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LAPLACIAN POLYNOMIAL AND LAPLACIAN ENERGY OF SOME BIPARTITE CLUSTER GRAPHS

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ABSTRACT. The graphs with large number of edges are referred as cluster graphs [1]. In [5] the ordinary energy of graphs obtained from complete bipartite graph is considered. In this article we consider the Laplacian polynomial of graphs obtained from complete bipartite graph by deleting the edges. Laplacian energy for these graphs in certain cases are also obtained. For a bipartite graph the Laplacian spectra and signless Laplacian spectra is same [2].

1. INTRODUCTION

The Laplacian matrix and its eigenvalues of a graph can be used in many research areas of mathematics and also have a physical interpretation in various physical and chemical theories [4].

Let G be a simple graph with n vertices and m edges. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ be the vertex set of G.

The matrix $L_p(G) = D(G) - A(G)$, is called the *Laplacian matrix*, where $A(G)$ is the adjacency matrix and $D(G) = diag[d_1, d_2, \dots, d_n]$ is the diagonal degree matrix where $d_i = deg(v_i)$. It is also called as the matrix of admittance due to its role in electrical theory [4]. The *Laplacian polynomial* [3] of graph G is defined as $L(G : \mu) = det(\mu I - L_p(G))$ where I is the identity matrix of order n. The

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roots $\mu_1, \mu_2, \ldots, \mu_n$ of $L(G : \mu) = 0$ are called the Laplacian eigenvalues of G and can be ordered as $\mu_1 \geq \mu_2 \geq, \ldots, \geq \mu_n$.

Laplacian energy of G is defined as $LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$.

Let $K_{p,q}$ denote the complete bipartite graph on $n = p+q$ vertices with partite set V_1 and V_2 where $|V_1| = p$ and $|V_2| = q$.

Definition 1.1. *([5]) Let e_i, $i = 1, 2, \ldots, k$, $1 \leq k \leq min\{p, q\}$, be independent edges of the complete bipartite graph $K_{p,q}$, $p, q \geq 1$. The graph $Ka_{p,q}(k)$ is obtained by deleting e_i, $i = 1, 2, \ldots, k$ from $K_{p,q}$. In addition $Ka_{p,q}(0) \cong K_{p,q}$.*

2. LAPLACIAN POLYNOMIAL AND LAPLACIAN ENERGY OF $Ka_{p,q}(k)$

Theorem 2.1. *For $p, q \geq 1$ and $0 \leq k \leq min\{p, q\}$,*

$$\begin{aligned} L(Ka_{p,q}(k) : \mu) &= \mu(\mu - p)^{q-k-1}(\mu - q)^{p-k-1}[(\mu - q + 1)(\mu - p + 1)]^{k-1} \\ &\quad \big\{\mu^3 - 2(p + q - 1)\mu^2 + [(p + q)(p + q - 3) + pq + 2k]\mu \\ &\quad +(p + q)(p + q - pq - k)\big\}. \end{aligned} \tag{2.1}$$

Proof. Without loss of generality we partition the vertex set of the complete bipartite graph $K_{p,q}$ into two disjoint sets $V_1 = \{u_1, u_2, \ldots, u_p\}$ and $V_2 = \{v_1, v_2, \ldots, v_q\}$ such that no two vertices in either sets are adjacent to each other. Assume that the independent edges e_i connect the vertices u_i and v_i where $i = 1, 2, \ldots, k$.

In order to make the following determinant more compact, the auxillary quantities X and Y are introduced:

$$X = \mu - q + 1, \qquad Y = \mu - p + 1.$$

Then the Laplacian polynomial of $K_{p,q}(k)$, $p, q \geq 1$ is the determinant of the following form.

$$
\begin{array}{c|cccccccccccccc|}
u_1 & X & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
u_2 & 0 & X & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
u_k & 0 & 0 & \cdots & X & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 & 1 & \cdots & 1 \\
u_{k+1} & 0 & 0 & \cdots & 0 & X-1 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
u_p & 0 & 0 & \cdots & 0 & 0 & \cdots & X-1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
v_1 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & Y & 0 & \cdots & 0 & 0 & \cdots & 0 \\
v_2 & 1 & 0 & \cdots & 1 & 1 & \cdots & 1 & 0 & Y & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
v_k & 1 & 1 & \cdots & 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & Y & 0 & \cdots & 0 \\
v_{k+1} & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & Y-1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
v_q & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & Y-1
\end{array} . \tag{2.2}
$$

