COEFFICIENT ESTIMATES FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH HORADAM POLYNOMIAL

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ABSTRACT. In this present article, we studied and examined the novel general subclasses of the function class $Σ$ of bi-univalent function defined in the open unit disk, which are associated with the Horadam polynomial. This study locates estimates on the Taylor - Maclaurin coefficients $|a_2|$ and $|a_3|$ in functions of the class which are considered. Additionally, Fekete-Szegö inequality of functions belonging to this subclasses are also obtained.

1. INTRODUCTION AND PRELIMINARIES

Let $𝒜$ be the family of functions $f(z)$ of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U) \]

which are analytic in the open unit open disk $U = \{ z : z \in \mathbb{C}, |z| < 1 \}$. Let $S$ be class of all functions in $𝒜$ which are univalent and normalized by the conditions $f(0) = 0 = f^\prime(0) - 1$ in $U$. Two of the most famous subclasses of univalent functions class $S$ are the class $S^\ast(\alpha)(0 \leq \alpha < 1)$ of starlike functions of order $\alpha$ and the class $C(\alpha)(0 \leq \alpha < 1)$ of convex functions of order $\alpha$. For two functions $f(z)$ and $g(z)$,

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are analytic in \( U \), we say that the function \( f(z) \) is subordinate to \( g(z) \) in \( U \), written as \( f(z) \prec g(z) \), if there exists an analytic function \( w(z) \) defined on \( U \) with
\[
 w(0) = 0 \quad \text{and} \quad |w(z)| < 1, \quad (z \in U),
\]
such that \( f(z) = g(w(z)) \) for all \( (z \in U) \). Also, it is known that
\[
 f(z) \prec g(z) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).
\]

The well-known Koebe one-quarter theorem [7] ensures that the image of \( U \) under every univalent function \( f \in A \) contains a disk of radius \( 1/4 \). Hence every univalent function \( f \) has an inverse \( f^{-1} \) satisfying \( f^{-1}(f(z)) = z, (z \in U) \) and
\[
 f^{-1}(f(w)) = w, (|w| < r_0(f), r_0(f) \geq 1/4),
\]
where
\[
(1.2) \quad g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_3^2 - 5a_2a_3 + a_4)w^4 + \ldots.
\]

A function \( f \in A \) is said to be bi-univalent in \( U \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( U \). Earlier, Lewin [12] investigated the bi-univalent functions and derived that \( |a_2| < 1.51 \). For the brief history of functions in the class \( \Sigma \), Brannan and Clunie [5], and Srivastava et al. [14] proved some results within these coefficient for different classes. Moreover, Brannan and Taha [6] introduced certain subclasses of the bi-univalent function class \( \Sigma \) for the familiar subclasses \( S^*(\alpha) \) and \( C(\alpha) \). More Recent studies inspired by Horcum and Kocer [10], Abirami et al. [1], Alamoush [2], [3], [4] considered Horadam polynomials \( h_n(x) \), which are given by the following recurrence relation
\[
(1.3) \quad h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x), \quad (n \in \mathbb{N} \geq 2),
\]
with \( h_1 = a, h_2 = bx, \) and \( h_3 = px^2 + q \) where \( (a, b, p, q) \) are some real constants.

By taking various values of \( a, b, p \) and \( q \) which leads to various polynomials
- when \( a = b = p = q = 1 \), we obtain the Fibonacci polynomials,
  \( F_n(x) = xF_{n-1}(x) + F_{n-2}(x), F_1(x) = 1, F_2(x) = x; \)
- when \( a = 2, b = p = q = 1 \), we have the Lucas polynomials,
  \( L_{n-1}(x) = xL_{n-2}(x) + L_{n-3}(x), L_0 = 2, L_1 = x; \)
- when \( a = q = 1, b = p = 2 \), we attain the Pell polynomials,
  \( P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), p_1 = 1, p_2 = 2x; \)
- when \( a = b = p = 2, q = 1 \), we get the Pell-Lucas polynomials,

\[
Q_{n-1}(x) = 2xQ_{n-2}(x) + Q_{n-3}(x), Q_0 = 2, Q_1 = 2x;
\]

- when \( a = 1, b = p = 2, q = 1 \), we obtain the Chebyshev polynomials of second kind sequence,

\[
U_{n-1}(x) = 2xU_{n-2}(x) + U_{n-3}(x), U_0 = 1, U_1 = 2x;
\]

- if \( a = 1, b = p = 2, q = 1 \), we have the Chebyshev polynomials of First kind sequence,

\[
T_{n-1}(x) = 2xT_{n-2}(x) + T_{n-3}(x), T_0 = 1, T_1 = x.
\]

One can refer [8], [9], [11] and [13] for more details connected with these polynomials succession.

