FIXED POINT THEOREM IN PROBABILISTICALLY CONVEX MANGER SPACE

K. SHRIVASTAVA1, S. SAXENA, AND VISHNU NARAYAN MISHRA1

ABSTRACT. The main target of this paper has been to apply the concept of probabilistically convexity on manger space and deal a common fixed point theorem by using the concept of compatibility between multi-valued mappings and self mappings in the above context.

1. INTRODUCTION

In 1972, Assad and Kirk in [2] gave sufficient conditions for non-self mappings to ensure the existence of fixed point by proving a result on multi-valued contraction mappings in complete metrically convex metric space. Pai and Veeramanji’s works, [11] seem to be the first to establish a probabilistic analogue of Nadler’s Banch contraction principle for multi-valued mappings, [10]. Hadzic and Gajic in [6], Imdad and Khan in [7], Rhoades in [12] and many others proved some fixed point theorems for non-self, multi-valued convex and sequence of set-valued mapping in metrically spaces. Our intention in this paper is to using the concept of compatibility between a multi-valued mapping and a single-valued mapping due to Kaneko and Sessa in [8] as a tool to produce some common fixed point theorems on complete probabilistically convex

1corresponding authors

2020 Mathematics Subject Classification. 47H10.
Key words and phrases. Menger space, probabilistic convex Menger space, compatible mapping.
menger space. The works of Som and Mukherjee in [15], Imdad and Khan in [7] and Ahmad and Assad in [1] are very useful to decisively establish our results.

2. Preliminaries

Definition 2.1. [13], A mapping \( F : R \to R^+ \) is called a distribution function if it is non decreasing left continuous with

\[
\inf\{F(t); t \in R\} = 0 \quad \text{and} \quad \sup\{F(t); t \in R\} = 1.
\]

We shall denote by \( L \) the set of all distribution function while \( H \) will always denote the specific distribution function defined by

\[
H(t) = \begin{cases} 
0; & t < 1 \\
1; & t > 0.
\end{cases}
\]

Definition 2.2. [13], A Probabilistic Menger Space (PM-space) is an ordered pair \( (X, F) \), where \( X \) is an abstract set of elements and \( F : X \times X \to L \), defined by \( (p, q) \to F_{p,q} \), where \( L \) is the set of all distribution function i.e. \( L = \{F_{p,q}|p, q \in X\} \), if the functions \( F_{p,q} \) satisfy:

1. \( F_{p,q}(x) = 1 \) for all \( x > 0 \), if and only if \( p = q \).
2. \( F_{p,q}(0) = 0 \).
3. \( F_{p,q} = F_{q,p} \).
4. if \( F_{p,q}(x) = 1 \), and \( F_{p,q}(y) = 1 \) then \( F_{p,q}(x + y) = 1 \)

Definition 2.3. [13], A mapping \( \Delta : [0, 1] \times [0, 1] \to [0, 1] \) is called a \( t \)-norm if

1. \( \Delta(a, 1) = a \),
2. \( \Delta(a, b) = \Delta(b, a) \),
3. \( \Delta(c, d) \geq \Delta(a, b) \) if \( c \geq a, d \geq b \),
4. \( \Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c)) \).

It follows that \( \Delta(a, 0) = 0, \forall a \in [0, 1] \) in particular \( \Delta(0, 0) = 0 \).

Definition 2.4. A Menger space is a triplet \( (X, F, \Delta) \), where \( (X, F) \) is a PM-space and \( \Delta \) is \( t \)-norm such that for all \( p, q, r \in X \) and \( \forall x, y \geq 0 \),

\[
F_{p,r}(x + y) \geq \Delta(F_{p,q}(x), F_{q,r}(y)).
\]
Schweizer and Sklar in [13] proved that if \((X, F, \Delta)\) is a menger space with 
\[ \sup_{0 < x < 1} \Delta(x, x) = 1, \]
then \((X, F, \Delta)\) is a Hausdorff topological space in the topology \(\tau\) introduced by the family of \((\epsilon, \lambda)\) neighborhoods.

\[
\{ U_p(\epsilon, \lambda) : p \in X, \epsilon > 0, \lambda > 0 \},
\]

where \(U_p(\epsilon, \lambda) = \{ x \in X ; f_{x,p}(\epsilon) > 1 - \lambda \}\)