Add the rows $2, 3, \ldots, n$ of (2.2) to its first row to obtain:

$$
\begin{array}{c|cccccccccccccc|}
u_1 & \mu & \mu & \cdots & \mu & \mu & \cdots & \mu & \mu & \mu & \cdots & \mu & \mu & \cdots & \mu \\
u_2 & 0 & X & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
u_k & 0 & 0 & \cdots & X & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 & 1 & \cdots & 1 \\
u_{k+1} & 0 & 0 & \cdots & 0 & X-1 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
u_p & 0 & 0 & \cdots & 0 & 0 & \cdots & X-1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
v_1 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & Y & 0 & \cdots & 0 & 0 & \cdots & 0 \\
v_2 & 1 & 0 & \cdots & 1 & 1 & \cdots & 1 & 0 & Y & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
v_k & 1 & 1 & \cdots & 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & Y & 0 & \cdots & 0 \\
v_{k+1} & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & Y-1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
v_q & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & Y-1
\end{array} . \tag{2.3}
$$

Subtract first column of (2.3) from its $2,3,\ldots,n$ columns to obtain

$$
\begin{array}{c}
u_1\\ u_2\\ \vdots\\ u_k\\ u_{k+1}\\ \vdots\\ u_p\\ v_1\\ v_2\\ \vdots\\ v_k\\ v_{k+1}\\ \vdots\\ v_q
\end{array}
\begin{vmatrix}
\mu & 0 & .. & 0 & 0 & .. & 0 & 0 & 0 & .. & 0 & 0 & .. & 0\\
0 & X & .. & 0 & 0 & .. & 0 & 1 & 0 & .. & 1 & 1 & .. & 1\\
\vdots & \vdots & .. & \vdots & \vdots & .. & \vdots & \vdots & \vdots & .. & \vdots & \vdots & .. & \vdots\\
0 & 0 & .. & X & 0 & .. & 0 & 1 & 1 & .. & 0 & 1 & .. & 1\\
0 & 0 & .. & 0 & X-1 & .. & 0 & 1 & 1 & .. & 1 & 1 & .. & 1\\
\vdots & \vdots & .. & \vdots & \vdots & .. & \vdots & \vdots & \vdots & .. & \vdots & \vdots & .. & \vdots\\
0 & 0 & .. & 0 & 0 & .. & X-1 & 1 & 1 & .. & 1 & 1 & .. & 1\\
0 & 1 & .. & 1 & 1 & .. & 1 & Y & 0 & .. & 0 & 0 & .. & 0\\
1 & -1 & .. & 0 & 0 & .. & 0 & -1 & Y-1 & .. & -1 & -1 & .. & -1\\
\vdots & \vdots & .. & \vdots & \vdots & .. & \vdots & \vdots & \vdots & .. & \vdots & \vdots & .. & \vdots\\
1 & 0 & .. & -1 & 0 & .. & 0 & -1 & -1 & .. & Y-1 & -1 & .. & -1\\
1 & 0 & .. & 0 & 0 & .. & 0 & -1 & -1 & .. & -1 & Y-2 & .. & -1\\
\vdots & \vdots & .. & \vdots & \vdots & .. & \vdots & \vdots & \vdots & .. & \vdots & \vdots & .. & \vdots\\
1 & 0 & .. & 0 & 0 & .. & 0 & -1 & -1 & .. & -1 & -1 & .. & Y-2
\end{vmatrix}. \tag{2.4}
$$

