A NOTE ON VERTEX-EDGE DOMINATING COLORING

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\textbf{ABSTRACT.} A vertex $u$ in a graph is said to \textit{ve-dominate} an edge $e = vw$ if $u \in \{v, w\}$ or $uv \in E(G)$ or $uw \in E(G)$. An edge coloring is said to be a ve-dominating if no two edges ve - dominated by a single vertex receive the same color. The minimum number of colors required for a ve - dominating coloring of a graph $G$ is called \textit{ve - chromatic number} of $G$ and is denoted by $\chi_{ve}(G)$. In this paper we find ve - chromatic number for special type of graph, called necklace.

1. \textsc{Introduction}

Let $G = (V, E)$ be a non-trivial connected graph of finite order. A vertex $v$ in a graph is said to \textit{ve-dominate} an edge $e = uw$ if either $v \in \{u, w\}$ or $vu \in E(G)$ or $vw \in E(G)$. A subset $D \subseteq V(G)$ is said to be a \textit{ve - dominating set} of a graph $G$ if every edge in the graph is dominated by a vertex in $D$. The minimum cardinality of a \textit{ve - dominating set} of a graph is called \textit{ve - domination number} of the graph and is denoted by $\gamma_{ve}(G)$. The study of ve - domination number has been initiated in [1]. An \textit{edge coloring} of a graph is called \textit{ve}-dominating coloring if the edges ve - dominated by a single vertex receive different colors. The minimum number of colors required for a ve - dominating coloring of a graph $G$ is called \textit{ve - chromatic number} of $G$ and is denoted by $\chi_{ve}(G)$. For a vertex $v$ of a graph $G$, the ve - degree of $v$ is defined as the number of edges ve - dominated by the vertex $v$ and is denoted by $\text{deg}_{ve}(v)$. The minimum and maximum ve - degrees of the graph are defined

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as \( \delta_{ve}(G) = \min\{ \deg_{ve}(v) | v \in V(G) \} \) and \( \Delta_{ve}(G) = \max\{ \deg_{ve}(v) | v \in V(G) \} \), respectively.

**Observation 1.1.** [2] For any graph \( G \), \( \chi_{ve}(G) \geq \Delta_{ve}(G) \).

### 2. \( \chi_{ve} \) of Necklace Graph

**Definition 2.1.** A Halin graph \( G \) is a plane graph obtained from a planar embedding of a tree \( T \) of order at least 4, whose vertices are of degree one or at least 3 by joining all the vertices of degree 1 in tree \( T \) as a cycle \( C' \), so that \( C' \) is the boundary of the unbounded face.

**Definition 2.2.** The tree \( T \) and the cycle \( C' \) is called the characteristic tree and the adjoint cycle of \( G \) respectively.

**Definition 2.3.** A caterpillar is a tree such that the removal of the leaves becomes a path.

**Definition 2.4.** A Halin graph \( G \) is called a cubic Halin graph if \( \deg(v) = 3 \) for all \( v \in G \). A cubic Halin graph in which the characteristic tree is a caterpillar is called necklace.

**Lemma 2.1.** Let \( G \) be a graph with \( \chi_{ve}(G) \geq 9 \). Let \( A, B, C, D \) be four vertices of degree 3 on \( C_4 \). Let \( X, Y, Z, W \) be the other neighbors of \( A \) and \( B \), \( C \) and \( D \) respectively. Then \( LE(G) \) is a graph obtained from \( G \) by replacing edge-induced subgraph \( G[<A, B, C, D>] \) by a ladder of length 7. Then \( \chi_{ve}(LE(G)) \leq \chi_{ve}(G) \)

**Proof.**

**Notation 1.** Let \( G_i = LE(G_{i-1}) \) where \( i \geq 2 \) and \( G_1 = LE(G) \).
Theorem 2.1.

\[ \chi_{ve}(N_{e_h}) = \begin{cases} 
6, & \text{if } h = 1 \\
9, & \text{if } h \equiv 2 \pmod{3} \\
10, & \text{if } h \equiv 0 \text{ or } 1 \pmod{3} \text{ and } h \geq 15, h = 9, 12 \\
11, & \text{if } h = 6, 10, 13 \\
12, & \text{if } h = 3, 7 \\
14, & \text{if } h = 4
\end{cases} \]

Proof. Case 1: When \( h = 1 \), \(|E(N_{e_1})| = 6 \). A single vertex dominates all the six edges. Therefore, \( \chi_{ve}(N_{e_1}) = 6 \).

