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INCOMPARABILITY GRAPH OF THE SPECIAL LATTICE $L_N^{2^2}$

A. DABHOLE[1], K. GHADLE, AND G. ROKADE

ABSTRACT. In the present research paper, we have studied the incomparability graph of the lattice $L_n^{2^2}$. In said graphs, we found a dominating set and the order of a graph. We have expressed the cardinality of neighbourhood of an atom making use of a expansion formula. We have also found the largest independent set of the mentioned graph.

1. INTRODUCTION

Duffus and Rival in [2] considered the covering graphs of posets. This graph has vertices which contain the elements of P and are adjacent if satisfying p covers q or q covers p. Allan and Laskar in [1] studied a domination and independent dominating numbers in a graph. A graph considered as finite, undirected, no multiple edges and with no loops. They proved, I is a maximal independent set if and only if I is an independent dominating set.

Filipov in [5] discussed a graph of a poset by defining an edge between the vertices p, q, making use of the comparable relation that is p, q have an edge if either $p \leq q$ or $q \leq p$. Nimbhorkar et al. in [7] discussed the graphs of a lattices L with 0. Authors defined the adjacency between the elements $p, q \in L$ as $p \wedge q = 0$. For a finite bounded lattice L, E. Estaji and K. Khashyarmanesh in [4] studied a natural generalization of the concept of zero-divisor graph for

[1] *corresponding author*

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a Boolean algebra. Also they discussed the properties of L with graph theoretic properties of $G(L)$.

J. Foxa and J. Pach in [6] defined incomparability graph with vertex set P with adjacency of two vertices of P if and only if they are incomparable. They emphasized the applications to extremal problems for string graphs and edge intersection patterns in topological graphs. Wasadikar and Dabhole in [8] discussed the dominating set and order of incomparability graph of lattices L_n and $L_n^{1^2}$. A. Dabhole et al. in [3] introduced a graph on the lattice having square of two prime factor of a positive integer. Also discussed some properties of the these graph called as zero divisor graph.

The present research article deals with the study of an incomparability graph of the lattice $L_n^{2^2}$. Let n be a positive integer. All the divisors of n gives a lattice under the divisibility relation. Here n is of the form $n = p_1^2 \times p_2^2 \times p_3 \times \cdots \times p_k$. We have denoted the incomparability graph of lattice $L_n^{2^2}$ by $\Gamma_I(L_n^{2^2})$. Let $a_1, a_2 \in \Gamma_I(L)$ are adjacent iff $a_1,\ a_2$ are incomparable. An element $p \in L$ is called an incomparable if there exist $q \in L$ such that $p \nmid q$ and $q \nmid p$.

We found an order of $\Gamma_I(L_n^{2^2})$, minimal dominating set and expressed the cardinality of $N(P)$ by using binomial expression. Also we found the largest independent set of $\Gamma_I(L_n^{2^2})$.

2. Order of $\Gamma_I(L_n^{2^2})$

Here we have found the total number of vertices in the incomparability graph of the lattice $L_n^{2^2}$. Order of graph is denoted by $\eta(\Gamma_I(L_n^{2^2}))$. We give a formula for the order of $\Gamma_I(L_n^{2^2})$ using binomial expression.

Theorem 2.1. *The order of* $\Gamma_I(L_n^{2^2})$ *is,*

$$\eta(\Gamma_I(L_n^{2^2})) = \sum_{r=1}^{k} {}^kC_r + 2\sum_{r=0}^{k-1} {}^{k-1}C_r + \sum_{r=0}^{k-3} {}^{k-2}C_r.$$

Proof. Let $p_i \in L_n^{2^2}$. As $p_i \nmid p_j$, for $i \neq j,\ \ i,j = 1, 2, \cdots, k$, we have $p_i - p_j$ are adjacent in $\Gamma_I(L_n^{2^2})$. So the vertex p_i or p_j is in $\Gamma_I(L_n^{2^2})$ and these are ${}^kC_1$ in numbers.

Now take the product of two primes from $\{p_1, p_2, ..., p_k\}$.

a) Let p_1, p_2 is not repeated, we have $p_i p_{q_1} \nmid p_j p_{q_1}$, for $i \neq q_1 \neq j$, $i, j, q_1 = 1, 2, \cdots, k$. Hence $p_i p_{q_1},\ p_j p_{q_1}$ are in $\Gamma_I(L_n^{2^2})$ and all these are ${}^kC_2$.

b) As $p_t^2 \nmid p_{z_1}p_{z_2}$, for $t \neq z_1 \neq z_2,\ z_1, z_2 = 3, 4, \cdots, k,\ t = 1, 2,$ we have these vertices p_t^2 is in $\Gamma_I(L_n^{2^2})$ and it is ${}^{k-1}C_0$ in number.