Expression (2.4) is equivalent to the following

$$
\mu
\begin{array}{c}
u_2\\ \vdots\\ u_k\\ u_{k+1}\\ \vdots\\ u_p\\ v_1\\ v_2\\ \vdots\\ v_k\\ v_{k+1}\\ \vdots\\ v_q
\end{array}
\begin{vmatrix}
X & .. & 0 & 0 & .. & 0 & 1 & 0 & .. & 1 & 1 & .. & 1\\
\vdots & .. & \vdots & \vdots & .. & \vdots & \vdots & \vdots & .. & \vdots & \vdots & .. & \vdots\\
0 & .. & X & 0 & .. & 0 & 1 & 1 & .. & 0 & 1 & .. & 1\\
0 & .. & 0 & X-1 & .. & 0 & 1 & 1 & .. & 1 & 1 & .. & 1\\
\vdots & .. & \vdots & \vdots & .. & \vdots & \vdots & \vdots & .. & \vdots & \vdots & .. & \vdots\\
0 & .. & 0 & 0 & .. & X-1 & 1 & 1 & .. & 1 & 1 & .. & 1\\
1 & .. & 1 & 1 & .. & 1 & Y & 0 & .. & 0 & 0 & .. & 0\\
-1 & .. & 0 & 0 & .. & 0 & -1 & Y-1 & .. & -1 & -1 & .. & -1\\
\vdots & .. & \vdots & \vdots & .. & \vdots & \vdots & \vdots & .. & \vdots & \vdots & .. & \vdots\\
0 & .. & -1 & 0 & .. & 0 & -1 & -1 & .. & Y-1 & -1 & .. & -1\\
0 & .. & 0 & 0 & .. & 0 & -1 & -1 & .. & -1 & Y-2 & .. & -1\\
\vdots & .. & \vdots & \vdots & .. & \vdots & \vdots & \vdots & .. & \vdots & \vdots & .. & \vdots\\
0 & .. & 0 & 0 & .. & 0 & -1 & -1 & .. & -1 & -1 & .. & Y-2
\end{vmatrix}. \tag{2.5}
$$

Subtract column v_1 of (2.5) from columns $v_2, v_3, \ldots, v_q$ to obtain (2.6). (Throughout this row(column) v_i means row(column) corresponding to vertex v_i.)

$$\mu \quad \begin{array}{c|ccccccccccccc|}
u_2 & X & \cdots & 0 & 0 & \cdots & 0 & 1 & -1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
u_k & 0 & \cdots & X & 0 & \cdots & 0 & 1 & 0 & \cdots & -1 & 0 & \cdots & 0 \\
u_{k+1} & 0 & \cdots & 0 & X-1 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
u_p & 0 & \cdots & 0 & 0 & \cdots & X-1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
v_1 & 1 & \cdots & 1 & 1 & \cdots & 1 & Y & -Y & \cdots & -Y & -Y & \cdots & -Y \\
v_2 & -1 & \cdots & 0 & 0 & \cdots & 0 & -1 & Y & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
v_k & 0 & \cdots & -1 & 0 & \cdots & 0 & -1 & 0 & \cdots & Y & 0 & \cdots & 0 \\
v_{k+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & Y-1 & \cdots & 0 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
v_q & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 & \cdots & Y-1
\end{array} . \tag{2.6}$$

Multiply columns $v_{k+1}, \ldots, v_q$ by $\dfrac{1}{Y-1}$ of (2.6) and add to column v_1. The resulting expression is equivalent to:

$$\mu(Y-1)^{q-k} \quad \begin{array}{c|cccccccccc|}
u_2 & X & .. & 0 & 0 & .. & 0 & 1 & -1 & .. & 0 \\
\vdots & \vdots & .. & \vdots & \vdots & .. & \vdots & \vdots & \vdots & .. & \vdots \\
u_k & 0 & .. & X & 0 & .. & 0 & 1 & 0 & .. & -1 \\
u_{k+1} & 0 & .. & 0 & X-1 & .. & 0 & 1 & 0 & .. & 0 \\
\vdots & \vdots & .. & \vdots & \vdots & .. & \vdots & \vdots & \vdots & .. & \vdots \\
u_p & 0 & .. & 0 & 0 & .. & X-1 & 1 & 0 & .. & 0 \\
v_1 & 1 & .. & 1 & 1 & .. & 1 & \left(1-\frac{(q-k)}{Y-1}\right)Y & -Y & .. & -Y \\
v_2 & -1 & .. & 0 & 0 & .. & 0 & -1 & Y & .. & 0 \\
\vdots & \vdots & .. & \vdots & \vdots & .. & \vdots & \vdots & \vdots & .. & \vdots \\
v_k & 0 & .. & -1 & 0 & .. & 0 & -1 & 0 & .. & Y
\end{array} . \tag{2.7}$$