A complete metric space can be treated as a complete menger space in the following way: Throughout this paper, we assume that \((X, F, \Delta)\) is a manger space with \((\epsilon, \lambda)\) – topology \(\tau\). Let,

\[
CB(X) = \{ A : A \text{ is non empty closed and bounded subset of } X \}
\]

\[
C(X) = \{ A : A \text{ is non empty closed and compact subset of } X \}.
\]

**Definition 2.5.** [4], Let \((X, F, \Delta)\) be a Menger space. \(A, B \in CB(X)\) and \(x \in X\) we define \(F_{x,A}\) and \(F_{A,B}\) by

\[
F_{x,A}(t) = \sup_{y \in A} F_{x,y}(t) \text{ and } F_{A,B}(t) = \sup_{s < t} \Delta \{ \inf_{x \in A} \sup_{y \in B} F_{x,y}(t), \inf_{y \in B} \sup_{x \in A} F_{x,y}(t) \}, \forall t \in R.
\]

We say that \(F_{x,A}\) is the probabilistic distance from \(x\) to \(A\) and \(F_{A,B}\) is the probabilistic distance from \(A\) to \(B\) induced by \(F\).

**Lemma 2.1.** [5], Let \((X, F, \Delta)\) be a Menger space \(\Delta\) be a left continuous \(t\)- norm, \(A \in CB(X)\) and \(x, y \in X\). then we have the following

1. For any \(B \in CB(X)\) and \(x \in A\)

\[
\inf_{x \in A} \sup_{y \in B} F_{x,y}(t) \leq F_{x,B}(t), \text{ for all } t \in R,
\]

2. \(F_{x,A}(t) = 1\) for all \(t > 0\) if and only if \(x \in A\)

\[
F_{x,A}(t_1 + t_2) \geq \Delta(F_{x,Y}(t_1), F_{Y,A}(t_2)) \text{ for all } t_1, t_2 > 0,
\]

3. \(F_{x,A}(t)\) is left continuous function at \(t\),

Now, we first consider the properties of an induced manger space.

**Theorem 2.1.** [14], Let \((X, d)\) be a complete metric space and define \(F : X \times X \rightarrow D^+\) (set of all distribution function)

\[
F_{x,y}(t) = H(t - d(x,y)), \text{ for } x, y \in E
\]

then the space \((X, F, \min)\) with a left continuous \(t\)- norm \(\Delta = \min\) is a \(\tau\)- complete menger space and topology \(\tau\) induced by the metric \(d\) coincides with the topology
And, for \( x \in X, K, C \in \text{CB}(X) \) we can easily obtain.

\[
F_{x,K}(t) = H(t - d(x, K)) \quad \text{and} \\
F_{K,C}(t) = H(t - d_H(K, C)).
\]

**Proposition 2.1.** Let \((X, F, \Delta)\) be \(\tau\)-complete Menger space induced by the metric \(d\) as follows:

\[
F_{x,y}(t) = H(t - d(x, y)), \text{ for } x, y \in X,
\]

where \(\Delta\) is a left-continuous \(t\)-norm such that \(\Delta(a, b) = \min\{a, b\}\).

Let \( T : X \rightarrow \text{CB}(X) \) a multi-valued mapping, then for each \( x, y \in X \) and \( u_x \in T_x \) there exist a \( v_y \in T_y \) such that

\[
F_{u_x, v_y}(t) \geq F_{T_x, T_y}(t), \quad t \geq 0
\]

**Proof.** From the compactness of \( T_y \), we can choose \( v_y \in T_y \) such that

\[
d(u_x, v_y) \leq d_H(T_x, T_y).
\]

Hence

\[
F_{u_x, v_y}(t) = H(t - d(u_x, v_y)) \\
\geq H(t - d_H(u_x, v_y)) \\
= F_{T_x, T_y}(t), \quad t \geq 0.
\]

By proposition 2.1 we can easily obtain the following.

