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SUBMODULE INCLUSION GRAPH OF A MODULE

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ABSTRACT. Let R be a commutative ring with unity and M be an R-module. The Submodule inclusion graph of M, denoted by $I_S(M)$, is a (undirected) graph with vertices as all non-trivial submodules of M and two distinct vertices N and L are adjacent if and only if $N \subset L$ or $L \subset N$. In this paper, it has been proved that $I_S(M)$ is not connected if and only if M is a direct sum of two simple R-modules. Moreover, it has been shown that $I_S(M)$ is a complete graph if and only if M is a uniserial module. The diameter, girth, clique number, and chromatic number of $I_S(M)$ have been studied. Finally, it has been shown that Beck's Conjecture holds in $I_S(M)$ under certain condition.

1. INTRODUCTION

The characterization of algebraic structures through association of graphs has become an exciting research topic in the last two decades, leading to many fascinating results and questions. Many fundamental papers assigning graphs to a ring have appeared recently, for instance see, [3,4,6,7,10]. The study of behaviour of the ideals and their properties is an important aspect in a ring. Observing this phenomenon Akbari and his co-researchers ([1,2]) have recently introduced the notion of inclusion ideal graph of a ring. The investigation of graphs associated to a module over a commutative ring is also an important and interesting area of research in algebraic graph theory as a module is a generalized algebraic structure of a vector space. Many fundamental papers assigning

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graphs to a module have appeared recently, for instance see, [5,12,13]. In this paper an attempt has been made to provide a module theoretic analogue of the recent work of Akbari ([1,2]) by introducing submodule inclusion graph of a module.

Throughout this paper R denotes a commutative ring with unity, M denotes an R-module and all graphs are simple.

In this paper, $J(M)$ denotes the Jacobson radical of M. A module M is called uniserial if all the submodules of M are linearly ordered by inclusion. A module is serial if it is a direct sum of uniserial modules. A module M is said to be simple if $M \neq 0$ and it has no non-trivial submodules. A module M is called semisimple if it is sum of its simple submodules. A chain of submodules of M of the type $0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$ such that each factor $M_t/M_{t-1}(t = 1, 2, \ldots, n)$ is a simple module is defined as a composition series of length n for M. By Jordan-Holder Theorem ([15], Theorem 6.3.11) any two composition series of M must have the same length. The length of a composition series of M is denoted by $\ell(M)$. Any other undefined terminology related to rings and modules can be found in [8,15].

Let $G = (V, E)$ be a graph where $V(G)$ is the vertex set and $E(G)$ is the edge set of G. For a subset $X \subset V(G)$, the induced subgraph of G on X is denoted by $\langle X\rangle_G$. The vertices u and v of G are said to be adjacent if they are joined by an edge and we denote it by $u \sim v$. A path in G is an alternating sequence of vertices and edges. We denote a path with n vertices in G by P_n. A graph G is connected if there is a path between every two distinct vertices. A graph which is not connected is called a disconnected graph. For two vertices x and y in G, the length of the shortest path from x to y is denoted by $d(x, y)$. If no such path exists, then we define $d(x, y) = \infty$. The diameter of G denoted by $diam(G)$ is defined as $diam(G)$=sup$\{d(x, y)|x$ and y are two vertices of $G\}$. The degree of $v \in V(G)$, denoted by $d(v)$ is the number of edges incident with v. A graph G with n vertices is said to be a complete graph if any two of its distinct vertices are adjacent and it is denoted by K_n. The chromatic number of a graph G, denoted by $\chi(G)$, is defined to be the minimum number of colors which can be assigned to the vertices of G such that no two adjacent vertices have the same color. A maximal complete subgraph of G is called a clique. The size of the largest clique in G is called the clique number of G and is denoted by $\omega(G)$. Clearly we have $\chi(G) \geq \omega(G)$. But, Beck [9] conjectured for a graph

G, $\chi(G) = \omega(G)$. A closed path in G is called a cycle. The length of the shortest cycle in G is called the girth of G and is denoted by $gr(G)$. If G has no cycle, then we define $gr(G) = \infty$. A tree is a connected graph without any cycle. A caterpillar is a tree for which removing the leaves and incident edges produces a path graph. Any other undefined terminology related to graph theory can be found in [11,14].

