INCOMPARABILITY GRAPH OF THE SPECIAL LATTICE $L_{2n}^2$

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ABSTRACT. In the present research paper, we have studied the incomparability graph of the lattice $L_{2n}^2$. In said graphs, we found a dominating set and the order of a graph. We have expressed the cardinality of neighbourhood of an atom making use of an expansion formula. We have also found the largest independent set of the mentioned graph.

1. INTRODUCTION

Duffus and Rival in [2] considered the covering graphs of posets. This graph has vertices which contain the elements of $P$ and are adjacent if satisfying $p$ covers $q$ or $q$ covers $p$. Allan and Laskar in [1] studied a domination and independent dominating numbers in a graph. A graph considered as finite, undirected, no multiple edges and with no loops. They proved, $I$ is a maximal independent set if and only if $I$ is an independent dominating set.

Filipov in [5] discussed a graph of a poset by defining an edge between the vertices $p, q$, making use of the comparable relation that is $p, q$ have an edge if either $p \leq q$ or $q \leq p$. Nimbhorkar et al. in [7] discussed the graphs of a lattices $L$ with 0. Authors defined the adjacency between the elements $p, q \in L$ as $p \land q = 0$. For a finite bounded lattice $L$, E. Estaji and K. Khashyarmanesh in [4] studied a natural generalization of the concept of zero-divisor graph for

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a Boolean algebra. Also they discussed the properties of $L$ with graph theoretic properties of $G(L)$.

J. Foxa and J. Pach in [6] defined incomparability graph with vertex set $P$ with adjacency of two vertices of $P$ if and only if they are incomparable. They emphasized the applications to extremal problems for string graphs and edge intersection patterns in topological graphs. Wasadikar and Dabhole in [8] discussed the dominating set and order of incomparability graph of lattices $L_n$ and $L_n^{12}$. A. Dabhole et al. in [3] introduced a graph on the lattice having square of two prime factor of a positive integer. Also discussed some properties of these graph called as zero divisor graph.

The present research article deals with the study of an incomparability graph of the lattice $L_n^{2^2}$. Let $n$ be a positive integer. All the divisors of $n$ gives a lattice under the divisibility relation. Here $n$ is of the form $n = p_1^2 \times p_2^2 \times p_3 \times \cdots \times p_k$. We have denoted the incomparability graph of lattice $L_n^{2^2}$ by $\Gamma_I(L_n^{2^2})$. Let $a_1, a_2 \in \Gamma_I(L)$ are adjacent iff $a_1, a_2$ are incomparable. An element $p \in L$ is called an incomparable if there exist $q \in L$ such that $p \nmid q$ and $q \nmid p$.

We found an order of $\Gamma_I(L_n^{2^2})$, minimal dominating set and expressed the cardinality of $N(P)$ by using binomial expression. Also we found the largest independent set of $\Gamma_I(L_n^{2^2})$.

2. ORDER OF $\Gamma_I(L_n^{2^2})$

Here we have found the total number of vertices in the incomparability graph of the lattice $L_n^{2^2}$. Order of graph is denoted by $\eta(\Gamma_I(L_n^{2^2}))$. We give a formula for the order of $\Gamma_I(L_n^{2^2})$ using binomial expression.

Theorem 2.1. The order of $\Gamma_I(L_n^{2^2})$ is,

$$\eta(\Gamma_I(L_n^{2^2})) = \sum_{r=1}^{k} k^r C_r + 2 \sum_{r=0}^{k-1} (k-1)^r C_r + \sum_{r=0}^{k-3} (k-3)^r C_r.$$ 

Proof. Let $p_i \in L_n^{2^2}$. As $p_i \nmid p_j$, for $i \neq j$, $i, j = 1, 2, \cdots, k$, we have $p_i - p_j$ are adjacent in $\Gamma_I(L_n^{2^2})$. So the vertex $p_i$ or $p_j$ is in $\Gamma_I(L_n^{2^2})$ and these are $kC_1$ in numbers.

