PACKING COLORING ON SUBDIVISION-VERTEX AND SUBDIVISION-EDGE JOIN OF CYCLE $C_M$ WITH PATH $P_N$

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ABSTRACT. The packing chromatic number $\chi_p$ of a graph $G$ is the smallest integer $k$ for which there exists a mapping $\pi$ from $V(G)$ to $\{1, 2, ..., k\}$ such that any two vertices of color $i$ are at distance at least $i + 1$. In this paper, the authors find the packing chromatic number of subdivision vertex join of cycle graph with path graph and subdivision edge join of cycle graph with path graph.

1. INTRODUCTION

A set of connected graphs are contemplated in this paper, which are undirected, loops less and without multiple edges. Let $G = (V,E)$ be a graph. No two adjacent vertices receive the same color as the vertex coloring of graph $G$ is the assignment of colors to the vertices of $G$.

Let $G(p,q)$ [7] be a graph with $p = |V|$ and $q = |E|$ correspondingly indicate the count of vertices and edges of a graph $G$.

A packing $k$-coloring of a graph $G$ [1,3,12] is a mapping $\pi$ from $V(G)$ to $\{1, 2, ..., k\}$ such that any two vertices of color $i$ are at distance at least $i + 1$. The packing chromatic number $\chi_p$ of a graph $G$ is the smallest integer $k$ for which $G$ has packing $k$-coloring.

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2010 Mathematics Subject Classification. 05C15, 05C70, 05C12, 05C76.

Key words and phrases. Packing coloring, Subdivision graph, Subdivision-vertex join, Subdivision-edge join.

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Mr. Goddard et al. [5] have justified that the packing coloring problem is NP-complete for general graphs. Fiala and Golovach [4] has confirmed that NP-complete is even for trees.

By inserting a new vertex of degree a subdivision graph $S(G)$ [11,6] of the graph $G$ is obtained. From $G$ of degree 2 on each edge of $G$. For $k \geq 1$, the $k$-th subdivision graph $S_k(G)$ is acquired. By connecting each vertex of $V(G_1)$ with every vertex of $V(G_2)$ a subdivision-vertex join [8] of two vertex disjoint graphs $G_1$ and $G_2$ is represented by $G_1 \dot{V}G_2$, has been procured from $S(G_1)$ and $(G_2)$.

Let $C_m$ and $P_n$ denote the cycle and path graph with $m$ and $n$ vertices, respectively. By definition of subdivision vertex join of graphs [2, 9, 13] we subdivide each edge of the cycle graph and join each vertex of the cycle graph with every vertex of the path graph [10, 11, 14].

Throughout this paper, \{v_k : 1 \leq k \leq m\}, \{u_k : 1 \leq k \leq m\} and \{s_p : 1 \leq p \leq n\} denote the vertices of cycle, the subdivided vertices of the cycle and the vertices of the path, respectively. The total number of vertices of the subdivision vertex join graph is $2m + n$.

2. PADDING COLORING OF SUBDIVISION-VERTEX JOIN AND SUBDIVISION-EDGE JOIN GRAPH

Theorem 2.1. $\chi_p[C_m \dot{v}P_n]$ is packing chromatic number of the subdivision-vertex join of a cycle graph $C_m$ and a path graph $P_n$ for $m \geq 3$ and $n \geq 2$. Then

$$\chi_p(C_m \dot{v}P_n) = \begin{cases} \frac{2m+n+1}{2} & \text{if } m \text{ is odd or even, } n \text{ is odd} \\ \frac{2m+n+2}{2} & \text{if } m \text{ is odd or even, } n \text{ is even} \end{cases}$$

Proof. Let us assume $V(C_m \dot{v}P_n) = \{v_k, u_k, s_p : 1 \leq k \leq m, 1 \leq p \leq n\}$. For $1 \leq i \leq m - 1$

- Each edge $v_kv_{k+1}$ is subdivided by $u_i$ of $C_m \dot{v}P_n$

For $1 \leq p \leq m$

- Each edge $v_1v_m$ subdivided by $u_m$ of $C_m \dot{v}P_n$

Case (i): $m$ is odd or even, $n$ is odd.