Similarly consider three primes from $\{p_1, p_2, \cdots, p_k\}$.

a) Consider p_1, p_2 is not repeated, $p_ip_{q_1}p_{q_2} \nmid p_jp_{q_1}p_{q_2}$, for $i \neq q_1 \neq q_2 \neq j,\ i, q_1, q_2, j = 1, 2, \cdots, k$. Hence $p_ip_{q_1}p_{q_2},\ p_jp_{q_1}p_{q_2}$ are in $\Gamma_I(L_n^{2^2})$ and are ${}^kC_3$ in number.

b) since $p_1^2p_{s_1} \nmid p_1^2p_{s_2}$, for $s_1 \neq s_2,\ s_1, s_2 = 2, 3, \cdots, k,$ we have these vertices $p_1^2p_{s_1},\ p_1^2p_{s_2}$ are adjacent in $\Gamma_I(L_n^{2^2})$. These are ${}^{k-1}C_1$ in number and the same we have for p_2^2. So the number of such vertices are $2^{k-1}C_1$.

Also consider the product of four elements from $\{p_1, p_2, \cdots, p_k\}$.

a) Consider p_1, p_2 is not repeated, $p_ip_{q_1}p_{q_2}p_{q_3} \nmid p_jp_{q_1}p_{q_2}p_{q_3}$, for $i \neq q_1 \neq q_2 \neq q_3 \neq j,\ i, q_1, q_2, q_3, j = 1, 2, \cdots, k$. Hence $p_ip_{q_1}p_{q_2}p_{q_3},\ p_jp_{q_1}p_{q_2}p_{q_3}$ are in $\Gamma_I(L_n^{2^2})$. These are ${}^kC_4$ in number.

b) since $p_1^2p_{s_1}p_{s_2} \nmid p_1^2p_{s_1}p_{s_3}$, for $s_1 \neq s_2 \neq s_3,\ \ s_1, s_2, s_3 = 2, 3, \cdots, k,$ we have these vertices $p_1^2p_{s_1}p_{s_2},\ p_1^2p_{s_1}p_{s_3}$ are adjacent in $\Gamma_I(L_n^{2^2})$. These are ${}^{k-1}C_2$ in number and the same we have for p_2^2. So the number of such vertices are $2^{k-1}C_2$.

c) As $p_1^2p_2^2 \nmid p_{s_1}p_{s_2}p_{s_3}p_{s_4}$, for $s_1 \neq s_2 \neq s_3 \neq s_4,\ s_1, s_2, s_3, s_4 = 3, 4, \cdots, k,$ we have $p_1^2p_2^2$ is in $\Gamma_I(L_n^{2^2})$ and are ${}^{k-2}C_0$ in number.

Continue the same for $k+1$ elements from $\{p_1, p_2, \cdots, p_k\}$.

a) Since $(p_1^2p_{s_1}p_{s_2}\cdots p_{s_j}) \nmid (p_1^2p_{s_1}p_{s_2}\cdots p_{s_{j-1}}p_{s_{j+1}})$, all s_j are distinct, $s_j = 2, 3, \cdots, k,\ j = 1, 2, \cdots, k-1,$ we have these vertices are adjacent in $\Gamma_I(L_n^{2^2})$. These are ${}^{k-1}C_{k-1}$ in number and same we have for p_2^2. So the number of such vertices are $2^{k-1}C_{k-1}$.

b) Also $(p_1^2p_2^2p_{s_1}p_{s_2}\cdots p_{s_j}) \nmid (p_1^2p_2^2p_{s_1}p_{s_2}\cdots p_{s_{j-1}}p_{s_{j+1}})$, all s_j are distinct, $s_j = 3, 4, \cdots, k,\ j = 1, 2, \cdots, k-3,$ we have these vertices $p_1^2p_2^2p_{s_1}p_{s_2}\cdots p_{s_j}$, $p_1^2p_2^2p_{s_1}p_{s_2}\cdots p_{s_{j-1}}p_{s_{j+1}}$ are in $\Gamma_I(L_n^{2^2})$ and these are ${}^{k-2}C_{k-3}$.