Add columns $u_2, \ldots, u_k$ of (2.7) and subtract from column v_1 to obtain:

$$\mu(Y-1)^{q-k} \quad \begin{array}{c|cccccccccc|}
u_2 & X & .. & 0 & 0 & .. & 0 & 1-X & -1 & .. & 0 \\
\vdots & \vdots & .. & \vdots & \vdots & .. & \vdots & \vdots & \vdots & .. & \vdots \\
u_k & 0 & .. & X & 0 & .. & 0 & 1-X & 0 & .. & -1 \\
u_{k+1} & 0 & .. & 0 & X-1 & .. & 0 & 1 & 0 & .. & 0 \\
\vdots & \vdots & .. & \vdots & \vdots & .. & \vdots & \vdots & \vdots & .. & \vdots \\
u_p & 0 & .. & 0 & 0 & .. & X-1 & 1 & 0 & .. & 0 \\
v_1 & 1 & .. & 1 & 1 & .. & 1 & R & -Y & .. & -Y \\
v_2 & -1 & .. & 0 & 0 & .. & 0 & 0 & Y & .. & 0 \\
\vdots & \vdots & .. & \vdots & \vdots & .. & \vdots & \vdots & \vdots & .. & \vdots \\
v_k & 0 & .. & -1 & 0 & .. & 0 & 0 & 0 & .. & Y
\end{array} \tag{2.8}$$

where $R = \left(1 - \dfrac{(q-k)}{Y-1}\right) Y - (k-1)$.

Multiply columns $v_2, \ldots, v_k$ by $\dfrac{1}{Y}$ of (2.8) and add to columns $u_2, \ldots, u_k$ respectively resulting expression is equivalent to

$$(2.9)\quad \mu(Y-1)^{q-k}(Y)^{k-1} \begin{array}{c} u_2 \\ \vdots \\ u_k \\ u_{k+1} \\ \vdots \\ u_p \\ v_1 \end{array} \begin{vmatrix} X-\frac{1}{Y} & .. & 0 & 0 & .. & 0 & 1-X \\ \vdots & .. & \vdots & \vdots & .. & \vdots & \vdots \\ 0 & .. & X-\frac{1}{Y} & 0 & .. & 0 & 1-X \\ 0 & .. & 0 & X-1 & .. & 0 & 1 \\ \vdots & .. & \vdots & \vdots & .. & \vdots & \vdots \\ 0 & .. & 0 & 0 & .. & X-1 & 1 \\ 1 & .. & 1 & 1 & .. & 1 & R \end{vmatrix}.$$

Multiply rows $u_{k+1}, \ldots, u_p$ by $\dfrac{1}{X-1}$ of (2.9) and subtract from row v_1 to obtain:

$$(2.10)\qquad \mu(Y-1)^{q-k}(Y)^{k-1}(X-1)^{p-k}(X-\frac{1}{Y})^{k-1}\left[R - \frac{p-k}{X-1}\right].$$

Substituting the values of auxillary quantities X, Y and R in (2.10) we get, the following expression

$$\mu(\mu-p)^{q-k}(\mu-p+1)^{k-1}(\mu-q)^{p-k}\left(\mu-q+1-\frac{1}{\mu-p+1}\right)^{k-1}$$

$$(2.11) = \left[\left(1-\frac{(q-k)}{\mu-p}\right)\mu - p + 1 - (k-1) - \frac{p-k}{\mu-q}\right].$$

On simplification, expression (2.11) leads to the expression (2.1) as

$$\begin{aligned} L(Ka_{p,q}(k):\mu) &= \mu(\mu-p)^{q-k-1}(\mu-q)^{p-k-1}[(\mu-q+1)(\mu-p+1)]^{k-1} \\ &\quad \{\mu^3 - 2(p+q-1)\mu^2 + [(p+q)(p+q-3)+pq+2k]\mu \\ &\quad +(p+q)(p+q-pq-k)\}. \end{aligned}$$

□

Example 1. *Consider the graph $Ka_{4,5}(2)$ of Figure 1. Here $p=4$, $q=5$ and $k=2$.*

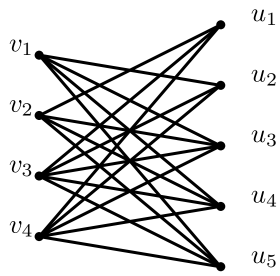

FIGURE 1. $Ka_{4,5}(2)$.

