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A NOTE ON STRONGLY PERFECT GROUPS

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ABSTRACT. In this work, we have introduced the notion of strongly perfect group. Let G be a finite group and $\delta(G) = \sum_{H \leq G} |H|$ be the sum of the orders of the subgroups of G. We define G to be strongly perfect if $\delta(G) = 2|G|$. Clearly, this group is a generalization of perfect group introduced by Leinster [1]. We have investigated some properties of this group.

1. INTRODUCTION

A natural number n is called a perfect number if n is equal to the sum of all its positive divisors, excluding n itself. For example, 6 is perfect as $6 = 1 + 2 + 3$. Since its inception, the notion of perfect number has been the center of attention for many mathematicians and the same has been studied in tandem with various other concepts from time to time. For instance, Leinster [1] extended the study of perfect numbers to finite groups. He defined a finite group to be perfect (or Leinster) if its order is equal to the sum of the orders of all proper normal subgroups of the group. As an interesting consequence of this approach, one may observe that a finite cyclic group is a Leinster group if and only if its order is a perfect number. The connection between perfect numbers and finite groups have also been studied by Medts [2]. In spite of all the endeavours made so far, a natural question that still remains open is whether there are odd perfect numbers or not.

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As an obvious generalization of perfect numbers, the notion of hyperperfect numbers have been introduced in [3]. A natural number n is called a k -hyperperfect number if there exist an integer k such that $\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k}$, where $\sigma(n)$ is the sum of the positive divisors of n. It is often referred to as a hyperperfect number for simplicity. Clearly, a number is perfect if and only if it is 1-hyperperfect. Thus, for example, 6 is hyperperfect, since $\sigma(6) = \frac{k+1}{k}6 + \frac{k-1}{k} \Rightarrow k = 1$. The notion of hyperperfect numbers has been extended to finite groups in [4].

Based on Leinster's concept to extend the study from perfect number to finite group, in this work we have introduced the notion of strongly perfect group as a generalization of perfect group and investigated some of its properties.

All the algebraic and number theoretic contents of this work can be found in [5] and [6] respectively.

2. Strongly Perfect Group And Its Properties

In this section we define strongly perfect group and investigate some of its properties. We start with the following definition.

Definition 2.1. *Let G be a finite group and $\delta(G) = \sum_{H \leq G} |H|$ be the sum of the orders of the subgroups of G. Then G is said to be strongly perfect if $\delta(G) = 2|G|$.*

Example 1. *Let us consider C_6, the cyclic group of order 6. .Here, $|C_6| = 6$ and $\delta(C_6) = 1 + 2 + 3 + 6 = 12 = 2|C_6|$. Therefore, C_6 is strongly perfect.*

Example 2. *Let us consider the Quaternion group Q_8. Here, $|Q_8| = 8$. But Q_8 has 3 subgroup of order 4 and one subgroup of order 2. Therefore, $\delta(Q_8) = (3 \times 4) + (1 \times 2) + 1 + 8 = 23 \neq 2|Q|$. Hence, Q_8 is not a strongly perfect group.*

Definition 2.2. *(Leinster [1]) Let G be a finite group and $D(G) = \sum_{n(G)} |N|$ be the sum of the orders of the normal subgroups of G. Then G is called a perfect group if $D(G) = 2|G|$.*

Remark 2.1. *It is clear from definitions 2.1 and 2.2 that every perfect group is strongly perfect.But the converse is not true in general. For example, let us consider the symmetric group S_3. We know that $|S_3| = 6$. But, $\delta(S_3) = 1 + 2 + 3 + 6 = 12 = 2 \times 6 = 2|S_3|$. Therefore S_3 is strongly perfect. However $D(S_3) = 1 + 3 + 6 = 10$ and so $D(S_3) \neq 2|S_3|$. Thus, S_3 is not perfect.*

Proposition 2.1. *Let G be an Abelian group. Then G is strongly perfect if and only if G is perfect.*

Proof. Let G be an Abelian group and H be a subgroup of G. Then H is normal in G, which yields $\delta(G) = D(G)$. Therefore, G is strongly perfect