**Corollary 2.1.** Let \((X, F, \Delta)\) be a \(\tau\) complete menger space induced by the metric \(d\) as follows:

\[
F_{x,y}(t) = H(t - d(x, y)), \text{ for } x, y \in X,
\]

where \(\Delta\) is left-continuous \(t\)-norm such that \(\Delta(a, b) = \min\{a, b\}\) and \( T : X \rightarrow \text{CB}(X) \)

is a multi-valued mapping. If for each \( x, y \in X \)

\[
F_{T_x, T_y}(\phi(t)) \geq F_{x,y}(t), \quad t \geq 0.
\]

Then for \( u_x \in T_x \) there exists \( v_y \in T_y \) such that

\[
F_{u_x, v_y}(\phi(t)) \geq F_{x,y}(t), \quad t \geq 0,
\]
where \( \phi : [0, +\infty) \to [0, +\infty) \) is a function.

**Definition 2.6.** A Menger space \((X, F, \Delta)\) is said to be probabilistically convex if for any \(x, y \in X\) with \(x \neq y\), there exist \(t\) a point \(z \in X, x \neq z \neq y\) such that

\[
\Delta(F_{x,z}(t_1), F_{z,y}(t_2)) = F_{x,y}(t_1 + t_2).
\]

**Lemma 2.2.** Let \((X, F, \Delta)\) is said to be complete probabilistically convex menger space. Let \(K\) be any non-empty closed subset of \(X\). Then for any \(x \in K\) and \(y \notin K\) there exists a point \(z \in \partial K\) (the boundary of \(K\)) such that

\[
\Delta(F_{x,z}(t_1), F_{z,y}(t_2)) = F_{x,y}(t_1 + t_2).
\]

Our main theorem is prefaced with the above lemma.

**Definition 2.7.** Let \(K\) be a non-empty subset of a menger space \((X, F, \Delta)\) and \(S, T : K \to X\) the pair \(\{S, T\}\) is said to be weakly commuting if for each \(x, y \in K\) such that \(X = Sy\) and \(Ty \in K\), we have

\[
F_{Tx,STy}(t) \geq F_{Sy,Ty}(t).
\]

**Definition 2.8.** Let \(K\) be a non-empty subset of a menger space \((X, F, \Delta)\) and \(S, T : K \to X\) the pair \(\{S, T\}\) is said to be compatible if for every sequence \(\{x_n\}\) from \(K\) and from relation

\[
\lim_{n \to \infty} F_{Tx_n,Sx_n}(t) = 1
\]

and \(Tx_n \in K, n \in \mathbb{N}\), it follows that

\[
\lim_{n \to \infty} F_{Ty_n,STx_n}(t) = 1,
\]

for every sequence \(\{y_n\}\) from \(K\) such that \(y_n = Sx_n, n \in \mathbb{N}\). Kaneko and Sessa in [8], extended the concept of compatibility for single-valued mapping to a multi-valued mapping as follows:

**Definition 2.9.** Let \((X, F, \Delta)\) be a menger space. The mappings \(A : X \to \text{CB}(X)\) and \(S : X \to X\) are compatible if \(SA(x) \in \text{CB}(x)\), \(\forall x \in X\) and

\[
\lim_{n \to \infty} F_{SAx_n,ASx_n}(t) = 1,
\]

whenever \(\{x_n\}\) is a sequence in \(X\) such that

\[
Ax_n \to M \in \text{CB}(x)\quad \text{and}\quad Sx_n \to t \in M.
\]
In [3] Chang defined the family of real function $\phi$ as follows:
Let $\Phi = \{ \phi : R^+ \to R^+, \phi \text{ is upper semi-continuous with } \phi(x) < x \text{ for each } x > 0 \text{ and } \phi(0) = 0 \}$, where $R^+$ is the set of all non-negative real numbers.

**Lemma 2.3.** [3], Let $\phi \in \Phi$, then there exists a strictly increasing continuous function $\psi : R^+ \to R^+$ such that $\phi(u) \leq \psi(u) < u$ for each $u > 0$, $\lim_{n \to \infty} \psi^{-n}(u) = \infty$ and $\psi(u) > 0$, for each $u > 0$.

**Remark 2.1.** In the above case the function $\psi$ is invertible if for each $u > 0$, we denote $\psi^0(u) = u$ and $\psi^{-n}(u) = \psi^\psi^{-n+1}(u)$ for each $n \in \mathbb{N}$, the $\lim_{n \to \infty} \psi^{-n}(u) = \infty$.