The characteristic equation of recurrence relation (1.3) is

\[
t^2 - pxt - q = 0.
\]

This equation has two real roots,

\[
\alpha = \frac{px + \sqrt{p^2x^2 + 4q}}{2},
\]

and

\[
\beta = \frac{px - \sqrt{p^2x^2 + 4q}}{2}.
\]

**Remark 1.1.** [10] Let \( \Omega(x,z) \) be the generating function of the Horadam polynomials \( h_n(x) \). Then

\[
\Omega(x, z) = \frac{a + (b - ap)xz}{1 - pxz - qz^2} = \sum_{n=1}^{\infty} h_n(x)z^{n-1}.
\]

In this present work, we introduce \( \mathcal{S}_\Sigma(\lambda, \gamma, x) \) and \( \mathcal{M}_\Sigma(\lambda, \gamma, x) \) are the class of bi-univalent functions. Within this, coefficient estimates \( |a_2| \) and \( |a_3| \). The Fekete-Szegö problem are also derived for the function \( f \in \Sigma \) belonging to the new defined subclasses.
2. Set of Main Results

We now define the new bi-univalent subclasses of analytic function.

**Definition 2.1.** For $0 \leq \lambda \leq 1$ and $\gamma \in \mathbb{C} \setminus \{0\}$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $S_{\Sigma}(\lambda, \gamma, x)$, if the following conditions are satisfied:

$$1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{(1 - \lambda)z + \lambda f(z)} - 1 \right) \prec \Omega(x, z) + 1 - a$$

and

$$1 + \frac{1}{\gamma} \left( \frac{wg'(w)}{(1 - \lambda)w + \lambda g(w)} - 1 \right) \prec \Omega(x, w) + 1 - a,$$

where $g = f^{-1}$ is given by (1.2) and $z, w \in \mathbb{U}$.

**Definition 2.2.** For $0 \leq \lambda \leq 1$ and $\gamma \in \mathbb{C} \setminus \{0\}$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $M_{\Sigma}(\lambda, \gamma, x)$, if the following conditions are satisfied:

$$1 + \frac{1}{\gamma} \left( \frac{zf'(z) + z^2f''(z)}{(1 - \lambda)z + \lambda zf'(z)} - 1 \right) \prec \Omega(x, z) + 1 - a$$

and

$$1 + \frac{1}{\gamma} \left( \frac{wg'(w) + w^2g''(w)}{(1 - \lambda)w + \lambda wg'(w)} - 1 \right) \prec \Omega(x, w) + 1 - a,$$

where $g = f^{-1}$ is given by (1.2) and $z, w \in \mathbb{U}$.

**Theorem 2.1.** Let the function $f \in \Sigma$ be given by (1.1) be in the class $S_{\Sigma}(\lambda, \gamma, x)$. Then

$$(2.2) \quad |a_2| \leq \frac{\gamma |bx| \sqrt{|bx|}}{\sqrt{|bx^2[(\lambda^2 - 3\lambda + 3)\gamma b - (2 - \lambda)^2p] - aq(2 - \lambda)^2|}},$$

$$(2.3) \quad |a_3| \leq \frac{\gamma |bx|}{(3 - \lambda)} + \frac{\gamma^2 (bx)^2}{(2 - \lambda)^2},$$

and for some $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|\gamma| |bx|}{(3 - \lambda)} & \text{if } |\eta - 1| \leq \sigma_1 \\ \frac{\gamma^2 |bx|^3(\eta - 1)}{|(\lambda^2 - 3\lambda + 3)\gamma b - p(2 - \lambda)^2|bx^2 - qa(2 - \lambda)^2|} & \text{if } |\eta - 1| \geq \sigma_1. \end{cases}$$