2. Submodule inclusion graph and its basic properties

Let R be a commutative ring with unity and M be an R-module. In this section, we introduce submodule inclusion graph of a module and investigate its basic properties.

We begin with the following definition.

Definition 2.1. *For an R-module M, submodule inclusion graph of M, $I_S(M)$, is a (undirected graph) whose vertices are all non-trivial submodules of M and two distinct vertices N and L are adjacent if and only if $N \subset L$ or $L \subset N$.*

Example 1.

(1) *If M is a simple R-module, then $I_S(M)$ is an empty graph.*
(2) *For a prime p, $I_S(\mathbb{Z}_p)$ is empty graph.*
(3) *$I_S(\mathbb{Z}_4)$ contains a single isolated vertex.*
(4) *$I_S(\mathbb{Z}_6)$ is the graph which is a disjoint union of two K_1's.*

Proposition 2.1. *Let M be an R-module. If K is a submodule of M, then $I_S(K)$ is a subgraph of $I_S(M)$.*

Proof. It is clear from the definition of $I_S(M)$ and the fact that every non-trivial submodule of K is also a non-trivial submodule of M. □

Proposition 2.2. *Let M be an R-module. Then $I_S(M)$ is a complete graph if and only if M is a uniserial module.*

Proof. First, assume that M is a uniserial module. Let P and Q be a pair of non-trivial submodules of M. then we have $P \subset Q$ or $Q \subset P$ which yields P and Q are adjacent in $I_S(M)$ and we are done.

Conversely, suppose that that $I_S(M)$ is a complete graph. Then any two non-trivial submodules of M are adjacent in $I_S(M)$. Thus, all submodules of M are in one chain of inclusion and the proof is complete. □

Corollary 2.1. *Let M be an R-module. If M is a serial module, then $I_S(M)$ is a disjoint union of complete graphs.*

Proof. It is clear from Proposition 2.2 and the fact that a serial module is a direct sum of uniserial modules. □

In the following two propositions we investigate some particular types of modules whose submodule inclusion graphs are complete.

Proposition 2.3. *Let M be an Artinian R-module. Then $I_S(M)$ is a complete graph if and only if M contains a unique minimal submodule.*

Proof. Since M is Artinian, so M has atleast one minimal submodule. Moreover, every non-trivial submodule of M contains a minimal submodule. Therefore, if M possesses a unique minimal submodule, say L, then L is contained in every non-trivial submodule of M. This implies that $I_S(M)$ is a complete graph. The converse is straightforward. □

Similarly, we have the following proposition.

Proposition 2.4. *Let M be a Noetherian R-module. Then $I_S(M)$ is a complete graph if and only if M contains a unique maximal submodule.*

we now study the connectedness and diameter of submodule inclusion graph of M.

Proposition 2.5. *Let M be an R-module. Then $I_S(M)$ is a disconnected graph if and only if M is a direct sum of two simple R-modules. Moreover, if $I_S(M)$ is a connected graph, then $diam(I_S(M)) \leq 3$.*

Proof. Suppose that $I_S(M)$ is disconnected. Let C_1 and C_2 be two components of $I_S(M)$ such that $P \in C_1$ and $Q \in C_2$ for a pair of non-trivial submodules P and Q of M. We first claim that $M = P \oplus Q$. Since$I_S(M)$ is disconnected so there is no $P - Q$ path which yields $d(P, Q) = \infty$. Therefore, we have $P \cap Q = 0$. Now, if $M \neq P + Q$, then $P \sim P + Q \sim Q$ is a path from P to Q, a contradiction. Hence, we have $M = P \oplus Q$.

