=2}^{\infty} \frac{(\lambda + k - 1)!}{\lambda!(k-1)!} a_k z^k, \quad (\lambda > -1).$$

Consider p is the class of functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + ...$ analytic in $\mathcal{U}$, $Re\{p(z)\} > 0$.

Consider the Polylogarithm function $E(n, \delta)$ given by

$$E(n, \delta) = \sum_{k=1}^{\infty} \frac{z^k}{[1 + (k-1)\delta]^n}.$$

Note that $E(-1, 1) = \frac{z}{(1-z)^2}$ for $k = 1, 2, 3, \ldots$ .is Koebe function. For further additional information about polylogarithms in theory of univalent functions see Ponnusamy and Sabapathy [7], K. Al Shaqsi and M. Daraus [4] and Ponnusamy [8].

Now we introduce a function $E^\kappa(n, \delta)$ given by

$$E(n, \delta) * E^\kappa(n, \delta) = \frac{z}{(1-z)^{\lambda+1}}, \lambda > -1, n \in Z,$$

thus obtaining the linear operator

$$\mathcal{R}_{\lambda,\delta}^n \mathsf{f}(z) = E^\kappa(n, \delta) * \mathsf{f}(z). \tag{1.2}$$

Now we come across the explicit form of the function

$$E^\kappa(n, \delta) = \sum_{k=1}^{\infty} [1 + (k-1)\delta]^n \frac{(\lambda + k - 1)!}{\lambda!(k-1)!} z^k.$$

From equation (1.2), we define

$$\mathcal{R}_{\lambda,\delta}^n \mathsf{f}(z) = \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n \frac{(\lambda + k - 1)!}{\lambda!(k-1)!} a_k z^k.$$

Note that $\mathcal{R}^n_{0,1} = \mathcal{D}^n, \mathcal{R}^0_{\lambda,\delta} = \mathcal{D}^\lambda$ which give the Salagean differential operator [10] and Ruscheweyh differential operator [9] respectively. It is obvious that the operator includes two well known operators. Also note that $\mathcal{R}^0_{0,\delta} = \mathsf{f}(z)$ and $\mathcal{R}^0_{1,\delta} = \mathcal{R}^1_{0,1} = z\mathsf{f}'(z)$.

Definition 1.1. *We define $\mathcal{P}^n_{\beta,\lambda,\delta,b}(\phi(z))$ be the class of the functions $\mathsf{f} \in \mathcal{A}$ for which*

$$1 + \frac{1}{b}\left(\frac{z(\mathcal{R}^n_{\lambda,\delta}\mathsf{f}(z))'}{\mathcal{R}^n_{\lambda,\delta}\mathsf{f}(z)} - 1\right) - \beta\left|\frac{z(\mathcal{R}^n_{\lambda,\delta}\mathsf{f}(z))'}{\mathcal{R}^n_{\lambda,\delta}\mathsf{f}(z)} - 1\right| \prec \phi(z),$$

where $n, \lambda \in N_0, \beta > 0, \delta > 0, b > 0, \phi \in p; z \in \mathcal{U}$.

Definition 1.2. *For $\phi(z) = \frac{1+(1-2\alpha)z}{(1-z)}$, we define $\mathcal{P}^n_{\beta,\lambda,\delta,b}(\phi(z)) \equiv \mathcal{P}^n_{\beta,\lambda,\delta,b}(\alpha)$, $\mathsf{f} \in \mathcal{A}$ for which*

$$1 + \frac{1}{b}\left(\frac{z(\mathcal{R}^n_{\lambda,\delta}\mathsf{f}(z))'}{\mathcal{R}^n_{\lambda,\delta}\mathsf{f}(z)} - 1\right) - \beta\left|\frac{z(\mathcal{R}^n_{\lambda,\delta}\mathsf{f}(z))'}{\mathcal{R}^n_{\lambda,\delta}\mathsf{f}(z)} - 1\right| > \alpha, \tag{1.3}$$

where $n, \lambda \in N_0, \beta > 0, \delta > 0, b > 0, \phi \in p, 0 \leq \alpha \leq 1; z \in \mathcal{U}$.