### 3. Main Result

**Theorem 3.1.** Let $(X, F, \Delta)$ be a complete probabilistically convex Menger space with $\Delta(a, a) \geq a$ and $K$ be a non-empty closed convex subset of $X$. Let $A, B : K \to \mathcal{CB}(X)$, and $S, T : K \to K$ satisfying the following conditions:

1. $\partial K \subseteq SK \cap K, AK \cap K \subseteq TK, BK \cap K \subseteq SK$,
2. $Sx \in \partial K \Rightarrow Ax \subseteq K$, $Tx \in \partial K \Rightarrow Bx \subseteq K$,
3. $(A, S)$ and $(B, T)$ are compatible mappings,
4. $A, B, S, T$ are continuous on $K$.

$$F_{Ax, By}(t) \geq \min(F_{Sx, Ty}(t), F_{Ax, Ax}(t), F_{Ty, By}(t))$$

then there exists a point $z$ in $X$ such that $Sz = Tz \in AZ \cap Gz$.

**Proof.** Let $x \in \partial K$, since $\partial K \subseteq SK$, there exists a point $x_0 \in K$ such that $x = Sx_0$ that is $Sx_0 \in \partial K \Rightarrow Ax_0 \subseteq K$ (from 2). Since $Ax_0 \in AK \Rightarrow Ax_0 \subseteq K \cap AK \subseteq TK$.

Let $x_1 \in K$ be such that $y_1 = Tx_1 \in Ax_0 \subseteq K$. Since $y_1 \in Ax_0$, there exists a point $y_2 \in Bx_1$ such that

$$F_{y_1, y_2}(t) \geq F_{Ax_0, Bx_1}(t).$$

Suppose $y_2 \in K$, then $y_2 \in K \cap BK \subset SK$ which implies that there exists a point $x_2 \in K$ such that $y_2 = Sx_2$. Otherwise if $y_2 \notin K$, then there exists a point $u \in \partial K$ such that

$$\Delta(F_{Ax_1, u}(t_1), F_{u, y_2}(t)) = F_{Ax_1, y_2}(t_1 + t_2), \forall t > 0.$$ 

since $u \in \partial K \subseteq SK$, then there exist a point $x_2 \in K$ such that $u = Sx_2$ and

$$\Delta(F_{Tx_1, Sx_2}(t_1), F_{Sx_2, y_2}(t_2)) = F_{Tx_1, y_2}(t_1 + t_2), \forall t > 0.$$
Let $y_3 \in Ax_2$ be such that
\[ F_{y_2,y_3}(t) \geq F_{Bx_1,Ax_2}(t). \]

Repeating the above argument, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that

(i) $y_{2n} \in Bx_{2n-1}$, $y_{2n+1} \in Ax_{2n}$,
(ii) $y_{2n} \in K \Rightarrow y_{2n} = Sx_{2n}$ or $y_{2n} \notin K \Rightarrow Sx_{2n} \in \partial K$ and

\[ \Delta(F_{Tx_{2n-1},Sx_{2n}}(t_1), F_{Sx_{2n},y_{2n}}(t_2)) = F_{Tx_{2n-1},y_{2n}}(t_1 + t_2). \]

(iii) $y_{2n+1} \in K$, $y_{2n+1} = Tx_{2n+1}$ or $y_{2n+1} \notin K$, $Sx_{2n+1} \in \partial K$

\[ \Delta(F_{Sx_{n},Tx_{2n+1}}(t_1), F_{Tx_{2n+1},y_{2n+1}}(t_2)) = F_{Sx_{2n},y_{2n+1}}(t_1 + t_2) \]

(iv) $F_{y_{2n-1},y_{2n}}(t) \geq F_{Bx_{2n-1},Ax_{2n-1}}(t)$, $F_{y_{2n},y_{2n+1}}(t) \geq F_{Ax_{2n},Bx_{2n-1}}(t)$. We denote

\[
\begin{align*}
P_0 &= \{Sx_{2i} \in \{Sx_{2n}\}; Sx_{2i} = y_{2i}\}, \\
P_1 &= \{Sx_{2i} \in \{Sx_{2n}\}; Sx_{2i} \neq y_{2i}\}, \\
Q_0 &= \{Tx_{2i+1} \in \{Tx_{2n+1}\}; Tx_{2i+1} = y_{2i+1}\}, \\
Q_1 &= \{Tx_{2i+1} \in \{Tx_{2n+1}\}; Tx_{2i+1} \neq y_{2i+1}\} \\
\end{align*}
\]

First we show that $(Sx_{2n}, Tx_{2n+1}) \notin P_1 \times Q_1$ and $(Tx_{2n-1}, Sx_{2n}) \notin Q_1 \times P_1$.

