SUFFICIENT CONDITIONS FOR STARLIKENESS USING SUBORDINATION METHOD

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Abstract. Let $f$ be analytic in the unit disk and normalized by $f(0) = f'(0) - 1 = 0$. In this paper using a method from the theory of first order differential subordination we investigate the sufficient conditions over the differential subordination

$$p(z) + zp'(z) < \frac{1 + Az(2 + Bz)}{(1 + Bz)^2}$$

that implies $p(z) < \frac{1 + Az}{1 + Bz}$, $-1 \leq B < A \leq 1$, and further use it for obtaining inequalities over the function $f$.

1. Introduction and Preliminaries

Analytic function $f$ defined in the domain $D$ is univalent if it is injective. Let $A$ denotes the class of functions $f$ that are analytic in the unit disk $D = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$, i.e., such that $f(z) = z + a_2z^2 + \cdots$.

A function $f \in A$ is said to be starlike if, and only if

$$\text{Re} \left[ \frac{zf'(z)}{f(z)} \right] > 0, \quad z \in D.$$ 

We denote by $S^*$ the class of all such functions which are at the same time univalent. Their geometrical characterisations is the following: $f$ is starlike if, and only

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if \( \omega \in f(\mathbb{D}) \) for all \( \omega \in f(\mathbb{D}) \) and all \( t \in [0,1] \), i.e., for all \( z \in \mathbb{D} \), \( f(z) \) is visible from the origin. For details see [1,7].

A special subclass of \( S^* \) is the class of starlike function of order \( \alpha \) with \( 0 \leq \alpha < 1 \), given by
\[
S^*(\alpha) = \left\{ f \in \mathcal{A} : \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \ z \in \mathbb{D} \right\}.
\]

Further, a function \( f \) is said to be subordinate to \( F \), written \( f \prec F \) or \( f(z) \prec F(z) \), if there exists a function \( w \) analytic in \( \mathbb{D} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \), and such that \( f(z) = F(w(z)) \). If \( F \) is univalent, then \( f \prec F \) if, and only if, \( f(0) = F(0) \) and \( f(\mathbb{D}) \subset F(\mathbb{D}) \). For details see [2].

Using subordination, another generalisation is defined by
\[
S^*[A,B] = \left\{ f \in \mathcal{A} : \frac{zf''(z)}{f'(z)} < \frac{1 + Az}{1 + Bz}, \ z \in \mathbb{D} \right\},
\]
\(-1 \leq B < A \leq 1 \). Geometrically, this means that the image of \( \mathbb{D} \) by \( \frac{zf''(z)}{f'(z)} \) is inside the open disk centered on the real axis with diameter endpoints \( (1-A)/(1-B) \) and \( (1+A)/(1+B) \). In [5] it is given that special selections of \( A \) and \( B \) lead us to the following:
- \( S^*[1,-1] \equiv S^* \);
- \( S^*[1-2\alpha,-1] \equiv S^*(\alpha), \ 0 \leq \alpha < 1 \).

Next, we denote by \( \mathcal{K} \) the class of convex functions, i.e., the class of function \( f(z) \in \mathcal{A} \) for which
\[
\Re \left[ 1 + \frac{zf''(z)}{f'(z)} \right] > 0, \ z \in \mathbb{D}.
\]
and its generalization, the class of convex functions of order \( \alpha \), with \( 0 \leq \alpha < 1 \), given by
\[
\mathcal{K}(\beta) = \left\{ f \in \mathcal{A} : \Re \left[ 1 + \frac{zf''(z)}{f'(z)} \right] > \alpha, \ z \in \mathbb{D} \right\}.
\]
Both these classes (\( S^* \) and \( \mathcal{K} \)) are subclasses of univalent function in \( \mathbb{D} \) and even more \( \mathcal{K} \subset S^* \). For details see [1,7].

In this paper we study the differential subordination of the form
\[
p(z) + zp'(z) < \frac{1 + Az(2 + Bz)}{(1 + Bz)^2}, \ -1 \leq B < A \leq 1,
\]
and conditions when it implies the subordination $p(z) \prec (1 + Az)/(1 + Bz)$, where $p(z)$ is analytic function and $p(0) = 1$. For special selection of the function $p(z)$, for example for $p(z) = zf'(z)/f(z)$, $p(z) = f(z)/z$ and $p(z) = f'(z)$ the left hand side of this subordination will give special cases that will imply results over inequalities involving the function $f$.