Note that $\mathcal{P}^n_{0,\lambda,1,1}\phi(z) = \mathcal{K}^n_\lambda\phi(z), \mathcal{P}^n_{0,\lambda,1,1}(\alpha) = \mathcal{R}^n_\lambda(\alpha)$ studied by K. AlShaqsi and M. Darus [4], $\mathcal{P}^0_{0,0,1,1}\phi(z) = \mathcal{S}^*\phi(z)$ studied by Ma and Minda [6], $\mathcal{P}^0_{0,\lambda,1,1}(\alpha) = \mathcal{R}_\lambda(\alpha)$ introduced and studied by Ahuja [1] and $\mathcal{P}^n_{0,0,1,1}(\alpha) = \mathcal{R}_n(\alpha)$ introduced and studied by Kadioglu [3].

2. Main results

Theorem 2.1. *Let $\mathsf{f}(z)$ be defined by* (1.1). *Then $\mathsf{f} \in \mathcal{A}$ if and only if*

$$\sum_{k=2}^{\infty}(kb\beta - b\beta - k + 1 - b + b\alpha)[1 + (k-1)\delta]^n\mathcal{C}(\lambda)\,|a_k| \leq (1-\alpha)b, \tag{2.1}$$

where $\mathcal{C}(\lambda) = \frac{(k+\lambda-1)!}{\lambda!(k-1)!}, 0 \leq \alpha < 1, n \in N_0 = N \cup \{0\}, \delta > 0, b > 0, \lambda \geq 0; z \in \mathcal{U}$.

Proof. Suppose that the inequality (2.1) is true and $|z| < 1$. Then it is proved that the values of (1.3) lies in a circle centered at $w = 1$ whose radius is $(1-\alpha)b$. It is sufficient to show that $\left|1 + \frac{1}{b}\left(\frac{z(\mathcal{R}^n\lambda,\delta\mathsf{f}(z))'}{\mathcal{R}^n_{\lambda,\delta}\mathsf{f}(z)} - 1\right) - \beta\left|\frac{z(\mathcal{R}^n\lambda,\delta\mathsf{f}(z))'}{\mathcal{R}^n_{\lambda,\delta}\mathsf{f}(z)} - 1\right| - \alpha + 1\right| < 1$, which gives $\sum_{k=2}^{\infty}(kb\beta - b\beta - k + 1 - b + b\alpha)[1 + (k-1)\delta]^n\mathcal{C}(\lambda)\,|a_k| \leq (1-\alpha)b$. Hence the condition (2.1) holds.

Conversely, let us assume that the function f defined by (1.1) is in the class $\mathcal{P}^n_{\beta,\lambda,\delta,b}(\alpha)$, then $Re\left(1+\frac{1}{b}\left(\frac{z(\mathcal{R}^n_{\lambda,\delta}\mathsf{f}(z))'}{\mathcal{R}^n_{\lambda,\delta}\mathsf{f}(z)}-1\right)-\beta\left|\frac{z(\mathcal{R}^n_{\lambda,\delta}\mathsf{f}(z))'}{\mathcal{R}^n\lambda,\delta\mathsf{f}(z)}-1\right|\right)>\alpha$, by the value of z on the real axis, let $z\to 1^-$ through real values, we obtain the result

$$\sum_{k=2}^{\infty}(kb\beta-b\beta-k+1-b+b\alpha)[1+(k-1)\delta]^n\mathcal{C}(\lambda)\,|a_k|\leq(1-\alpha)b.$$

Hence the result is sharp for the function $\mathsf{f}(z)=z+\frac{(1-\alpha)b}{(kb\beta-b\beta-k+1-b+b\alpha)[1+(k-1)\delta]^n\mathcal{C}(\lambda)}$ □

Theorem 2.2. *Let*

$$\mathsf{f}_1(z)=z,\mathsf{f}_k(z)=z+\sum_{k=2}^{\infty}\eta_k\frac{(1-\alpha)b}{\psi(\lambda)}z^k,k=2,3,...,$$

where $\psi(\lambda)=\sum_{k=2}^{\infty}(kb\beta-b\beta-k+1-b+b\alpha)[1+(k-1)\delta]^n\mathcal{C}(\lambda)$. *Then* $\mathsf{f}\in\mathcal{P}^n_{\beta,\lambda,\delta,b}(\alpha)$ *if and only if it can be expressed in the form* $\mathsf{f}(z)=\sum_{k=1}^{\infty}\eta_k\mathsf{f}_k(z)$ *where* $\eta_k>0$ *and* $\sum_{k=1}^{\infty}\eta_k=1$.