Now take the product of two primes from $\{p_1, p_2, \ldots, p_k\}$.

a) Let $p_1, p_2$ is not repeated, we have $p_ip_{q_1} \nmid p_jp_{q_1}$, for $i \neq q_1 \neq j$, $i, j, q_1 = 1, 2, \cdots, k$. Hence $p_ip_{q_1}$, $p_jp_{q_1}$ are in $\Gamma_I(L_n^{2^2})$ and all these are $kC_2$. 

(b) As $p_i^2 \nmid p_z p_{z_2}$, for $t \neq z_1 \neq z_2$, $z_1, z_2 = 3, 4, \ldots, k$, $t = 1, 2$, we have these vertices $p_i^2$ is in $\Gamma_i(L_n^{2^k})$ and it is $k^{-1}C_0$ in number.

Similarly consider three primes from $\{p_1, p_2, \ldots, p_k\}$.

a) Consider $p_1, p_2$ is not repeated, $p_1 p_1 p_2 \nmid p_1 p_1 p_2$, for $i \neq q_1 \neq q_2 \neq j$, $i, q_1, q_2, j = 1, 2, \ldots, k$. Hence $p_1 p_1 p_2, p_2 p_1 p_2$ are in $\Gamma_i(L_n^{2^k})$ and are $k^{-1}C_0$ in number.

b) since $p_i^2 p_{s_1} \nmid p_i^2 p_{s_2}$, for $s_1 \neq s_2$, $s_1, s_2 = 2, 3, \ldots, k$, we have these vertices $p_i^2 p_{s_1}, p_i^2 p_{s_2}$ are adjacent in $\Gamma_i(L_n^{2^k})$. These are $k^{-1}C_1$ in number and the same we have for $p_i^2$. So the number of such vertices are $2k^{-1}C_2$.

Also consider the product of four elements from $\{p_1, p_2, \ldots, p_k\}$.

a) Consider $p_1, p_2$ is not repeated, $p_1 p_1 p_2 p_2 \nmid p_1 p_1 p_2 p_2$, for $i \neq q_1 \neq q_2 \neq q_3 \neq j$, $i, q_1, q_2, q_3, j = 1, 2, \ldots, k$. Hence $p_1 p_1 p_2 p_2, p_2 p_1 p_2 p_2$ are in $\Gamma_i(L_n^{2^k})$. These are $k^{-1}C_0$ in number.

b) since $p_i^2 p_{s_1} p_{s_2} \nmid p_i^2 p_{s_1} p_{s_3}$, for $s_1 \neq s_2 \neq s_3$, $s_1, s_2, s_3 = 2, 3, \ldots, k$, we have these vertices $p_i^2 p_{s_1} p_{s_2}, p_i^2 p_{s_1} p_{s_3}$ are adjacent in $\Gamma_i(L_n^{2^k})$. These are $k^{-1}C_2$ in number and the same we have for $p_i^2$. So the number of such vertices are $2k^{-1}C_2$.

c) As $p_i^2 p_j^2 \nmid p_i p_j p_{s_1} p_{s_2}$, for $s_1 \neq s_2 \neq s_3 \neq s_4$, $s_1, s_2, s_3, s_4 = 3, 4, \ldots, k$, we have $p_i^2 p_j^2$ is in $\Gamma_i(L_n^{2^k})$ and are $k^{-2}C_0$ in number.

Continue the same for $k + 1$ elements from $\{p_1, p_2, \ldots, p_k\}$.

a) Since $(p_i^2 p_{s_1} \cdots p_{s_j}) \nmid (p_i^2 p_{s_1} \cdots p_{s_{j-1}} p_{s_{j+1}})$, all $s_j$ are distinct, $s_j = 2, 3, \ldots, k$, $j = 1, 2, \ldots, k - 1$, we have these vertices are adjacent in $\Gamma_i(L_n^{2^k})$. These are $k^{-1}C_{k-1}$ in number and same we have for $p_i^2$. So the number of such vertices are $2k^{-1}C_{k-2}$.

b) Also $(p_i^2 p_{s_1} p_{s_2} \cdots p_{s_j}) \nmid (p_i^2 p_{s_1} p_{s_2} \cdots p_{s_{j-1}} p_{s_{j+1}})$, all $s_j$ are distinct, $s_j = 3, 4, \ldots, k$, $j = 1, 2, \ldots, k - 3$, we have these vertices $p_i^2 p_j^2 p_{s_1} p_{s_2} \cdots p_{s_j}$, $p_j^2 p_{s_1} p_{s_2} \cdots p_{s_{j-1}} p_{s_{j+1}}$ are in $\Gamma_i(L_n^{2^k})$ and these are $k^{-2}C_{k-3}$.