Therefore, the number of vertices in $\Gamma_I(L_n^{2^2})$ is,

$$\begin{aligned}{}^kC_1 + {}^kC_2 + \cdots + {}^kC_k + 2[{}^{k-1}C_0 + {}^{k-1}C_1 + \cdots + {}^{k-1}C_{k-1}] \\ + {}^{k-2}C_0 + {}^{k-2}C_1 + \cdots + {}^{k-2}C_{k-3}.\end{aligned}$$

Hence,

$$\eta(\Gamma_I(L_n^{2^2})) = \sum_{r=1}^{k} {}^kC_r + 2\sum_{r=0}^{k-1} {}^{k-1}C_r + \sum_{r=0}^{k-3} {}^{k-2}C_r.$$

□

3. NEIGHBOURHOODS OF ATOMS OF THE LATTICE IN THE INCOMPARABILITY GRAPH

In this section, we have found the neighbourhoods of a vertex in the incomparability graph of the lattice $L_n^{2^2}$. We consider the atoms p_1, p_2 of the lattice $L_n^{2^2}$. We find the number of elements in the neighbourhood of an atom in $\Gamma_I(L_n^{2^2})$. We prove that the cardinality of the neighbourhood of p_1, p_2, denoted by $N(p_1), N(p_2)$ can be expressed using a binomial expansion.

Theorem 3.1. *In the graph* $\Gamma_I(L_n^{2^2})$*,*

$$|N(p_1)| = \sum_{r=1}^{k-1} {}^{k-1}C_r + \sum_{r=0}^{k-2} {}^{k-2}C_r.$$

Proof. Let $p_1 \in L_n^{2^2}$, since $p_1 \nmid p_j$, for $j = 2, 3, \cdots, k$, we have $p_1 - p_j$ is an edge in $\Gamma_I(L_n^{2^2})$, i.e. each p_j is in the neighbourhood of p_1 and the number of such elements is ${}^{k-1}C_1$.

Now we take the product of two elements from $\{p_2, p_3, ..., p_k\}$. As $p_1 \nmid p_{j_1}p_{j_2}$, $j_1, j_2 = 2, 3, \cdots, k$, we have $p_1 - (p_{j_1}p_{j_2})$ is an edge in $\Gamma_I(L_n^{2^2})$. So $p_{j_1}p_{j_2}$ is in the neighbourhood of p_1.

a) If $p_{j_1} \neq p_{j_2}$ then $p_{j_1}p_{j_2}$ is in the neighbourhood of p_1 and the number such elements is ${}^{k-1}C_2$,

b) otherwise $p_{j_1} = p_{j_2}$ i.e. p_2^2 is in the neighbourhood of p_1 and the number such elements is ${}^{k-1}C_0$.

Similarly consider the product of three elements from $\{p_2, p_3, \cdots, p_k\}$. Since $p_1 \nmid p_{j_1}p_{j_2}p_{j_3}$, $j_1, j_2, j_3 = 2, 3, \cdots, k$, we have $p_1 - (p_{j_1}p_{j_2}p_{j_3})$ is an edge in $\Gamma_I(L_n^{2^2})$. So $p_{j_1}p_{j_2}p_{j_3}$ is in the neighbourhood of p_1.

a) If $p_{j_1}p_{j_2}p_{j_3}$ are distinct, then $p_{j_1}p_{j_2}p_{j_3}$ is in the neighbourhood of p_i and the number such elements is ${}^{k-1}C_3$,

b) otherwise $p_2^2 p_j$ $j = 3, 4, ..., k$ is in the neighbourhood of p_1 and the number such elements is ${}^{k-2}C_1$.

Continuing for the product of $k-1$ elements from $\{p_2, p_3, \cdots, p_k\}$. Since $p_1 \nmid p_{j_1}p_{j_2}\cdots p_{j_r}$, for $j_r = 2, 3, \cdots, k$, $r = 1, 2, \cdots, k-1$, we have $p_1 - (p_{j_1}p_{j_2}\cdots p_{j_r})$ is an edge in $\Gamma_I(L_n^{2^2})$. So $p_{j_1}p_{j_2}\cdots p_{j_r}$ is in the neighbourhood of p_1.

a) If all p_{j_r} are distinct, then number of $p_{j_1}p_{j_2}\cdots p_{j_r}$ element is ${}^{k-1}C_{k-1}$,

b) otherwise, it is ${}^{k-2}C_{k-3}$.