Proof. Let

$$\begin{aligned}\mathsf{f}(z)&=\sum_{k=1}^{\infty}\eta_k\mathsf{f}_k(z)\\&=z+\sum_{k=2}^{\infty}\eta_k\frac{(1-\alpha)b}{\psi(\lambda)}z^k\\&=\sum_{k=2}^{\infty}\eta_k\frac{(1-\alpha)b}{\psi(\lambda)}(\psi(\lambda)\\&=(1-\alpha)b\sum_{k=1}^{\infty}\eta_k\\&=(1-\alpha)b(1-\eta_1)<(1-\alpha)b\end{aligned}$$

which shows that $\mathsf{f}\in\mathcal{P}^n_{\beta,\lambda,\delta,b}(\alpha)$.

Conversely, suppose that $\mathsf{f}\in\mathcal{P}^n_{\beta,\lambda,\delta,b}(\alpha)$. Since $|a_k|\leq\frac{(1-\alpha)b}{\psi(\lambda)},k=2,3,....$ Let $\eta_k\leq\frac{\psi(\lambda)}{(1-\alpha)b},\eta_1=1-\sum_{k=2}^{\infty}\eta_k$. Then we obtain $\mathsf{f}(z)=\sum_{k=1}^{\infty}\eta_k\mathsf{f}_k(z)$. □

Theorem 2.3. *Let* $\mathsf{f}(z) = z + \sum_{k=2}^{\infty} |a_k| z^k$, $\mathsf{f} \in \mathcal{P}^n_{\beta,\lambda,\delta,b}(\alpha)$*, then for* $|z| = r$*, we have*

$$r - \frac{(1-\alpha)b}{(\beta b + \alpha b - b - 1)(1+\delta)^n(\lambda+1)} r^2 \leq |\mathsf{f}(z)| \leq r + \frac{(1-\alpha)b}{((\beta b + \alpha b - b - 1)(1+\delta)^n(\lambda+1)} r^2$$

and

$$1 - \frac{2(1-\alpha)b}{(\beta b + \alpha b - b - 1)(1+\delta)^n(\lambda+1)} r \leq |\mathsf{f}'(z)| \leq 1 + \frac{2(1-\alpha)b}{(\beta b + \alpha b - b - 1)(1+\delta)^n(\lambda+1)} r.$$

Putting $\beta = 0, \delta = 1$ in the above theorem the result obtained is analogue the results of M. Thirucheran, M. Vinothkumar and T. Stalin [5].

Corollary 2.1. *Let* $\mathsf{f}(z) = z + \sum_{k=2}^{\infty} |a_k| z^k$, $\mathsf{f} \in \mathcal{P}^n_{\beta,\lambda,\delta,b}(\alpha)$*, then for* $|z| = r$ *we have*

$$r - \frac{(1-\alpha)b}{(\alpha b - b - 1)(2)^n(\lambda+1)} r^2 \leq |\mathsf{f}(z)| \leq r + \frac{(1-\alpha)b}{(\alpha b - b - 1)(2)^n(\lambda+1)} r^2$$

and

$$1 - \frac{2(1-\alpha)b}{(\alpha b - b - 1)(2)^n(\lambda+1)} r \leq |\mathsf{f}'(z)| \leq 1 + \frac{2(1-\alpha)b}{(\alpha b - b - 1)(2)^n(\lambda+1)} r.$$

Putting $\beta = 0, \delta = 1, b = 1$ in the above theorem the result obtained is analogue to the results of K. AlShaqsi and M. Darus [4].