We assume $\chi_p[C_m \dot{v}P_n] < \frac{2m+n+1}{2}$ to get $\chi_p[C_m \dot{v}P_n] \geq \frac{2m+n+1}{2}$ as lower bound of the packing chromatic number. We choose ($\frac{2m+n-1}{2}$) colors for each valid
vertex in $C_m \hat{v} P_n$. As per the definition, we have two rules for subdivision-vertex join graph of cycle and path graph. The rules that $c(u_k) = c(s_{2p-1}) = c_1$ and $d(v_k, s_p) = 1, d(v_k, v_{k+1}) = 2$. We remaining of $(\frac{2m+n+3}{2})$ colors after select $(\frac{2m+n-1}{2})$ colors. We get $c(v_k) \neq c(v_{k+1})$ and $(\frac{2m+n-3}{2})$ colors are required for each $v_k$ and $v_{k+1}$, as per the definition of packing coloring that two statement of $\chi_{\rho}[C_m \hat{v} P_n] < \frac{2m+n+1}{2}$ is wrong because it is a contrary value compared to the expected output. Then we accept the statement of $\chi_{\rho}[C_m \hat{v} P_n] \geq \frac{2m+n+1}{2}$. We get the upper bound of packing chromatic number $\chi_{\rho}[C_m \hat{v} P_n] \leq \frac{2m+n+1}{2}$ has to be calculated as follows.

The function of color $c : V[C_m \hat{v} P_n] \rightarrow \{c_1, c_2, ... c_{\frac{2m+n+1}{2}}\}$ is defined by,

\[
c(u_k) = c_1 \quad \text{for} \ 1 \leq k \leq m; \\
c(s_{2p-1}) = c_1 \quad \text{for} \ 1 \leq p \leq n; \\
c(v_k) = c_{k+1} \quad \text{for} \ 1 \leq k \leq m; \\
c(s_{2p}) = c_{m+p+1} \quad \text{for} \ 1 \leq p \leq n.
\]

Therefore, $\chi_{\rho}[C_m \hat{v} P_n] \leq \frac{2m+n+1}{2}$. It is not difficult to prove that this coloring is a packing coloring. Hence, $\chi_{\rho}[C_m \hat{v} P_n] = \frac{2m+n+1}{2}$.

Case (ii) : $m$ is odd or even, $n$ is even.

We assume $\chi_{\rho}[C_m \hat{v} P_n] < \frac{2m+n+2}{2}$ to get $\chi_{\rho}[C_m \hat{v} P_n] \geq \frac{2m+n+2}{2}$ as lower bound of the packing chromatic number. We choose $(\frac{2m+n}{2})$ colors for each valid vertex in $C_m \hat{v} P_n$. As per the definition, we have two rules for subdivision-vertex join graph of cycle and path graph. The rules that $c(u_k) = c(s_{2p-1}) = c_1$ and $d(v_k, s_p) = 1, d(v_k, v_{k+1}) = 2$. We remaining of $(\frac{2m+n-2}{2})$ colors after select $(\frac{2m+n}{2})$ colors. We get $c(v_k) \neq c(v_{k+1})$ and $(\frac{2m+n-2}{2})$ colors are required for each $v_k$ and $v_{k+1}$, as per the definition of packing coloring that two statement of $\chi_{\rho}[C_m \hat{v} P_n] < \frac{2m+n+2}{2}$ is wrong because it is a contrary value compared to the expected output. Then we accept the statement of $\chi_{\rho}[C_m \hat{v} P_n] \geq \frac{2m+n+2}{2}$. We get the upper bound of packing chromatic number $\chi_{\rho}[C_m \hat{v} P_n] \leq \frac{2m+n+2}{2}$ has to be calculated as follows.