Here,

$$\sigma_1 = \frac{|(\lambda^2 - 3\lambda + 3)\gamma b - p(2 - \lambda)^2|bx^2 - qa(2 - \lambda)^2|}{(bx)^2(3 - \lambda)}.$$
Proof. Let \( f \in \Sigma \) be given by the Taylor-Maclaurin expansion (1.1). Then, for some analytic functions \( \Psi \) and \( \Phi \) such that \( \Phi(0) = \psi(0) = 0 \), \( |\psi(z)| < 1 \) and \( |\phi(w)| < 1 \), \( z, w \in U \) and using Definition 2.1, we can write

\[
1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1 \right) = \Omega(x, \Phi(z)) + 1 - a
\]

and

\[
1 + \frac{1}{\gamma} \left( \frac{wg'(w)}{(1-\lambda)w + \lambda g(w)} - 1 \right) = \Omega(x, \psi(w)) + 1 - a.
\]

Equivalently,

\[
1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1 \right) = 1 + h_1(x) - a + h_2(x)\Phi(z) + h_3(x)[\Phi(z)]^3 + \ldots
\]

and

\[
1 + \frac{1}{\gamma} \left( \frac{wg'(w)}{(1-\lambda)w + \lambda g(w)} - 1 \right) = 1 + h_1(x) - a + h_2(x)\psi(w) + h_3(x)[\psi(w)]^2 + \ldots.
\]

From (2.4) and (2.5), we obtain

\[
1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1 \right) = 1 + h_2(x)p_1z + [h_2(x)p_2 + h_3(x)p_3^2]z^2 + \ldots
\]

and

\[
1 + \frac{1}{\gamma} \left( \frac{wg'(w)}{(1-\lambda)w + \lambda g(w)} - 1 \right) = 1 + h_2(x)q_1w + [h_2(x)q_2 + h_3(x)q_3^2]w^2 + \ldots.
\]

Notice that if

\[
|\Phi(z)| = |p_1z + p_2z^2 + p_3z^3 + \ldots| < 1 \quad (z \in U)
\]

and

\[
|\psi(w)| = |q_1w + q_2w^2 + q_3w^3 + \ldots| < 1 \quad (w \in U),
\]

then

\[
|p_i| \leq 1 \quad \text{and} \quad |q_i| \leq 1 \quad (i \in \mathbb{N}).
\]

Thus, upon comparing the corresponding coefficients in (2.6) and (2.7), we have

\[
\frac{(2-\lambda)}{\gamma} a_2 = h_2(x)p_1,
\]

and

\[
\frac{(3-\lambda)a_3 - \lambda(2-\lambda)a_2^2}{\gamma} = h_2(x)p_2 + h_3(x)p_3^2.
\]
\[
(2.10) \quad \frac{-(2 - \lambda)}{\gamma} a_2 = h_2(x)q_1, \\
\text{and} \quad \frac{(3 - \lambda)(2a_2^2 - a_3) - \lambda(2 - \lambda)a_2^2}{\gamma} = h_2(x)q_2 + h_3(x)q_1^2.
\]
From (2.8) and (2.10), we find that
\[
(2.12) \quad p_1 = -q_1
\]
and
\[
(2.13) \quad a_2^2 = \frac{\gamma^2 h_2^2(x)(p_1^2 + q_1^2)}{2(2 - \lambda)^2}.
\]
Adding (2.9) and (2.11), we obtain
\[
(2.14) \quad \frac{2(\lambda^2 - 3\lambda + 3)}{\gamma} a_2^2 = h_2(x)(p_2 + q_2) + h_3(x)(p_1^2 + q_1^2).
\]
By using (2.13) in (2.14), we get
\[
(2.15) \quad a_2^2 = \frac{\gamma^2 h_3^2(x)(p_2 + q_2)}{2(\lambda^2 - 3\lambda + 3)h_2^2(x) - 2h_3(x)(2 - \lambda)^2}.
\]
From (1.3), we have the desired inequality (2.2).

Next, by subtracting (2.11) from (2.9) and in view of (2.12), we have
\[
\frac{2(3 - \lambda)a_3 - 2(3 - \lambda)a_2^2}{\gamma} = h_2(x)(p_2 - q_2) + h_3(x)(p_1^2 - q_1^2)
\]
and
\[
a_3 = a_2^2 + \frac{\gamma h_2(x)(p_2 - q_2)}{2(3 - \lambda)}.
\]
Hence using (2.13) and applying (1.3), we get desired inequality (2.3).