\[ \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6
\end{array} \]

\( N_{e_1} \)

Case 2: If \( h \equiv 2 \pmod{3} \), then \( N_{e_h} = G_i \) for some \( i \). By Lemma 2.1, \( \chi_{ve}(N_{e_h}) \leq \chi_{ve}(N_{e_5}) \leq 9 \). Since \( \chi_{ve}(N_{e_h}) \geq \Delta_{ve} = 9 \), \( \chi_{ve}(N_{e_h}) = 9 \).

\[ \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8
\end{array} \]

\( N_{e_5} \)

Case 3: If \( h \equiv 0 \pmod{3} \). Then \(|E(N_{e_h})| = 3(3k) + 3 = 9k + 3 \). Using 9 colors to color its edges, each color class contains atmost \( k \) edges. Therefore, the remaining 3 edges cannot be colored with 9 colors. Hence, \( \chi_{ve}(N_{e_h}) \geq 10 \). By Lemma 2.1 \( 10 \leq \chi_{ve}(N_{e_h}) \leq \chi_{ve}(N_{e_9}) \leq 10 \). Therefore, \( \chi_{ve}(N_{e_h}) = 10 \).
Case 4: If \( h \equiv 1 \pmod{3} \). Then \( |E(Ne_h)| = 3(3k+1) + 3 = 9k+6 \). Using 9 colors to color its edges, each color class contains atmost \( k \) edges. Therefore, the remaining 6 edges cannot be colored with 9 colors. Hence, \( \chi_{ve}(Ne_h) \geq 10 \). By Lemma 2.1, \( 10 \leq \chi_{ve}(Ne_h) \leq \chi_{ve}(Ne_{16}) \leq 10 \). Therefore, \( \chi_{ve}(Ne_h) = 10 \).

Case 5: When \( h = 6, 10, 13 \).
When \( h = 6 \).
For $N_{e_6}$, $|E(N_{e_6})| = 21$. If we use 9 colors, then each ve-color class contains at most 2 edges and at least 3 edges are uncolored. If we use 10 colors, then each ve-color class contains at most 2 edges and at least one edge is uncolored. Therefore, $\chi_{ve}(N_{e_6}) \geq 11$.

When $h = 10$.

For $N_{e_{10}}$, $|E(N_{e_{10}})| = 33$. If we use 9 colors, then each ve-color class contains at most 3 edges and at least 6 edges are uncolored. If we use 10 colors, then each ve-color class contains at most 3 edges and at least 3 edges are uncolored. Therefore, $\chi_{ve}(N_{e_{10}}) \geq 11$.

When $h = 13$.

For $N_{e_{13}}$, $|E(N_{e_{13}})| = 42$. If we use 9 colors, then each ve-color class contains at most 4 edges and at least 6 edges are uncolored. If we use 10 colors, then each ve-color class contains at most 4 edges and at least 2 edges are uncolored. Therefore, $\chi_{ve}(N_{e_{13}}) \geq 11$.

Case 6: When $h = 3, 7$.

When $h = 3$. 
For $Ne_3$, $|E(Ne_3)| = 12$. Any two edges are ve-dominated by a vertex. Therefore, $\chi_{ve}(Ne_3) = |E(Ne_3)| = 12$.

When $h = 7$.

For $Ne_7$, $|E(Ne_7)| = 24$. If we use 9 colors, then each ve-color class contains atmost 2 edges and atleast 6 edges are uncolored. If we use 10 colors, then each ve-color class contains atmost 2 edges and atleast 4 edges are uncolored. If we use 11 colors, then each ve-color class contains atmost 2 edges and atleast 2 edges are uncolored. Therefore, $\chi_{ve}(Ne_7) \geq 12$.

Case 7: When $h = 4$

For $Ne_4$, $|E(Ne_4)| = 15$. If we use 9 colors, then each ve-color class contains atmost 1 edge and atleast 6 edges are uncolored. If we use 10 colors, then each ve-color class contains atmost 1 edge and atleast 5 edges are uncolored. If we use 11 colors, then each ve-color class contains atmost 1 edge and atleast 4 edges are uncolored.
uncolored. If we use 12 colors, then each ve-color class contains atmost 1 edge and atleast 3 edges are uncolored. If we use 13 colors, then each ve-color class contains atmost 1 edge and atleast 2 edges are uncolored. Therefore, $\chi_{ve}(N_4) \geq 14$. □

REFERENCES


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