The function of color $c : V[C_m \hat{v} P_n] \rightarrow \{c_1, c_2, ... c_{\frac{2m+n+2}{2}}\}$ is defined by,

\[
c(u_k) = c_1 \quad \text{for} \ 1 \leq k \leq m; \\
c(s_{2p-1}) = c_1 \quad \text{for} \ 1 \leq p \leq n; \\
c(v_k) = c_{k+1} \quad \text{for} \ 1 \leq k \leq m; \\
c(s_{2p}) = c_{m+p+1} \quad \text{for} \ 1 \leq p \leq n.
\]
Therefore, $\chi_\rho(C_m \uplus P_n) \leq \frac{2m+n+2}{2}$. It is not difficult to prove that this coloring is a packing coloring. Hence, $\chi_\rho(C_m \uplus P_n) = \frac{2m+n+2}{2}$. □

**Theorem 2.2.** $\chi_\rho(C_m \uplus P_n)$ is packing chromatic number of the subdivision-edge join of a cycle graph $C_m$ and a path graph $P_n$ for $m \geq 3$ and $n \geq 2$. Then

$$
\chi_\rho(C_m \uplus P_n) = \begin{cases} 
\frac{2m+n+1}{2} & \text{if } m \text{ is odd or even, } n \text{ is odd;} \\
\frac{2m+n+2}{2} & \text{if } m \text{ is odd or even, } n \text{ is even.}
\end{cases}
$$

**Proof.** Let us assume $V(C_m \uplus P_n) = \{v_k, u_k, s_p : 1 \leq k \leq m, 1 \leq p \leq n\}$

For $1 \leq k \leq m - 1$

- Each edge $v_kv_{k+1}$ is subdivided by $u_k$ of $C_m \uplus P_n$.

For $1 \leq p \leq m$

- Each edge $v_1v_m$ subdivided by $u_m$ of $C_m \uplus P_n$.

Case (i) : $m$ is odd or even, $n$ is odd.

We assume $\chi_\rho(C_m \uplus P_n) < \frac{2m+n+1}{2}$ to get $\chi_\rho(C_m \uplus P_n) \geq \frac{2m+n+1}{2}$, as lower bound of the packing chromatic number. We choose $(\frac{2m+n-1}{2})$ colors for each valid vertex in $C_m \uplus P_n$. As per the definition, we have two rules for subdivision-edge join graph of cycle and path graph. The rules are $c(u_k) = c(s_{2p-1}) = c_1$ and $d(v_k, s_p) = 1, d(v_k, v_{k+1}) = 2$. We remaining of $(\frac{2m+n-3}{2})$ colors after select $(\frac{2m+n-1}{2})$ colors. We get $c(v_k) \neq c(v_{k+1})$ and $(\frac{2m+n-1}{2})$ colors are required for each $v_k$ and $v_{k+1}$, as per the definition of packing coloring that two vertices of color $i$ are at distance at least $i + 1$ apart and $d(v_k, v_{k+1}) = 2$. The statement $\chi_\rho(C_m \uplus P_n) < \frac{2m+n+1}{2}$ is wrong because it is a contrary value compared to the expected output. Then, we accept the statement of $\chi_\rho(C_m \uplus P_n) \geq \frac{2m+n+1}{2}$. We get the upper bound of packing chromatic number, $\chi_\rho(C_m \uplus P_n) \leq \frac{2m+n+1}{2}$ has to be calculated as follows.

The function $c : V(C_m \uplus P_n) \rightarrow \{c_1, c_2, ... c_{\frac{2m+n+1}{2}}\}$ defined by,

- $c(v_k) = c_1$ for $1 \leq k \leq m$
- $c(s_{2p-1}) = c_1$ for $1 \leq p \leq n$
- $c(u_k) = c_{k+1}$ for $1 \leq k \leq m$
- $c(s_{2p}) = c_{m+p+1}$ for $1 \leq p \leq n$