For that purpose we will use a method from the theory of first order differential subordinations. If $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ is analytic in the domain $D$, if $h(z)$ is univalent in $D$, and if $p(z)$ is analytic in $D$ with $(p(z), zp'(z)) \in D$ when $z \in \mathbb{D}$, then we say that $p(z)$ satisfies the (first-order) differential subordination

$$(1.1) \quad \psi (p(z), zp'(z)) \prec h(z).$$

The function $p(z)$ is called the solution of differential subordination (1.1). The univalent function $q(z)$ is called dominant of the solution of differential equation (1.1) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.1). The dominant $\tilde{q}(x)$ satisfies $\tilde{q}(x) \prec q(z)$ for all dominants $q(z)$ of (1.1) is said to be the best dominant of (1.1).

From this theory we will make use of the following lemma due to Miller and Mocanu [2].

**Lemma 1.1.** [2] Let $q$ be univalent in the unit disk $\mathbb{D}$, and let $\theta(w)$ and $\phi(w)$ be analytic in a domain $D$ containing $q(\mathbb{D})$, with $\phi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that:

(i) $Q$ is starlike in the unit disk $\mathbb{D}$,

(ii) $\Re \frac{zh'(z)}{Q(z)} = \Re \left[ \frac{\theta'(q(z)) + zQ'(z)}{\phi(q(z))} \right] > 0$, $z \in \mathbb{D}$.

If $p$ is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subseteq D$ and

$$(1.2) \quad \theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z)$$

then $p(z) \prec q(z)$, and $q$ is the best dominant of (1.2).

2. **Main results and consequences**

First we will prove a lemma that will later lead to the main result.
Lemma 2.1. Let $p(z)$ be analytic in the unit disk $D$, $p(0) = 1$, $0 \notin p(D)$. Also, let $A$, $B$ be a real number with $-1 \leq B < A \leq -1$. If
\begin{equation}
(2.1) \quad p(z) + zp'(z) \preceq \frac{1 + Az(2 + Bz)}{(1 + Bz)^2},
\end{equation}
then $p(z) \prec q(z) = \frac{1 + Az}{1 + Bz}$ and $q(z)$ is the best dominant of (2.1).

Proof. In Lemma 1.1 we choose $\theta(\omega) = \omega$ and $\phi(\omega) = 1$, which are analytic in domain $D = \mathbb{C}$. Then $q(z)$ is univalent in $\mathbb{D}$ and $\phi(\omega)$ and $\theta(\omega)$ are analytic in domain $D = \mathbb{C}$ containing $q(z) = \frac{1 + Az}{1 + Bz}$ with $\phi(\omega) \neq 0$ when $\omega \in q(D)$. Further, set
\[ Q(z) = zq'(z)\phi(q(z)) = (A - B)z \]
which is starlike because
\[ \frac{zQ'(z)}{Q(z)} = \frac{1 - Bz}{1 + Bz}, \]
and for $z = e^{i\theta}$, $\theta \in [-\pi, \pi]$,
\[ \text{Re} \left\{ \frac{zQ'(z)}{Q(z)} \right\} = \frac{1 - B^2}{|1 + Be^{i\theta}|^2} \geq 0. \]
Next,
\[ h(z) = \theta(q(z)) + Q(z) = \frac{1 + Az(2 + Bz)}{(1 + Bz)^2} \]
and
\[ \frac{zh'(z)}{Q(z)} = \frac{2}{1 + Bz}. \]
For $z = e^{i\theta}$, $\theta \in [-\pi, \pi]$ we have
\[ \text{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \frac{2 + 2B \cos \theta}{1 + 2B \cos \theta + B^2} = \frac{2 + 2B \cos \theta}{|1 + Be^{i\theta}|^2} \geq 0. \]
So, from $p(0) = q(0) = 1$ and from (1.2) we receive that $p(z) \prec q(z)$ and $q(z) = \frac{1 + Az}{1 + Bz}$ is the best dominant of (2.1). $\Box$

Putting $p(z) = \frac{zf'(z)}{f(z)}$ in Lemma 1 we obtain the main result.

Theorem 2.1. Let $f \in \mathcal{A}$, and let $A$, $B$ be a real numbers, $-1 \leq B < A \leq -1$. If
\begin{equation}
(2.2) \quad \frac{zf'(z)}{f(z)} \left[ \frac{1 + zf''(z)}{f'(z)} + \left( 1 - \frac{zf'(z)}{f(z)} \right) \right] \preceq \frac{1 + Az(2 + Bz)}{(1 + Bz)^2} \equiv h(z)
\end{equation}
then

\[ \frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz}. \]

The right hand side of (2.3) is the best dominant of (2.2).