If $Sx_{2n} \in P_1$ then $y_{2n} \neq Sx_{2n}$ and we have $Sx_{2n} \in \partial K$ which implies that $y_{2n+1} \in Ax_{2n} \subseteq K$. Hence $y_{2n+1} = Tx_{2n+1} \in Q_0$. Similarly we have argue that $(Tx_{2n-1}, Sx_{2n}) \notin Q_1 \times P_1$.

**Case-1** If $(Sx_{2n}, Tx_{2n+1}) \in P_0 \times Q_0$ then

\[
\begin{align*}
F_{Sx_{2n},Tx_{2n+1}}(\phi t) &= F_{y_{2n},y_{2n+1}}(\phi t) \\
&\geq F_{Ax_{2n},Bx_{2n+1}}(\phi t) \\
&\geq \min(F_{Sx_{2n},Tx_{2n-1}}(t), F_{Sx_{2n},Ax_{2n}}(t), F_{Tx_{2n-1},Bx_{2n-1}}(t)) \\
&\geq \min(F_{Sx_{2n},Tx_{2n-1}}(t), F_{Sx_{2n},Tx_{2n+1}}(t), F_{Tx_{2n-1},Sx_{2n}}(t)). \\
&\geq \min(F_{Sx_{2n},Tx_{2n-1}}(t), F_{Sx_{2n},Tx_{2n+1}}(t)).
\end{align*}
\]
Thus in all cases, we put $z_{2n} = Sx_{2n}, z_{2n+1} = Tx_{2n+1}$, we have

\[
F_{z_{2n},z_{2n+1}}(\phi t) \geq \min(F_{z_{2n},z_{2n+1}}(t), F_{z_{2n},z_{2n+1}}(\phi^{-1}t)), \\
F_{z_{2n},z_{2n+1}}(t) \geq \min(F_{z_{2n},z_{2n+1}}(t), F_{z_{2n},z_{2n+1}}(\phi^{-1}t)),
\]

\[
= \min(F_{z_{2n},z_{2n+1}}(t), F_{z_{2n},z_{2n+1}}(\phi^{-1}t)), \\
= \min(F_{z_{2n},z_{2n+1}}(t), F_{z_{2n},z_{2n+1}}(\phi^{-2}t))
\]

since $x < \phi^{-1}(x) < \phi^{-2}(x) \ldots$

---

**Case-2** If $(Sx_{2n}, Tx_{2n+1}) \in P_0 \times Q_1$ then from (iii), we get

\[
F_{Sx_{2n},Tx_{2n+1}}(\phi t) = F_{Sx_{2n},y_{2n+1}}(2\phi t) \\
= F_{y_{2n},y_{2n+1}}(2\phi t) \\
\geq \min(F_{Sx_{2n-1},Ty_{2n}}(t), F_{Sx_{2n},Tx_{2n+1}}(t)). \quad [\text{from case 1}]
\]

Similarly, if $(Tx_{2n-1}, Sx_{2n}) \in Q_1 \times P_0$ then we show that

\[
F_{Tx_{2n-1},Sx_{2n}}(\phi t) \geq \min(F_{Sx_{2n-2},Tx_{2n-1}}(t), F_{T_{2n-1},Sx_{2n}}(t))
\]

**Case-3** If $(Sx_{2n}, Tx_{2n+1}) \in P_1 \times Q_0$ then $Tx_{2n-1} = y_{2n-1}$. Hence proceeding as in case 1, we have

\[
F_{Sx_{2n},Tx_{2n+1}}(2\phi t) = F_{Sx_{2n},y_{2n+1}}(2\phi t) \geq \Delta(F_{Sx_{2n},y_{2n}}(\phi t), F_{y_{2n},y_{2n+1}}(\phi t)) \\
\geq \Delta(F_{Sx_{2n},y_{2n}}(\phi t), F_{Ax_{2n},Bx_{2n-1}}(\phi t)) \\
\geq \Delta(F_{Sx_{2n},y_{2n}}(\phi t), \min(F_{Sx_{2n},Tx_{2n-1}}(t), F_{Sx_{2n},Ax_{2n}}(t), F_{Tx_{2n-1},Bx_{2n-1}}(t))) \\
\geq \Delta(F_{Sx_{2n},y_{2n}}(\phi t), \min(F_{Sx_{2n},Tx_{2n-1}}(t), F_{Sx_{2n},Tx_{2n+1}}(t)))
\]

since $\Delta(F_{Tx_{2n-1},Sx_{2n}}(t), F_{Sx_{2n},y_{2n}}(t)) = F_{Tx_{2n-1},y_{2n}}(2t)$.