Next, we claim that both P and Q are minimal submodules of M. Assume that $0 \neq R \subset P$ is another submodule of M. Then R and P are adjacent vertices in $I_S(M)$, which implies that $R \in C_1$. Hence there is no $R - Q$ path and so by the same argument as above we have $M = R + Q$. Now, let $x \in P$. Then $x = y + z$ for some $y \in R$ and $z \in Q$ which yields $x - y \in P \cap Q = 0$ and so $x = y$, which implies that $P \subseteq R$, a contradiction. Thus, P is a minimal submodule of M. Similarly, Q is also a minimal submodule of M. The same argument shows that P and Q are also maximal submodules of M. But, the minimality of P and Qimplies that they are simple R-modules and since $M = P \oplus Q$ we are done.

Conversely, assume that $M = N \oplus L$, where N and L are simple R-modules. Then each non-trivial submodule of M is minimal, and so no pair of non-trivial submodules are adjacent in $I_S(M)$. Therefore, $I_S(M)$ is a union of isolated vertices, and we are done.

Now, assume that $I_S(M)$ is a connected graph. Then $M \neq N \oplus L$ for any two simple R-modules N and L. By the above argument, for every two non-trivial submodules P and Q of M, we obtain an $P - Q$ path $P \sim P + Q \sim Q$ or $P \sim P \cap Q \sim Q$ in $I_S(M)$. So, $diam(I_S(M)) \leq 3$, and the proof is complete. □

Next, we study the girth of submodule inclusion graph of M. We start with the following proposition.

Proposition 2.6. *Let M be an R-module. If $I_S(M)$ contains a cycle of length 4 or 5, then $I_S(M)$ contains a triangle.*

Proof. Suppose that $C_1 : N_1 \sim N_2 \sim N_3 \sim N_4 \sim N_5 \sim N_1$ is a cycle of length 5 in $I_S(M)$. This produces a chain $N_i \subset N_j \subset N_k$ in M, where $1 \leq i, j, k \leq 5$ and hence $I_S(M)$ possesses a triangle.

Again, let $C_2 : N_1 \sim N_2 \sim N_3 \sim N_4 \sim N_1$ be a cycle of length 4 in $I_S(M)$. Suppose that N_1 is not adajcent to N_3 and N_2 is not adjacent to N_4.

So, $N_2, N_4 \supset N_1 + N_3$ or $N_2, N_4 \subset N_1 \cap N_3$. Hence $N_1 \cap N_3, N_1 + N_3 \notin \{N_1, N_2, N_3, N_4\}$. Thus, $N_2 \sim N_1 \sim N_1 + N_3 \sim N_2$ or $N_2 \sim N_1 \sim N_1 \cap N_3 \sim N_2$ is a triangle in $I_S(M)$, and we are done. □

Lemma 2.1. *([8], Proposition 10.15) For an R-module M, M is semisimple and finitely generated if and only if M is Artinian and $J(M) = 0$.*

Lemma 2.2. *([8], Corollary 10.16) For a semisimple R-module M, M is Artinian if and only if M is finitely generated.*

Proposition 2.7. *Let M be an R-module such that $I_S(M)$ is a tree. Then the following hold:*

(1) *$I_S(M)$ is a caterpillar with $diam(I_S(M)) \leq 3$.*
(2) *$J(M) \neq 0$.*
(3) *If M has a unique maximal submodule, then $I_S(M)$ is a star graph.*

Proof.
(1) It follows from Proposition 2.5.

(2) First, we show that M is an Artinian R-module. Note that, if a chain of non-trivial submodules of M has more than two elements, then $I_S(M)$ has a triangle, a contradiction. So, M is an Artinian R-module. Now, we claim that $J(M) \neq 0$. To the contrary, suppose that $J(M) = 0$. Then by Lemma 2.1 and 2.2, we have M is a semisimple module. Therefore by Proposition 2.5, we have $I_S(M)$ is not connected and so $I_S(M)$ is not a tree, a contradiction. Hence, we must have $J(M) \neq 0$.