Therefore, the number of vertices in $\Gamma_i(L_n^{2^k})$ is,

$$k C_1 + k C_2 + \cdots + k C_k + 2^{[k^{-1}C_0 + k^{-1}C_1 + \cdots + k^{-1}C_{k-1}]}
\quad + k^{-2}C_0 + k^{-2}C_1 + \cdots + k^{-2}C_{k-3}.$$ 

Hence,

$$\eta(\Gamma_i(L_n^{2^k})) = \sum_{r=1}^{k} k^r C_r + 2 \sum_{r=0}^{k-1} k^{-1}C_r + \sum_{r=0}^{k-3} k^{-2}C_r.$$

\qed
3. **Neighbourhoods of atoms of the lattice in the incomparability graph**

In this section, we have found the neighbourhoods of a vertex in the incomparability graph of the lattice $L_n^{2^2}$. We consider the atoms $p_1, p_2$ of the lattice $L_n^{2^2}$. We find the number of elements in the neighbourhood of an atom in $\Gamma(L_n^{2^2})$. We prove that the cardinality of the neighbourhood of $p_1, p_2$, denoted by $N(p_1), N(p_2)$ can be expressed using a binomial expansion.

**Theorem 3.1.** *In the graph $\Gamma(L_n^{2^2})$,*

$$|N(p)| = \sum_{r=1}^{k-1} C_r^k + \sum_{r=0}^{k-2} C_r^k.$$

**Proof.** Let $p_1 \in L_n^{2^2}$, since $p_1 \not\models p_j$, for $j = 2, 3, \ldots, k$, we have $p_1 - p_j$ is an edge in $\Gamma(L_n^{2^2})$, i.e. each $p_j$ is in the neighbourhood of $p_1$ and the number such elements is $kC_1$.

Now we take the product of two elements from $\{p_2, p_3, \ldots, p_k\}$. As $p_1 \not\models p_j, p_j$, $j = 2, 3, \ldots, k$, we have $p_1 - (p_j, p_j)$ is an edge in $\Gamma(L_n^{2^2})$. So $p_j, p_j$ is in the neighbourhood of $p_1$.

a) If $p_j \neq p_j$, then $p_j, p_j$ is in the neighbourhood of $p_1$ and the number such elements is $kC_2$.

b) otherwise $p_j = p_j$ i.e. $p_j^2$ is in the neighbourhood of $p_1$ and the number such elements is $kC_0$.

Similarly consider the product of three elements from $\{p_2, p_3, \ldots, p_k\}$. Since $p_1 \not\models p_j, p_j, p_j$, $j = 2, 3, \ldots, k$, we have $p_1 - (p_j, p_j, p_j)$ is an edge in $\Gamma(L_n^{2^2})$. So $p_j, p_j, p_j$ is in the neighbourhood of $p_1$.

a) If $p_j, p_j, p_j$ are distinct, then $p_j, p_j, p_j$ is in the neighbourhood of $p_1$ and the number such elements is $kC_3$.

b) otherwise $p_j^2, p_j$ $j = 3, 4, \ldots, k$ is in the neighbourhood of $p_1$ and the number such elements is $kC_3$.

Continuing for the product of $k - 1$ elements from $\{p_2, p_3, \ldots, p_k\}$. Since $p_1 \not\models p_j, p_j, \ldots, p_j$, for $j = 2, 3, \ldots, k$, we have $p_1 - (p_j, p_j, \ldots, p_j)$ is an edge in $\Gamma(L_n^{2^2})$. So $p_j, p_j, \ldots, p_j$ is in the neighbourhood of $p_1$.

a) If all $p_j$ are distinct, then number of $p_j, p_j, \ldots, p_j$ element is $k^{-1}C_{k-1}$,

b) otherwise, it is $k^{-2}C_{k-3}$. 
Lastly, as \( p_1 \nmid (p_2^2 p_3 p_4 \cdots p_k) \), we have \( p_2^2 p_3 p_4 \cdots p_k \) which is one neighbourhood of \( p_1 \) and it is counted by \( k-2 C_{k-2} \) way.