Therefore, $\chi_\rho(C_m \uplus P_n) \leq \frac{2m+n+1}{2}$. It is not difficult to prove that this coloring is a packing coloring. Hence, $\chi_\rho(C_m \uplus P_n) = \frac{2m+n+1}{2}$. 
Therefore, we assume $\chi_p[C_m \sqcup P_n] < \frac{2m+n+2}{2}$ to get $\chi_p[C_m \sqcup P_n] \geq \frac{2m+n+2}{2}$, as lower bound of the packing chromatic number. We choose $(\frac{2m+n}{2})$ colors for each valid vertex in $C_m \sqcup P_n$. As per the definition, we have two rules for subdivision-edge join graph of cycle and path graph. The rules are $c(u_k) = c(s_{2p-1}) = c_1$ and $d(v_k, s_p) = 1, d(v_k, v_{k+1}) = 2$. We remaining of $(\frac{2m+n-2}{2})$ colors after select $(\frac{2m+n}{2})$ colors. We get $c(v_k) \neq c(v_{k+1})$ and $(\frac{2m+n-2}{2})$ colors are required for each $v_k$ and $v_{k+1}$, as per the definition of packing coloring that two vertices of color $i$ are at distance at least $i+1$ apart and $d(v_k, v_{k+1}) = 2$. The statement $\chi_p[C_m \sqcup P_n] < \frac{2m+n+2}{2}$ is wrong because it is a contrary value compared to the expected output. Then, we accept the statement of $\chi_p[C_m \sqcup P_n] \geq \frac{2m+n+2}{2}$. We get the upper bound of packing chromatic number, $\chi_p[C_m \sqcup P_n] \leq \frac{2m+n+2}{2}$ has to be calculated as follows.

The function $c : V[C_m \sqcup P_n] \rightarrow \{c_1, c_2, ..., c_{\frac{2m+n+2}{2}}\}$ is defined by,

- $c(v_k) = c_1$ for $1 \leq k \leq m$;
- $c(s_{2p-1}) = c_1$ for $1 \leq p \leq n$;
- $c(u_k) = c_{k+1}$ for $1 \leq k \leq m$;
- $c(s_{2p}) = c_{m+p+1}$ for $1 \leq p \leq n$.

Therefore, $\chi_p[C_m \sqcup P_n] \leq \frac{2m+n+2}{2}$. It is not difficult to prove that this coloring is a packing coloring. Hence, $\chi_p[C_m \sqcup P_n] = \frac{2m+n+2}{2}$.

**Theorem 2.3.** $\chi_p[C_m \sqcup P_n]$ is packing chromatic number of the join of a cycle graph $C_m$ and a path graph $P_n$ for $m \geq 3$ and $n \geq 2$. Then

$$\chi_p(C_m \sqcup P_n) = \begin{cases} \frac{2m+n+1}{2} & \text{if } m \text{ is odd or even, } n \text{ is odd} \\ \frac{2m+n+2}{2} & \text{if } m \text{ is odd or even, } n \text{ is even} \end{cases}$$

**Proof.** Let us assume $V(C_m \sqcup P_n) = \{v_k, s_p : 1 \leq k \leq m, 1 \leq p \leq n\}$. For $1 \leq k \leq m - 1$

- Each edge $v_kv_{k+1}$ is subdivided by $u_k$ of $C_m \sqcup P_n$.

For $1 \leq p \leq m$

- Each edge $v_1v_m$ subdivided by $u_m$ of $C_m \sqcup P_n$.

**Case (i) :** $m$ is odd or even, $n$ is odd.
We assume $\chi_\rho[C_m v P_n] < \frac{2m+n+1}{2}$ to get $\chi_\rho[C_m v P_n] \geq \frac{2m+n+1}{2}$ as lower bound of the packing chromatic number. We choose $(\frac{2m+n-1}{2})$ colors for each valid vertex in $C_m v P_n$. As per the definition, we have two rules for join graph of cycle and path graph. The rules are $c(s_{2p-1}) = c_1$ and $d(v_k, s_p) = 1, d(v_k, v_{k+1}) = 2$. We remaining of $(\frac{2m+n-3}{2})$ colors after select $(\frac{2m+n+1}{2})$ colors. We get $c(v_k) \neq c(v_{k+1})$ and $(\frac{2m+n-3}{2})$ colors are required for each $v_k$ and $v_{k+1}$, as per the definition of packing coloring that two vertices of color $i$ are at distance at least $i + 1$ apart and $d(v_k, v_{k+1}) = 2$, the statement $\chi_\rho[C_m v P_n] < \frac{2m+n+1}{2}$ is wrong because it is a contrary value compared to the expected output. Then, we accept the statement of $\chi_\rho[C_m v P_n] \geq \frac{2m+n+1}{2}$. We get the upper bound for the packing chromatic number, $\chi_\rho[C_m v P_n] \leq \frac{2m+n+1}{2}$ has to be calculated as follows.