(3) Note that if $N_1 \subset N_2 \subset N_3$ are three non-trivial submodules of M, then $I_S(M)$ contains a triangle, a contradiction. Therefore, we can assume that every submodule of M is a minimal submodule or a maximal submodule of M. Now, if M has a unique maximal submodule then every non-trivial submodule of M will be adjacent to $J(M)$. On the other hand, since $I_S(M)$ has no cycle, each non-trivial submodule($\neq J(M)$) is minimal. Thus, $I_S(M)$ is a star graph. □

Lemma 2.3. *([8], Proposition 9.16) If M is a semisimple R-module, then $J(M) = 0$.*

Proposition 2.8. *Let M be an R-module. Then $gr(I_S(M)) \in \{3, 6, \infty\}$.*

Proof. First we see that, if there exist three non-trivial submodules N_1,N_2 and N_3 such that $N_1 \subset N_2 \subset N_3$, then $I_S(M)$ contains a triangle and so $gr(I_S(M)) = 3$. Thus we can assume that every non-trivial submodule of M is minimal or maximal and so M is an Artinian R-module.

Now, let us assume that $J(M) \neq 0$. Then by Lemma 2.3, we have M is not semisimple. Therefore, by Proposition 2.5, we have $I_S(M)$ is connected. If M has a unique maximal submodule, then $J(M)$ is the unique maximal submodule of M. Thus every other non-trivial submodule of M is minimal. Therefore, $I_S(M)$ is a star graph and so $gr(I_S(M)) = \infty$. If M has no unique maximal submodule, then $J(M)$ is a minimal submodule of M. If $J(M)$ is the unique minimal submodule of M, then $I_S(M)$ is a star graph and so $gr(I_S(M)) = \infty$. Hence suppose that $N \neq J(M)$ is a minimal submodule of M. If there are two maximal submodules of M containing N, say L_1 and L_2, then $L_1 = N + J(M) = L_2$, a contradiction. Thus each minimal submodule of M, except $J(M)$, is contained in a unique maximal submodule of M. Therefore, $I_S(M)$ is a tree and so $gr(I_S(M)) = \infty$.

Moreover, if $I_S(M)$ has a cycle of length 4 or 5, then by Proposition 2.6, $I_S(M)$ has a triangle and so $gr(I_S(M)) = 3$. Also, if $I_S(M)$ has a cycle of length ≥ 6, then $gr(I_S(M)) = 6$, and the proof is complete. □

3. Clique Number and Chromatic Number of submodule inclusion graph

In this section, the clique number and chromatic number of submodule inclusion graph have been studied.

Proposition 3.1. *Let M_1 and M_2 be two R-modules. Then*

$$\omega(I_S(M_1)) + \omega(I_S(M_2)) + 1 \leq \chi(I_S(M_1 \times M_2)) \leq \chi(I_S(M_1)) + \chi(I_S(M_2)) + 1.$$

Proof. Let $c_1 : V(I_S(M_1)) \to \{1, 2, \ldots, n\}$ be a proper coloring for $I_S(M_1)$, where $n = \chi(I_S(M_1))$ and X_1,X_2,...,X_n be the color classes. First, we claim that one can define c_1 such that for every edge $e = NL \in E(I_S(M_1)), N \subset L,$ where $N \in X_i, L \in X_j$ and $1 \leq i < j \leq n.$ To prove it, let us suppose that H is a connected component of the induced subgraph $\langle X_1 \cup X_2 \rangle$ and $N \supset L,$ where $N \in H_1 = X_1 \cap H$ and $L \in H_2 = X_2 \cap H$. Therefore, for every edge $e' = N'L' \in E(H)$ with $N' \in H_1$ and $L' \in H_2$, we have $N' \supset L'$. Otherwise, it is easy to see that there exists submodules $P_1, P_2 \in H_1$ and $Q \in H_2$ such that $P_1 \subset Q$ and $Q \subset P_2$, which implies that $P_1 \subset P_2$, a contradiction. So, one can replace H_1 and H_2. By continuing this procedure on $\langle X_2 \cup X_3 \rangle$, $\langle X_3 \cup X_4 \rangle$,...,$\langle X_{n-1} \cup X_n \rangle$, the claim is proved.