Hence,
\[
|N(p_1)| = \sum_{r=1}^{k-1} k-1 C_r + \sum_{r=0}^{k-2} k-2 C_r.
\]

\[ \square \]

4. A DOMINATING SET IN \( \Gamma_1(L_n^{2^2}) \)

We have given a minimum dominating set in \( \Gamma_1(L_n^{2^2}) \) of the lattice \( L_n^{2^2} \). We shown that any two vertices \( q_1, q_2 \) of the graph \( \Gamma_1(L_n^{2^2}) \) form a dominating set.

**Theorem 4.1.** The dominating set in \( \Gamma_1(L_n^{2^2}) \) is \( D = \{q_1, q_2\} \), where \( q_1 \times q_2 = n \) and \( q_1, q_2 \) have no prime factors common.

**Proof.** Let \( V(\Gamma_1(L_n^{2^2})) \) denote the vertex set in \( \Gamma_1(L_n^{2^2}) \). Now suppose that \( a \in V(\Gamma_1(L_n^{2^2})) \),

a) If \( q_1 \mid a \), then \( q_2 \mid | a \).

- If not then i) \( a \mid q_2 \) or ii) \( q_2 \mid a \) i.e. \( a \leq q_2 \) or \( q_2 \leq a \)

  i) if \( a \mid q_2 \) and \( q_1 \mid a \), then \( q_1 \mid q_2 \) a contradiction.

  ii) if \( q_2 \mid a \) and \( q_1 \mid a \) which implies that \( (q_1q_2) \mid a \) i.e. \( n \mid a \) which is a contradiction to \( a < n \). Thus \( a - n_q \) is an edge in \( \Gamma_1(L_n^{2^2}) \).

b) If \( a \mid q_1 \), then \( a \nmid q_2 \).

As if \( q_2 \mid a \) and \( a \mid q_1 \implies q_2 \mid q_1 \) becomes a contradiction, hence \( q_2 \nmid a \).

Thus \( a \mid q_2 \) then \( a - q_2 \) is an edge in \( \Gamma_1(L_n^{2^2}) \). If both a) and b) fail, then \( a \mid q_1 \) then \( a - q_1 \) is an edge in \( \Gamma_1(L_n^{2^2}) \). \[ \square \]

5. A INDEPENDENT SET IN \( \Gamma_1(L_n^{2^2}) \)

In this section we found the largest independent set of \( \Gamma_1(L_n^{2^2}) \).

**Theorem 5.1.** The largest independent set in \( \Gamma_1(L_n^{2^2}) \) contains \( k + 1 \) elements.

**Proof.** Let \( p_q \in L_n^{2^2} \), \( q = 1, 2, \ldots, k \). As \( p_{q_1} \mid (p_{q_1} p_{q_2}), (p_{q_1} p_{q_2}) \mid (p_{q_1} p_{q_2} p_{q_3}) \) continuing like this for \( k - 2 \) times
\(\{p_{q_1}p_{q_2}\ldots p_{q_m}\} \mid \{p_{q_1}p_{q_2}\ldots p_{q_m}p_{q_{m+1}}\}, \quad q_m = 1, 2, \ldots, k,\) we have these vertices 

\(p_{q_1}, p_{q_1}p_{q_2}, p_{q_1}p_{q_2}p_{q_3}, \ldots p_{q_1}p_{q_2}p_{q_3}\cdots p_{q_{m+1}}\) are not adjacent to each other in \(\Gamma_I(L_n^{2^2})\).

So all these vertices \((p_{q_1}), (p_{q_1}p_{q_2}), \ldots(p_{q_1}p_{q_2}\ldots p_{q_{m+1}})\) form an independent set containing \(k + 1\) vertices.

Hence, the largest independent set in \(\Gamma_I(L_n^{2^2})\) contains \(k + 1\) elements. \(\square\)

**References**