For some \(\eta \in \mathbb{R}\), we write
\[
(2.16) \quad a_3 - \eta a_2^2 = \frac{\gamma h_2(x)(p_2 - q_2)}{2(3 - \lambda)} + (1 - \eta)a_2^2.
\]
Now, by using (2.15) and (2.16), we get
\[
a_3 - \eta a_2^2 = \frac{\gamma^2 h_2^3(x)(1 - \eta)(p_2 + q_2)}{2(\lambda^2 - 3\lambda + 3)\gamma h_2^2(x) - 2(2 - \lambda)^2 h_3(x)} + \frac{\gamma h_2(x)(p_2 - q_2)}{2(3 - \lambda)}
\]
\[
= \gamma h_2(x) \left[ \left( \Theta(\eta, x) + \frac{1}{2(3 - \lambda)} \right) p_2 + \left( \Theta(\eta, x) - \frac{1}{2(3 - \lambda)} \right) q_2 \right],
\]
where

\[ \Theta(\eta, x) = \frac{\gamma [h_2(x)]^2 (1 - \eta)}{2(\lambda^2 - 3\lambda + 3)\gamma [h_2(x)]^2 - 2(2 - \lambda)^2 h_3(x)}. \]

So, we conclude that

\[ |a_3 - \eta a_2^2| \leq \begin{cases} 
\frac{|\gamma||h_2(x)|}{3 - \lambda} & \text{if } 0 \leq |\Theta(\eta, x)| \leq \frac{1}{2(3 - \lambda)} \\
2|\gamma||h_2(x)||\Theta(\eta, x)| & \text{if } |\Theta(\eta, x)| \geq \frac{1}{2(3 - \lambda)}. 
\end{cases} \]

This proves Theorem 2.1. \(\square\)

For \(\lambda = 1\), Theorem 2.1 readily yields the following coefficient estimates:

**Corollary 2.1.** Let the function \(f \in \Sigma\) given by (1.1) be in the class \(S_{\Sigma}(1, \gamma, x)\). Then

\[ |a_2| \leq \frac{|\gamma||bx|\sqrt{|bx|}}{\sqrt{|bx^2[b - p] - qa|}}, \]

\[ |a_3| \leq \frac{|\gamma||bx|}{2} + \gamma^2 b^2 x^2 \]

and for some \(\eta \in \mathbb{R}\),

\[ |a_3 - \eta a_2^2| \leq \begin{cases} 
\frac{|\gamma||bx|}{3 - \lambda} & \text{if } |\eta - 1| \leq \sigma_2 \\
\gamma^2 |bx|^2 (\eta - 1) & \text{if } |\eta - 1| \geq \sigma_2.
\end{cases} \]

Here,

\[ \sigma_2 = \frac{|[b - p]bx^2 - qa|}{2(bx)^2}. \]

In light of Remark 1.1, we have

**Corollary 2.2.** Let the function \(f \in \Sigma\) given by (1.1) be in the class \(S_{\Sigma}(\lambda, \gamma, x)\). Then

\[ |a_2| \leq \frac{2|\gamma||t|\sqrt{|2t|}}{\sqrt{|[\gamma^2 - 3\lambda + 3]2\gamma - 2(2 - \lambda)^2|2t^2 + (2 - \lambda)^2}|}}, \]

\[ |a_3| \leq \frac{2|\gamma||t|}{3 - \lambda} + \gamma^2 4t^2 \]

and for some \(\eta \in \mathbb{R}\),

\[ |a_3 - \eta a_2^2| \leq \begin{cases} 
\frac{|\gamma||2t|}{3 - \lambda} & \text{if } |\eta - 1| \leq \sigma_3 \\
\gamma^2 |2t|^3 (\eta - 1) & \text{if } |\eta - 1| \geq \sigma_3.
\end{cases} \]
Here
\[
\sigma_3 = \left\{ \frac{|(\lambda^2 - 3\lambda + 3)2\gamma - 2(2 - \lambda)^2|2\eta^2 + (2 - \lambda)^2}{4\lambda^2(3 - \lambda)} \right\}.
\]