Now, suppose that $c_2 : V(I_S(M_2)) \to \{1, 2, \ldots, m\}$ is a proper coloring for $I_S(M_2)$, where $m = \chi(I_S(M_2))$ and Y_1,Y_2,...,Y_m are the color classes. Similar to the previous argument, we can assume that for every edge $e = NL \in E(I_S(M_2)), N \subset L,$ where $N \in Y_i, L \in Y_j$ and $1 \leq i < j \leq m.$

Now, we define a coloring $c : V(I_S(M_1 \times M_2)) \to \{1, 2, \ldots, n+m+1\}$ as follows:

$$c(N \times L) = \begin{cases} c_1(N) + c_2(L) & N < M_1 and L < M_2, \\ c_1(N) & N < M_1 and L = 0, \\ c_2(L) & N = 0 and L < M_2, \\ c_1(N) + m + 1 & N < M_1 and L = M_2, \\ c_2(L) + n + 1 & N = M_1 and L < M_2. \end{cases}$$

It is easy to check that c is a proper coloring and so $\chi(I_S(M_1 \times M_2)) \leq \chi(I_S(M_1)) + \chi(I_S(M_2)) + 1$. To prove the lower bound of the chromatic number, suppose that $N_1 \subset N_2 \subset \ldots \subset N_n$ and $L_1 \subset L_2 \subset \ldots \subset L_m$ are the chains of submodules of M_1 and M_2 respectively. Then $N_1 \times 0 \subset \ldots \subset N_n \times 0 \subset N_n \times L_1 \subset \ldots \subset N_n \times L_m \subset N_n \times M_2$ is a chain of $M_1 \times M_2$, and we are done. □

Proposition 3.2. *Let M be an R-module. Then $\omega(I_S(M)) = \ell(M)$.*

Proof. It is obvious that every chain of submodules corresponds to a clique in $I_S(M)$. Let us assume that H is the largest clique in $I_S(M)$. Then we have a tournament from H concerning inclusion which implies that H has a Hamiltonian path corresponding to a chain of M. □

The following proposition shows that Beck's conjecture holds for the submodule inclusion graph $I_S(M)$.

Proposition 3.3. *Let M be an R-module with a finite number of submodules. Then $\chi(I_S(M)) = \omega(I_S(M))$.*

Proof. We show that $I_S(M)$ and its complement do not contain induced odd cycle of length at least 5. To the contrary, first assume that

$$N_1 \sim N_2 \sim \ldots \sim N_{2n+1} \sim N_1,$$

is an induced subgraph of G and $n \geq 2$.It is easy to see that there are three submodules N_i,N_j and N_k such that $N_i \subset N_j \subset N_k$,where $1 \leq i, j, k \leq 2n + 1$.So,$\langle N_i, N_j, N_k \rangle_G$ forms a triangle, a contradiction. Now, assume that $H = \langle N_1, N_2, \ldots, N_{2n+1} \rangle_G$ be the complement of the above cycle.Consider the cycle

$$N_1 \sim N_{n+1} \sim N_{2n+1} \sim N_n \sim \ldots \sim N_2 \sim N_{n+2} \sim N_1,$$

as a subgraph of H.Similarly, it is easy to see that there are three submodules N_i,N_j and N_k such that $N_i \subset N_j \subset N_k$,where $1 \leq i, j, k \leq 2n+1$,$j \equiv i + n(mod 2n+1)$ and $k \equiv j + n \equiv i - 1(mod 2n+1)$.So, $\langle N_i, N_j, N_k \rangle_G$ is a triangle and $N_i \sim N_k$, a contradiction. Thus the result follows. □

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