The function of color $c : V[C_m v P_n] \rightarrow \{c_1, c_2, ...c_{\frac{2m+n+1}{2}}\}$ is defined by,

$c(s_{2p-1}) = c_1$ for $1 \leq p \leq n$;
$c(v_k) = c_{k+1}$ for $1 \leq k \leq m$;
$c(s_{2p}) = c_{m+p+1}$ for $1 \leq p \leq n$.

Therefore, $\chi_\rho[C_m v P_n] \leq \frac{2m+n+1}{2}$. It is not difficult to prove that this coloring is a packing coloring. Hence, $\chi_\rho[C_m v P_n] = \frac{2m+n+1}{2}$.

Case (ii) : $m$ is odd or even, $n$ is even.

We assume $\chi_\rho[C_m v P_n] < \frac{2m+n+2}{2}$ to get $\chi_\rho[C_m v P_n] \geq \frac{2m+n+2}{2}$ as lower bound of the packing chromatic number. We choose $(\frac{2m+n}{2})$ colors for each valid vertex in $C_m v P_n$. As per the definition, we have two rules for join graph of cycle and path graph. The rules are $c(s_{2p-1}) = c_1$ and $d(v_k, s_p) = 1, d(v_k, v_{k+1}) = 2$. We remaining of $(\frac{2m+n-2}{2})$ colors after select $(\frac{2m+n}{2})$ colors. We get $c(v_k) \neq c(v_{k+1})$ and $(\frac{2m+n-2}{2})$ colors are required for each $v_k$ and $v_{k+1}$, as per the definition of packing coloring that two vertices of color $i$ are at distance at least $i + 1$ apart and $d(v_k, v_{k+1}) = 2$, the statement $\chi_\rho[C_m v P_n] < \frac{2m+n+2}{2}$ is wrong because it is a contrary value compared to the expected output. Then, we accept the statement of $\chi_\rho[C_m v P_n] \geq \frac{2m+n+2}{2}$. We get the upper bound for the packing chromatic number, $\chi_\rho[C_m v P_n] \leq \frac{2m+n+2}{2}$ has to be calculated as follows.

The function of color $c : V[C_m v P_n] \rightarrow \{c_1, c_2, ...c_{\frac{2m+n+2}{2}}\}$ defined by,

$c(s_{2p-1}) = c_1$ for $1 \leq p \leq n$;
$c(v_k) = c_{k+1}$ for $1 \leq k \leq m$;
$c(s_{2p}) = c_{m+p+1}$ for $1 \leq p \leq n$. 
Therefore, $\chi_\rho[C_mvP_n] \leq \frac{2m+n+2}{2}$. It is not difficult to prove that this coloring is a packing coloring. Hence, $\chi_\rho[C_mvP_n] = \frac{2m+n+2}{2}$.

At the end of this paper we can conclude that for any cycle and path graph $m \geq 3$ and $n \geq 2$, we have:

$$\chi_\rho(C_mvP_n) = \chi_\rho(C_mvP_n) = \chi_\rho(C_mvP_n) = \left\{ \begin{array}{ll} \frac{2m+n+1}{2} & \text{if } m \text{ is odd or even, } n \text{ is odd}, \\ \frac{2m+n+2}{2} & \text{if } m \text{ is odd or even, } n \text{ is even}. \end{array} \right.$$

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