**Theorem 2.2.** Let the function \( f \in \Sigma \) be given by (1.1) be in the class \( \mathcal{M}_\Sigma(\lambda, \gamma, x) \). Then
\[
|a_2| \leq \frac{|\gamma||bx|\sqrt{|bx|}}{\sqrt{|bx^2|((4\lambda^2 - 11\lambda + 9)\gamma b - (2 - \lambda)^24p - aq4(2 - \lambda)^2)}},
\]
and for some \( \eta \in R \),
\[
|a_3 - \eta a_2^2|
\]
\[
\leq \begin{cases} 
\frac{|\gamma||bx|}{3(3 - \lambda)} & \text{if } |\eta - 1| \leq \sigma_4 \\
\frac{\gamma^2|bx|^3(\eta - 1)}{((4\lambda^2 - 11\lambda + 9)\gamma b - 4p(2 - \lambda)^2)bx^2 - 4qa(2 - \lambda)^2} & \text{if } |\eta - 1| \geq \sigma_4.
\end{cases}
\]

Here,
\[
\sigma_4 = \left\{ \frac{|(4\lambda^2 - 11\lambda + 9)\gamma b - 4p(2 - \lambda)^2|bx^2 - 4qa(2 - \lambda)^2}{(bx)^23(3 - \lambda)} \right\}.
\]

**Proof.** Let \( f \in \mathcal{M}_\Sigma(\lambda, \gamma, x) \). be given by Taylor-Maclaurin expansion (1.1). Then for all \( z, w \in \mathbb{U} \) with \( \Phi(0) = \psi(0) = 0, |\Phi(z)| < 1, |\psi(w) < 1 | \) such that
\[
1 + \frac{1}{\gamma} \left( \frac{zf'(z) + z^2f''(z)}{(1 - \lambda)z + \lambda z f'(z) - 1} \right) = \Omega(x, \Phi(z)) + 1 - a
\]
and
\[
1 + \frac{1}{\gamma} \left( \frac{wg'(w) + w^2g''(w)}{(1 - \lambda)w + \lambda wg'(w) - 1} \right) = \Omega(x, \psi(w)) + 1 - a.
\]
Equivalently it can be written as
\[
1 + \frac{1}{\gamma} \left( \frac{zf'(z) + z^2f''(z)}{(1 - \lambda)z + \lambda z f'(z) - 1} \right) = 1 + h_1(x) - a + h_2(x)\Phi(z) + h_3(x)[\Phi(z)]^3 + \ldots
\]
and
\[
1 + \frac{1}{\gamma} \left( \frac{wg'(w) + w^2g''(w)}{(1 - \lambda)w + \lambda wg'(w) - 1} \right) = 1 + h_1(x) - a + h_2(x)\psi(w) + h_3(x)[\psi(w)]^3 + \ldots.
\]
Making use of the inequality \( |\Phi(z)| < 1 \) and \( |\psi(z)| < 1 \), we have
\[
1 + \frac{1}{\gamma} \left( \frac{zf'(z) + z^2f''(z)}{(1 - \lambda)z + \lambda z f'(z) - 1} \right)
= 1 + h_2(x)p_1z + [h_2(x)p_2 + h_3(x)p_1^2]z^2 + \ldots
\]

(2.17)
and
\[
1 + \frac{1}{\gamma} \left( \frac{w g'(w) + w^2 g''(w)}{(1 - \lambda)w + \lambda wg'(w)} - 1 \right)
= 1 + h_2(x)q_1 w + [h_2(x)q_2 + h_3(x)q_1^2]w^2 + \ldots.
\]

(2.18)

Now comparing the like coefficients of (2.17) and (2.18), we have
\[
\frac{2(2 - \lambda)}{\gamma} a_2 = h_2(x)p_1,
\]
(2.19)

\[
\frac{3(3 - \lambda)a_3 - 4\lambda(2 - \lambda)a_2^2}{\gamma} = h_2(x)p_2 + h_3(x)p_1^2,
\]
(2.20)

\[
\frac{-2(2 - \lambda)}{\gamma} a_2 = h_2(x)q_1,
\]
(2.21)

and
\[
\frac{3(3 - \lambda)(2a_2^2 - a_3) - 4\lambda(2 - \lambda)a_2^2}{\gamma} = h_2(x)q_2 + h_3(x)q_1^2.
\]
(2.22)

From (2.19) and (2.21), we can observe that
\[
p_1 = -q_1
\]
(2.23)

and
\[
a_2^2 = \frac{\gamma^2 h_2^3(x)(p_1^2 + q_1^2)}{2(1 + 2\lambda + 6\delta)[h_2(x)]^2 - 2h_3(x)(1 + \lambda + 2\delta)^2}.
\]
(2.24)