Then

\[
F_{Sx_{2n},Tx_{2n+1}}(2\phi t) \geq \Delta(F_{Tx_{2n-1},y_{2n}}(2\phi t), \min(F_{Sx_{2n},Tx_{2n-1}}(t), F_{Sx_{2n},Tx_{2n+1}}(t))) \\
= \Delta(F_{Tx_{2n-1},y_{2n}}(2\phi t), \min(F_{Sx_{2n},Tx_{2n-1}}(t), F_{Sx_{2n},Tx_{2n+1}}(t))) \\
\geq \min(F_{Sx_{2n},Tx_{2n-1}}(t), F_{Sx_{2n},Tx_{2n+1}}(t)).
\]

[from case 1 and $\Delta(a, a) \geq a$]
By repeated applications of above inequality, we obtain
\[ F_{z_{2n},z_{2n+1}}(t) \geq \min(F_{z_{2n},z_{2n-1}}(\phi^{-1}t), F_{z_{2n},z_{2n+1}}(\phi^{-i}t)). \]
Since \( F_{z_{2n},z_{2n+1}}(\phi^{-i}t) \to 1 \) as \( i \to \infty \) \( \forall t > 0 \) it follows that
\[ F_{z_{2n},z_{2n+1}}(t) \geq F_{z_{2n},z_{2n-1}}(\phi^{-1}t), \quad \forall n \in \mathbb{N} \quad \text{and} \quad \forall t > 0. \]
Therefore,
\[ F_{z_{2n},z_{2n+1}}(t) \geq F_{z_{2n},z_{2n-1}}(\phi^{-1}t) \geq F_{z_{2n-1},z_{2n-2}}(\phi^{-2}t) \geq \cdots \geq F_{z_{2n_0},z_{2n_1}}(\phi^{-n}t), \]
taking limit as \( n \to \infty \), we obtain that
\[ \lim_{n \to \infty} F_{z_{2n},z_{2n+1}}(t) = 1. \]

Now \( F_{z_{2n},z_{2n+p}}(t) \geq \Delta(F_{z_{2n},z_{2n+p}}(t/p), \ldots, F_{z_{2n+p-1},z_{2n+p}}(t/p)). \) Taking limit as \( n \to \infty \), we have
\[ \lim_{n \to \infty} F_{z_{2n},z_{2n+p}}(t) = 1. \]
This implies that \( z_n \) is a cauchy and hence converges to a point \( z \) consequently, the subsequences
\[ \{z_{2n}\} = \{Sx_{2n}\} \to z \]
\[ \{z_{2n+1}\} = \{Tx_{2n+1}\} \to z. \]
Since \((B, T)\) is compatible mappings
\[ \lim_{n \to \infty} F_{Bx_{2n-1},Tx_{2n-1}}(t) = 1 \]
\[ \Rightarrow \lim_{n \to \infty} F_{TSx_{2n},BTx_{2n-1}}(t) = 1. \]
By the continuity of \( B \) and \( T \) then \( F_{Tx, Bz}(t) = 1 \), i.e.,
\[ (3.1) \quad Tz \in Bz. \]
Similarly, the continuity and compatibility of \((A, S)\) lead to
\[ (3.2) \quad Sz \in Az. \]
Again
\[ F_{Sz, Tz}(\phi t) \geq F_{Ax, Bz}(\phi t) \]
\[ \geq \min(F_{Sz, Tz}(t), F_{Sz, Az}(t), F_{Tx, Bz}(t)) \]
\[ \geq \min(F_{Sz, Tz}(t), F_{Sz, Tz}(t), F_{Tx, Sz}(t)) \]
\[ \Rightarrow F_{Sz, Tz}(\phi t) \geq F_{Sz, Tz}(\phi t) \]
which implies that
\[(3.3) \quad Sz = Tz.\]
From (3.1), (3.2) and (3.3)
\[Sz = Tz \in Az \cap Bz.\]
This complete the proof. \(\square\)

Similarly, we can prove the following theorem.