Lastly, as $p_1 \nmid (p_2^2 p_3 p_4 \cdots p_k)$, we have $p_2^2 p_3 p_4 \cdots p_k$ which is one neighbourhood of p_1 and it is counted by ${}^{k-2}C_{k-2}$ way.

Hence,

$$|N(p_1)| = \sum_{r=1}^{k-1} {}^{k-1}C_r + \sum_{r=0}^{k-2} {}^{k-2}C_r.$$

□

4. A DOMINATING SET IN $\Gamma_I(L_n^{2^2})$

We have given a minimum dominating set in $\Gamma_I(L_n^{2^2})$ of the lattice $L_n^{2^2}$. We shown that any two vertices q_1, q_2 of the graph $\Gamma_I(L_n^{2^2})$ form a dominating set.

Theorem 4.1. *The dominating set in $\Gamma_I(L_n^{2^2})$ is $D = \{q_1, q_2\}$, where $q_1 \times q_2 = n$ and q_1, q_2 have no prime factors common.*

Proof. Let $V(\Gamma_I(L_n^{2^2}))$ denote the vertex set in $\Gamma_I(L_n^{2^2})$. Now suppose that $a \in V(\Gamma_I(L_n^{2^2}))$,

a) If $q_1 \mid a$, then $q_2 \mid\mid a.$

If not then i) $a \mid q_2$ or ii) $q_2 \mid a$ i.e. $a \leq q_2$ or $q_2 \leq a$

i) if $a \mid q_2$ and $q_1 \mid a$, then $q_1 \mid q_2$ a contradiction.

ii) if $q_2 \mid a$ and $q_1 \mid a$ which implies that $(q_1 q_2) \mid a$ i.e. $n \mid a$ which is a contradiction to $a < n$. Thus $a - n_q$ is an edge in $\Gamma_I(L_n^{2^2})$.

b) If $a \mid q_1,$ then $a \nmid q_2.$

As if $q_2 \mid a$ and $a \mid q_1 \implies q_2 \mid q_1$ becomes a contradiction, hence $q_2 \nmid a.$ Thus $a \mid\mid q_2$ then $a - q_2$ is an edge in $\Gamma_I(L_n^{2^2})$. If both a) and b) fail, then $a \mid\mid q_1$ then $a - q_1$ is an edge in $\Gamma_I(L_n^{2^2})$. □

5. AN INDEPENDENT SET IN $\Gamma_I(L_n^{2^2})$

In this section we found the largest independent set of $\Gamma_I(L_n^{2^2})$.

Theorem 5.1. *The largest independent set in $\Gamma_I(L_n^{2^2})$ contains $k + 1$ elements.*

Proof. Let $p_q \in L_n^{2^2}$, $q = 1, 2, ..., k$. As $p_{q_1} \,|\, (p_{q_1} p_{q_2}),\ (p_{q_1} p_{q_2}) \,|\, (p_{q_1} p_{q_2} p_{q_3})$ continuing like this for $k - 2$ times

$(p_{q_1}p_{q_2}...p_{q_m}) \mid (p_{q_1}p_{q_2}...p_{q_m}p_{q_{m+1}}),\ q_m = 1, 2, ..., k$, we have these vertices $p_{q_1}, p_{q_1}p_{q_2}, p_{q_1}p_{q_2}p_{q_3}, \cdots\ p_{q_1}p_{q_2}p_{q_3}\cdots p_{q_{m+1}}$ are not adjacent to each other in $\Gamma_I(L_n^{2^2})$.

So all these vertices $(p_{q_1}),\ (p_{q_1}p_{q_2}), ...(p_{q_1}p_{q_2}...p_{q_{m+1}})$ form an independent set containing $k+1$ vertices.

Hence, the largest independent set in $\Gamma_I(L_n^{2^2})$ contains $k+1$ elements. □

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DEPARTMENT OF MATHEMATICS
MIT ENGINEERING COLLEGE
BEED BYPASS ROAD, AURANGABAD, (M.S.) INDIA.
Email address: amit_dabhole22@rediffmail.com

DEPARTMENT OF MATHEMATICS
DR. BABASAHEB AMBEDKAR MARATHWADA UNIVERSITY
AURANGABAD-431004, (M.S.) INDIA.
Email address: drkp.ghadle@gmail.com

DEPARTMENT OF MATHEMATICS
J.E.S. COLLEGE
JALNA, (M.S.) INDIA.
Email address: ganeshrokade01@gmail.com