Adding (2.20) and (2.22), we get
\[
\frac{2(4\lambda^2 - 11\lambda + 9)}{\gamma} a_2 = h_2(x)(p_2 + q_2) + h_3(x)(p_1^2 + q_1^2).
\]
(2.25)

Substituting (2.24) in (2.25), we have
\[
a_2^2 = \frac{h_2(x)^3(u_2 + v_2)}{2(1 + 2\lambda + 6\delta)[h_2(x)]^2 - 2h_3(x)(1 + \lambda + 2\delta)^2}.
\]
(2.26)

Using (1.3), the above equation yields
\[
|a_2| \leq \frac{|\gamma||bx|\sqrt{|bx|}}{\sqrt{|bx^2|(4\lambda^2 - 11\lambda + 9)\gamma b - (2 - \lambda)^2 4p} - aq(2 - \lambda)^2}.
\]

Similarly, upon subtracting equation (2.22) from the equation (2.20) and in view of (2.23), we obtain
\[
\frac{3(3 - \lambda)a_3 - 3(3 - \lambda)(2a_2^2 - a_3)}{\gamma} = h_2(x)(p_2 - q_2) + h_3(x)(p_1^2 - q_1^2).
\]
Applying (1.3), we deduce that

$$|a_3| \leq |\gamma b| |x| + \frac{\gamma^2 (bx)^2}{3(3 - \lambda)}.$$  

For any $\eta \in \mathbb{R}$,

$$a_3 - \eta a_2^2 = \frac{\gamma h_2(x)(p_2 - q_2)}{6(3 - \lambda)} + (1 - \eta) a_2^2.$$  

Substituting (2.26) in (2.27), we have

$$a_3 - \eta a_2^2 = \frac{\gamma h_2(x)(p_2 - q_2)}{6(3 - \lambda)} + \left[\Theta(\eta, x) + \frac{1}{6(3 - \lambda)}p_2 + \left(\Theta(\eta, x) - \frac{1}{6(3 - \lambda)}\right)\right],$$

where

$$\Theta(\eta, x) = \frac{\gamma [h_2(x)]^2 (1 - \eta)}{2(4\lambda^2 - 11\lambda + 9)\gamma [h_2(x)]^2 - 8(2 - \lambda)^2 h_3(x)}.$$  

Hence in view of (1.3), we conclude that

$$|a_3 - \nu a_2^2| \leq \begin{cases} 
\frac{|\gamma||h_2(x)||}{3(3 - \lambda)} & 0 \leq |\Theta(\eta, x)| \leq \frac{1}{6(3 - \lambda)} \\
2|\gamma||h_2(x)||\Theta(\eta, x)| & |\Theta(\eta, x)| \geq \frac{1}{6(3 - \lambda)} 
\end{cases}$$

which completes the proof of the Theorem (2.2).  

For $\lambda = 1$, Theorem 2.2 readily yields the following corollaries:

**Corollary 2.3.** Let the function $f \in \Sigma$ given by (1.1) be in the class $\mathcal{M}_\Sigma(1, \gamma, x)$. Then for some $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \leq \begin{cases} 
\frac{|\gamma||b|}{6(3 - \lambda)} & \text{if } |\eta - 1| \leq \sigma_5 \\
\frac{\gamma^2 |b|^3 (\eta - 1)}{|2\gamma b - 4p|^2 b^2 - 4qa} & \text{if } |\eta - 1| \geq \sigma_5.
\end{cases}$$

Here

$$\sigma_5 = \frac{|2\gamma b - 4p|^2 b^2 - 4qa}{6(bx)^2}.$$  

In view of Remark 1.1,

**Corollary 2.4.** Let the function $f \in \Sigma$ given by (1.1) be in the class $\mathcal{M}_\Sigma(\lambda, \gamma, t)$ and for some $\eta \in \mathbb{R}$,
\[|a_3 - \eta a_2^2| \leq \left\{ \begin{array}{ll}
\frac{|\gamma|2|t|}{3(3 - \lambda)} & \text{if } |\eta - 1| \leq \sigma_6 \\
\gamma^2|2t|^3(\eta - 1) & \text{if } |\eta - 1| \geq \sigma_6.
\end{array} \right.\]

Where

\[\sigma_6 = \frac{|((4\lambda^2 - 11\lambda + 9)2\gamma - 8(2 - \lambda)^2)|2t^2 + 4(2 - \lambda)^2|}{12t^2(3 - \lambda)}.
\]

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