**Proof.** Let \( p(z) = \frac{zf'(z)}{f(z)} \) and \( q(z) = \frac{1 + Az}{1 + Bz} \). Then,

\[
p(z) + zp'(z) = \frac{zf'(z)}{f(z)} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right) + \left( 1 - \frac{zf'(z)}{f(z)} \right) \right],
\]

\[
q(z) + zq'(z) = \frac{1 + Az(2 + Bz)}{(1 + Bz)^2}
\]

and

\[ p(z) + zp'(z) \prec q(z) + zq'(z). \]

Since, \( p(0) = q(0) = 1 \) from Lemma 1.1 we have \( p(z) \prec q(z) \), i.e.,

\[ \frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz}, \]

where \( q(z) \) is best dominant. \( \square \)

**Corollary 2.1.** Let \( f \in A \).

(i) If \(-1 \leq B < A \leq 1 \) and

\[
\left| \frac{zf'(z)}{f(z)} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right) + \left( 1 - \frac{zf'(z)}{f(z)} \right) \right] - 1 \right| < \frac{(A - B)}{2(1 + |B|)^2},
\]

\( z \in \mathbb{D} \), then

\[ \left| \frac{zf'(z)}{f(x)} - 1 \right| < \frac{A - B}{1 - |B|}, \quad z \in \mathbb{D}. \]

(ii) If \( B = 0 \) and \( 0 < A \leq 1 \), then

\[
\left| \frac{zf'(z)}{f(z)} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right) + \left( 1 - \frac{zf'(z)}{f(z)} \right) \right] - 1 \right| \leq 2A, \quad z \in \mathbb{D},
\]

implies

\[ \left| \frac{zf'(z)}{f(x)} - 1 \right| < A, \quad z \in \mathbb{D}. \]

(iii) If \( B = -1 \) and \( A = 1 - 2\alpha \), for \( \alpha \in [0, 1) \), then:

\[
\left| \frac{zf'(z)}{f(z)} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right) + \left( 1 - \frac{zf'(z)}{f(z)} \right) \right] - 1 \right| < \frac{3}{2}(1 - \alpha), \quad z \in \mathbb{D},
\]

implies that \( f \in S^*(\alpha) \).
Proof.
(i) By the definition of subordination, we have that
\[
\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}
\]
is equivalent to
\[
\frac{zf'(z)}{f(z)} - 1 < \frac{(A - B)z}{1 + Bz}
\]
and implies that
\[
\sup_{z \in \mathbb{D}} \left| \frac{zf'(z)}{f(z)} - 1 \right| = \inf_{|z|=1} \frac{(A - B)z}{1 + Bz} = \frac{A - B}{1 - |B|},
\]
i.e.,
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{A - B}{1 - |B|} \quad (z \in \mathbb{D}).
\]

So, the conclusion in (i) will follow from Theorem 2.1 if we show that
\[
\min_{\theta \in [0,2\pi]} |h(e^{i\theta}) - 1| = (A - B) \frac{2 + |B|}{(1 + |B|)^2},
\]
where \(h\) is defined in Theorem 2.1,
\[
h(z) = \frac{1 + Az(2 + Bz)}{(1 + Bz)^2}.
\]
From
\[
|h(e^{i\theta}) - 1|^2 = \frac{(A - B)^2(4 + B^2 + 4B \cos t)}{(1 + B^2 + 2B \cos t)},
\]
using \(x = \cos t, -1 \leq x \leq 1\), we have
\[
\varphi(x) = \frac{(A - B)^2(4 + B^2 + 4Bx)}{(1 + B^2 + 2Bx)}
\]
and
\[
\varphi'(x) = -\frac{4(A - B)^2B(3 + 2Bx)}{(1 + B^2 + 2Bx)}. \]
The equation \(\varphi'(x) = 0\) is equivalent to \(-\frac{4(A - B)^2B(3 + 2Bx)}{(1 + B^2 + 2Bx)} = 0\) with solution \(x = -\frac{3}{2B}\). But, since \(-1 \leq B \leq 1\), we have \(-\frac{3}{2B} \notin [-1, 1]\), meaning that the extreme values occur for \(x = -1\) and \(x = 1\) (at the endpoints). Next, \(\varphi(\pm 1) = (A - B)^2 \frac{(2 \pm B)^2}{(1 \pm B)^4} = (A - B)^2 \left[ \frac{2 \pm B}{(1 \pm B)^2} \right]^2\).
and

\[ \varphi(1) > \varphi(-1) \iff B < 0. \]

This means that

\[ B < 0 \Rightarrow \min_{t \in [0, 2\pi]} |h(e^{it}) - 1|^2 = \varphi(-1) \]

and

\[ B > 0 \Rightarrow \min_{t \in [0, 2\pi]} |h(e^{it}) - 1|^2 = \varphi(1), \]

i.e., that

\[ \min_{t \in [0, 2\pi]} |h(e^{it}) - 1|^2 = (A - B)^2 \left( \frac{2 + |B|}{1 + |B|} \right)^2 \]

and

\[ \min_{t \in [0, 2\pi]} |h(e^{it}) - 1| = (A - B) \frac{2 + |B|}{1 + |B|}. \]

This completes the proof of (i).