**Theorem 3.2.** Let \((X, F, \Delta)\) be a complete probabilistically convex menger space and \(K\) be a non-empty closed convex subset of \(X\). Let \(A, B : K \to C(X)\) and \(S, T : k \to k\) satisfying the conditions.

1. \(\partial K \subseteq SK \cap K, AK \cap K \subseteq TK, BK \cap K \subseteq SK,\)
2. \(Sx \in \partial K \Rightarrow Ax \subseteq K, Tx \in \partial K \Rightarrow Bx \subseteq K,\)
3. \((A, S)\) and \((B, T)\) are compatible mappings,
4. \(A, B, S, T\) are continuous on \(K,\)
5. \(d_H(Ax, By) \leq \phi \max\{d(Sx, Ty), d_H(Sx, Ax), d_H(Ty, By)\},\) then there exists a point \(z\) in \(X\) such that
\[Sz = Tz \in Fz \cap Gz.\]

**Remark 3.1.** If we take \(S = T = 1\) (identity function) and \(A = B\) and complete menger space in theorem 3.1 one deduces a result due to Lee, [9].

**4. Application**

Here, we study the existence of fixed point for multi-valued mappings in a metric space \((X, d)\) using the results in the previous section.

**Theorem 4.1.** Let \((X, d)\) be a convex complete metric space and \(A, B : (K, d) \to (CB(X), d_H), S, T : (K, d) \to (K, d)\) satisfying the conditions.

1. \(\partial K \subseteq SK \cap K, AK \cap K \subseteq TK, BK \cap K \subseteq SK,\)
2. \(Sx \in \partial K \Rightarrow Ax \subseteq K,Tx \in \partial K \Rightarrow Bx \subseteq K,\)
3. \((A, S)\) and \((B, T)\) are compatible mappings,
4. \(A, B, S, T\) are continuous on \(K,\)
5. \(d_H(Ax, By) \leq \phi \max\{d(Sx, Ty), d_H(Sx, Ax), d_H(Ty, By)\},\) then there exists a point \(z\) in \(X\) such that
$S_z = T_z \in A_z \cap G_z$.

**Proof.** If we define $F : X \times X \to D^+$ such that

$$F_{A,B}(t) = H(t - d_H(A, B)), \forall A, B \in CB(X)$$

then the space $(X, F, \min)$ with $t-$ norm $\Delta = \min$ is a probabilistically convex $\tau-$ complete menger space and topology induced by the metric $d$ coincided with the topology $\tau$. For any $A, B \in CB(X)$, we have

$$F_{Ax, By}(\phi t) = H(\phi t - d_H(Ax, By))$$

$$\geq H[\phi t - \max\{d(Sx, Ty), d_x(Sx, Ax), d_H(Ty, By)\}]$$

$$= H[\min\{(t - d(Ax, By)), (t - d_H(Sx, Ax)), (t - d_H(Ty, By))\}]$$

$$= \min[\{H(t - d(Ax, By)), H(t - d_H(Sx, Ax)), H(t - d_H(Ty, By))\}]$$

$$= \min[F_{Ax, By}(t), F_{Sx, Ay}(t), F_{Tx, By}(t)].$$

Thus the Theorem 4.1 follows from theorem 3.1 immediately. Hence there exist a point $z \in X$ such that $S_z = T_z \in A_z \cap B_z$.  

**REFERENCES**


DEPARTMENT OF MATHEMATICS & STATISTICS
DR. HARISINGH GOUR UNIVERSITY
SAGAR, MADHYA PRADESH - 470003, INDIA.
Email address: kavita.rohit@rediffmail.com

SAGAR INSTITUTE OF SCIENCE AND, TECHNOLOGY
SISTec BHOPAL BHOPAL, MADHYA PRADESH - 462036, INDIA.
Address
Email address: swatisaxena@sistec.ac.in

DEPARTMENT OF MATHEMATICS
INDRA GANDHI NATIONAL TRIBAL UNIVERSITY LALPUR,
AMARKANTAK, ANUPPUR, MADHYA PRADESH - 484887, INDIA.
Address
Email address: vishnunarayanmishra@gmail.com