Therefore Laplacian polynomial of $Ka_{4,5}(2)$ is

$$LE(Ka_{4,5}(2):\mu) = \mu(\mu-4)^2(\mu-5)[(\mu-4)(\mu-3)-1][\mu^3-16\mu^2+78\mu-117]$$

$$= \mu(\mu-4)^2(\mu-5)(\mu-3)(\mu^2-7\mu+11)(\mu^2-13\mu+39).$$

Corollary 2.1. *For $p,q \geq 1$ and $0 \leq k \leq min(p,q)$, the Laplacian spectra of $Ka_{p,q}(k)$ consists of 0, p $(q-k-1\ times)$, q $(p-k-1\ times)$, $\frac{1}{2}[p+q-2\pm\sqrt{(p-q)^2+4}]$ $(k-1\ times)$ and the three roots, say μ_i, $i=1,2,3$ of the cubic equation*

$$\mu^3-2(p+q-1)\mu^2+[(p+q)(p+q-3)+pq+2k]\mu+(p+q)(p+q-pq-k)=0.$$

3. LAPLACIAN SPECTRA AND LAPLACIAN ENERGY OF $L(Ka_{p,q}(k))$ IN SOME SPECIAL CASES

Case(i): Let $p=q=k$. By equation (2.1)

$$L(Ka_{p,p}(p):\mu) = \mu(\mu-p)^{p-1}(\mu-p+2)^{p-1}(\mu-2(p-1)).$$

Therefore the Laplacian spectra of $Ka_{p,p}(p)$ consists of 0, p $(p-1\ times)$, $p-2$ $(p-1\ times)$ and $2(p-1)$ and so

$$LE(Ka_{p,p}(p)) = 4(p-1).$$

Case(ii): Let $p=q$ and $0\leq k\leq min\{p,q\}$. By equation (2.1)

$$L(Ka_{p,p}(k):\mu) = \mu(\mu-p)^{2p-k-3}(\mu-p+2)^{k-1}\{\mu^3-(4p-2)\mu^2 +(5p^2-6p+2k)\mu-(2p^3-4p^2+2pk)\}.$$

Therefore, the spectra of $L(Ka_{p,p}(k))$ consists of 0, p $(2p-k-2\ times)$, $p-2$ $(k-1\ times)$ and $\frac{(3p-2\pm\sqrt{(p+2)^2-8k})}{2}$. Therefore,

$$LE(Ka_{p,p}(k)) = \frac{1}{n}\left\{ \left((p+2k)^2-2(p+k)-6k^2\right) + \left|\frac{p^2-2(p-k)\pm p\sqrt{(p+2)^2-8k}}{2}\right| \right\}.$$

Case(iii): Let $p=k$ and $p\neq q$. By equation (2.1)

$$L(Ka_{p,q}(p):\mu) = \mu(\mu-p)^{q-p-1}\left[(\mu-q+1)(\mu-p+1)-1\right]^{p-1} \left[\mu^2-(2p+q-2)\mu+(p+q)(p-1)\right].$$

Therefore the Laplacian spectra of $Ka_{p,q}(p)$ consists of 0, p $(q-p-1\ times)$, $\left(\frac{(p+q-2)\pm\sqrt{(p-q)^2+4}}{2}\right)$ $(p-1\ times)$ and $\frac{(2p+q-2)\pm\sqrt{q^2-4p+4}}{2}$. The Laplacian energy of $Ka_{p,q}(p)$ is

$$\begin{aligned} LE(Ka_{p,q}(p)) = \frac{1}{p+q}\Bigg\{ & 2p(q-1)+(q-p-1)\left|q-p-2\right| \\ & +(p-1)\left|\frac{(p+q)\left[(p+q-2)\pm\sqrt{(p-q)^2+4}-4p(q-1)\right]}{2}\right| \\ & +\left|\frac{(p+q)\left[(2p+q-2)\pm\sqrt{q^2-4p+4}\right]-4p(q-1)}{2}\right|\Bigg\}. \end{aligned}$$

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