(ii) Follows from (i) for \( B = 0 \).

(iii) When \( B = -1 \) and \( A = 1 - 2\alpha \), for \( \alpha \in [0, 1) \), from

\[ \min_{t \in [0, 2\pi]} |h(e^{it}) - 1| = (A - B) \frac{2 + |B|}{1 + |B|} = \frac{3}{2}(1 - \alpha) \]

and

\[ \frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz} = \frac{1 + (1 - 2\alpha)z}{1 - z}, \]

we have that

\[ \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in \mathbb{D}, \]

or \( f \in \mathcal{S}^*(\alpha) \).

Putting \( p(z) = \frac{f(z)}{z} \) in Lemma 2.1 we obtain the following theorem.

**Theorem 2.2.** Let \( f \in \mathcal{A} \), and let \( A, B \) be a real numbers such that \( -1 \leq B < A \leq -1 \). If

\[ f'(z) < \frac{1 + Az(2 + Bz)}{(1 + Bz)^2} \]

(2.4)
then
\[ (2.5) \quad \frac{f(z)}{z} < \frac{1 + Az}{1 + Bz}. \]
And the right hand side of (2.5) is the best dominant of (2.4).

In a similar way as we obtained Corollary 2.1 from Theorem 2.1, Theorem 2.2 implies the following result.

**Corollary 2.2.** Let \( f \in A \).

(i) If \( -1 \leq B < A \leq 1 \), then
\[
|f'(z) - 1| \leq (A - B) \frac{2 + |B|}{(1 + |B|)^2}, \quad z \in \mathbb{D},
\]
implies
\[
\left| \frac{f(z)}{z} - 1 \right| < A - B, \quad z \in \mathbb{D}.
\]

(ii) If \( B = 0 \), and \( 0 < A \leq 1 \), then
\[
|f'(z) - 1| < 2A, \quad z \in \mathbb{D},
\]
implies
\[
\left| \frac{f(z)}{z} - 1 \right| < A, \quad z \in \mathbb{D}.
\]

(iii) If \( B = -1 \) and \( A = 1 - 2\alpha, \alpha \in [0, 1), (-1 < A \leq 1) \) then
\[
|f'(z) - 1| < \frac{3}{2}(1 - \alpha), \quad z \in \mathbb{D},
\]
implies
\[
\text{Re} \left( \frac{f(z)}{z} \right) > \alpha, \quad z \in \mathbb{D}.
\]

By specifying values for \( A \) and \( B \) in Corollary 2.2 we receive the following

**Example 1.**

(i) For \( A = 1 \) and \( B = \frac{-3 + \sqrt{17}}{4} \approx 0.28 \ldots \) we have that
\[
|f'(z) - 1| < 1, \quad z \in \mathbb{D},
\]
implies
\[
\left| \frac{f(z)}{z} - 1 \right| < 1, \quad z \in \mathbb{D}.
\]
(ii) For $A = 1$ and $B = 0$ we have that
$$|f'(z) - 1| < 2, \quad z \in \mathbb{D},$$
implies
$$\left| \frac{f(z)}{z} - 1 \right| < 1, \quad z \in \mathbb{D}.$$ (iii) For $B = -1$ and $\alpha = \frac{1}{2}$ we have that
$$|f'(z) - 1| < \frac{3}{4}, \quad z \in \mathbb{D},$$
implies
$$\text{Re} \left( \frac{f(z)}{z} \right) > \frac{1}{2}, \quad z \in \mathbb{D}.$$
(iii) If $B = -1$ and $A = 1 - 2\alpha$, $\alpha \in [0, 1)$, $(-1 < A \leq 1)$ then

$$|f'(z) + zf''(z) - 1| < \frac{3}{2}(1 - \alpha), \quad z \in \mathbb{D},$$

implies

$$\text{Re } f'(z) > \alpha, \quad z \in \mathbb{D}.$$

Specifying values for $A$, $B$ and $\alpha$ in Corollary 2.1 and Corollary 2.3, in a similar way as in Example 1, we can obtain other examples.

REFERENCES


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