$\Leftrightarrow \quad \delta(G) = 2|G|$

$\Leftrightarrow \quad D(G) = 2|G|$

$\Leftrightarrow \quad G$ is perfect. □

Corollary 2.1. *Let G be a cyclic group. Then G is strongly perfect if and only if G is perfect.*

Proof. Since every cyclic group is Abelian, so the result follows from proposition 2.1 immediately. □

Remark 2.2. *In view of corollary 2.1, we have a cyclic group C_n of order n is strongly perfect if and only if it is perfect. Also, we know that C_n has one and only one normal subgroup of order d for each divisor d of n. So, $D(C_n) = D(n)$ and C_n is perfect just when n is a perfect number. Thus we conclude that C_n is a strongly perfect group if and only if n is a perfect number. Hence, $C_6, C_{28}, C_{496}, \ldots$ are all strongly perfect groups.*

Proposition 2.2. *No p-group is strongly perfect for a prime p.*

Proof. Suppose G is a p-group having order p^r, r being any positive integer. Then by Sylow's first theorem, G has subgroups of orders $1, p, p^2, \ldots, p^{r-1}, p^r$. Then we have $\delta(G) = 1 + p + p^2 + \ldots + p^{r-1} + p^r$. If possible let us assume that G is strongly perfect. This implies $\delta(G) = 2|G|$

$$\begin{aligned}
&\Rightarrow \quad 1 + p + p^2 + \ldots + p^{r-1} + p^r = 2p^r \\
&\Rightarrow \quad 1 + p + p^2 + \ldots + p^{r-1} = p^r \\
&\Rightarrow \quad \frac{1 - p^r}{1 - p} = p^r \\
&\Rightarrow \quad 1 - p^r = p^r(1 - p) \\
&\Rightarrow \quad 1 = p^r(1 - p) + p^r \\
&\Rightarrow \quad p^r(2 - p) = 1
\end{aligned}$$

Which is impossible for any prime p. Therefore, G is not strongly perfect. □

Corollary 2.2. *No Abelian group of prime order is strongly perfect*

Proof. It follows directly from proposition 2.2. □

Proposition 2.3. *Let G be a group of order pq, p and q being primes such that $p < q$ and p does not divide $q - 1$. Then G is strongly perfect if and only if G is perfect.*

Proof. Let G be a group of order pq where p and q are primes such that $p < q$ and p does not divide $q - 1$. Then G is cyclic. Therefore, by corollary 2.1 we have G is strongly perfect if and only if G is perfect. □

We know that a Hamiltonian group is a non-abelian group in which all subgroups are normal. Immediately we have the following proposition.

Proposition 2.4. *Let G be a Hamiltonian group. Then G is strongly perfect if and only if G is perfect*

Proof. Let G be a Hamiltonian group and H be any subgroup of G. Then H is normal in G which yields $\delta(G) = D(G)$.

Therefore, G is strongly perfect

$\Leftrightarrow$ $\delta(G) = 2|G|$

$\Leftrightarrow$ $D(G) = 2|G|$

$\Leftrightarrow$ G is perfect. □

Proposition 2.5. *Let G and F be two groups such that $G \cong F$. G is strongly perfect if and only if F is strongly perfect.*

Proof. Let G and F be two groups such that $G \cong F$. Let $f : G \to F$ be an isomorphism between G and F. Then we have $|G| = |F|$ and $H \leq G$ if and only if $f(H) \leq F$.

Therefore, G is strongly perfect

$\Leftrightarrow$ $\delta(G) = 2|G|$

$\Leftrightarrow$ $\sum_{H \leq G} |H| = 2|G|$

$\Leftrightarrow$ $\sum_{f(H) \leq F} |f(H)| = 2|F|$

$\Leftrightarrow$ $\delta(F) = 2|F|$

$\Leftrightarrow$ F is strongly perfect. □

3. Conclusion

In this article, we have introduced the notion of strongly perfect group as a generalization of perfect group. We have shown that whenever a group is

Abelian or cyclic the notion of strongly perfectness and perfectness coincide but the same may not hold for non-abelian or non-cyclic groups. We have also characterized p-groups and groups of order pq for strongly perfectness. Finally, we have shown that strongly perfectness is preserved under group isomorphism.

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