Corollary 2.2. *Let* $\mathsf{f}(z) = z + \sum_{k=2}^{\infty} |a_k| z^k$, $\mathsf{f} \in \mathcal{P}^n_{\beta,\lambda,\delta,b}(\alpha)$*, then for* $|z| = r$ *we have*

$$r - \frac{(1-\alpha)}{(\alpha - 2)(2)^n(\lambda+1)} r^2 \leq |\mathsf{f}(z)| \leq r + \frac{(1-\alpha)}{(\alpha - 2)(2)^n(\lambda+1)} r^2$$

and

$$1 - \frac{2(1-\alpha)}{(\alpha - 2)(2)^n(\lambda+1)} r \leq |\mathsf{f}'(z)| \leq 1 + \frac{2(1-\alpha)}{(\alpha - 2)(2)^n(\lambda+1)} r.$$

Theorem 2.4. *The class* $\mathcal{P}^n_{\beta,\lambda,\delta,b}(\alpha)$ *is convex.*

Proof. Let the function $\mathsf{f}_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k, a_{k,j} \geq 0, j = 1, 2$ lies in the class $\mathsf{f} \in \mathcal{P}^n_{\beta,\lambda,\delta,b}(\alpha)$. It is sufficient to prove that $h(z) = (\gamma+1)\mathsf{f}_1(z) - \gamma \mathsf{f}_2(z), 0 \leq z \leq 1$, the class $\mathrm{f} \in \mathcal{P}^n_{\beta,\lambda,\delta,b}(\alpha)$. Since $h(z) = z + \sum_{k=2}^{\infty} \left[(1+\gamma)a_{k,1} - \gamma a_{k,2}\right] z^k$, which implies that

$$\begin{aligned}\sum_{k=2}^{\infty}&(kb\beta - b\beta - k + 1 - b + b\alpha)[1+(k-1)\delta]^n C(\lambda)(1+\gamma)a_{k,1}\\ &+ (kb\beta - b\beta - k + 1 - b + b\alpha)[1+(k-1)\delta]^n C(\lambda)(\gamma)a_{k,2}\\ &\leq (1+\gamma)(1-\alpha)b + \gamma(1-\alpha)b\\ &\leq (1-\alpha)b\end{aligned}$$

therefore $h \in \mathcal{P}^n_{\beta,\lambda,\delta,b}(\alpha)$. Hence $\mathcal{P}^n_{\beta,\lambda,\delta,b}(\alpha)$ is convex. □

Theorem 2.5. *Let* $\mathrm{f} \in \mathcal{P}^n_{\beta,\lambda,\delta,b}(\alpha)$*, then* f *is close-to-convex of order* $\sigma(0 \leq \sigma < 1)$ *in the disc* $|z| < r_1$*, where* $r_1 := \left(\frac{(1-\sigma)[(kb\beta - b\beta - k + 1 - b + b\alpha)[1+(k-1)\delta]^n C(\lambda)]}{(k)(1-\alpha)b}\right)^{\frac{1}{k-1}}$.

Theorem 2.6. *Let* $\mathrm{f} \in \mathcal{P}^n_{\beta,\lambda,\delta,b}(\alpha)$*, then* f *is starlike of order* $\sigma(0 \leq \sigma < 1)$ *in the disc* $|z| < r_2$*, where* $r_2 := inf \left(\frac{(1-\sigma)[(kb\beta - b\beta - k + 1 - b + b\alpha)[1+(k-1)\delta]^n C(\lambda)]}{(k-\sigma)(1-\alpha)b}\right)^{\frac{1}{k-1}}, (k \geq 2)$.

Theorem 2.7. *Let* $\mathrm{f} \in \mathcal{P}^n_{\beta,\lambda,\delta,b}(\alpha)$*, then* f *is convex of order* $\sigma(0 \leq \sigma < 1)$ *in the disc* $|z| < r_3$*, where* $r_3 := inf \left(\frac{(1-\sigma)[(kb\beta - b\beta - k + 1 - b + b\alpha)[1+(k-1)\delta]^n C(\lambda)]}{k(k-\sigma)(1-\alpha)b}\right)^{\frac{1}{k-1}}, (k \geq 2)$.

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