NON-EXISTENCE OF SKOLEM MEAN LABELING
FOR FOUR STAR GRAPHS

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ABSTRACT. In this paper, we prove if \( r > s < t \), the four star graph \( G = k_{1,r} \cup k_{1,s} \cup k_{1,t} \) is not a skolem mean graph if \( |s-t| > 4 + 2r \) for \( r = 2, 3, \ldots \); \( s = 1, 2, \ldots \) and \( t \geq 2r + s + 5 \).

1. Introduction

In [1], V. Balaji and et al. proved that the three star graph \( K_{1,\ell} \cup K_{1,m} \cup K_{1,n} \), \( \ell \leq m < n \) is skolem mean graph if \( |m-n| \leq \ell + 4 \). In [2], they have proved that the four star graph \( K_{1,\ell} \cup K_{1,\ell} \cup K_{1,m} \cup K_{1,n} \), \( \ell \leq m < n \) is skolem mean graph if \( |m-n| \leq 2\ell + 4 \). In [3], V. Balaji and et al. proved that the three star graph \( K_{1,p} \cup K_{1,q} \cup K_{1,r} \), \( p > q < r \) is skolem mean graph if and only if \( |p-q| \leq r + 4 \).

**Definition 1.1.** A graph \( G \) with \( p \) nodes and \( q \) links is said to be a skolem mean graph if there exists a function \( f \) from the node set of \( G \) to \( \{1, 2, \ldots, p\} \) such that the induced map \( f^* \) from the link set of \( G \) to \( \{2, 3, \ldots, p\} \) defined by

\[
    f^*(e = uv) = \begin{cases} 
    \frac{f(u)+f(v)}{2} & \text{if } f(u) + f(v) \text{ is even;} \\
    \frac{f(u)+f(v)+1}{2} & \text{if } f(u) + f(v) \text{ is odd,}
    \end{cases}
\]

then the resulting links get distinct labels from the set \( \{2, 3, \ldots, p\} \).

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2. MAIN RESULT

Theorem 2.1. If \( r > s < t \), the four star \( K_{1,r} \cup K_{1,r} \cup K_{1,s} \cup K_{1,t} \) is not a skolem mean graph if \( |s-t| > 4 + 2r \) with \( r-s = 1 \), for \( r = 2, 3 \cdots \) and \( s = 1, 2 \cdots \) and \( t \geq 2r + s + 5 \).

Proof. Let \( G = K_{1,r} \cup K_{1,r} \cup K_{1,s} \cup K_{1,t} \) where,

\[
V(G) = \{v_{a,b} : 1 \leq a \leq 2, 0 \leq b \leq 3\} \cup \{v_{3,b} : 0 \leq b \leq 2\} \cup \{v_{4,b} : 0 \leq b \leq 13\}
\]

\[
E(G) = \{v_{a,0}v_{a,b} : 1 \leq a \leq 2, 1 \leq b \leq 3\} \cup \{v_{3,0}v_{3,b} : 1 \leq b \leq 2\} \cup \{v_{4,0}v_{4,b} : 1 \leq b \leq 13\}.
\]

Then, \( p = 25 \) and \( q = 21 \).

Suppose \( G \) is a skolem mean graph, then there exists a function \( f \) from the node set of \( G \) to \( 1, 2 \cdots \) such that the induced map \( f^* \) from the link set of \( G \) to \( 2, 3 \cdots \) defined by

\[
f^*(e = uv) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even;} \\ \frac{f(u) + f(v)+1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases}
\]

Then, the resulting links get distinct labels from the set \( \{2, 3 \cdots p\} \). Let \( x_{a,b} \) be the label given to the node \( v_{a,b} \) for \( 1 \leq a \leq 2, \ 0 \leq b \leq 3 \), \( v_{3,b} \) for \( 0 \leq b \leq 2 \) and \( v_{4,b} \) for \( 0 \leq b \leq 13 \).

Let \( y_{a,b} \) be the respective link label of the link \( v_{a,0}v_{a,b} \) for \( 1 \leq a \leq 2, \ 0 \leq b \leq 3 \), \( v_{3,0}v_{3,b} \) for \( 1 \leq b \leq 2 \) and \( v_{4,0}v_{4,b} \) for \( 0 \leq b \leq 13 \).

Let us first consider the case that \( x_{4,0} = 24 \). If \( v_{4,b} = 2t - 1 \) and \( v_{4,c} = 2t \) for some \( n \) and for some \( b \) and \( c \), then,

\[
f^*(v_{4,0}v_{4,b}) = \frac{24 + 2t}{2} = \frac{24 + 2t - 1}{2} = 12 + t = f^*(v_{4,0}v_{4,b})
\]

This is not possible as \( f^* \) is a bijection. Therefore, the 13 nodes \( x_{4,b} \) for \( 1 \leq b \leq 13 \) are among the 13 numbers \( (1/2), (3/4), (5/6), (7/8), (9/10), (11/12), (13/14), (15/16), (17/18), (19/20), (21/22), 23 \) and \( 25 \). Since \( x_{4,0} = 24 \), first let us consider all the biggest link labels possible for \( K_{1,13} \). That is, for 13 nodes \( x_{4,b} \) for \( 1 \leq b \leq 13 \). Consider the 13 choices that may induce the larger link values.

Therefore, the 13 choices are \( (1/2), (3/4), (5/6), (7/8), (9/10), (11/12), (13/14), (15/16), (17/18), (19/20), (21/22), 23 \) and \( 25 \). The respective link
labels are 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24 and 25. Then, the set 
\[ \{y_{4,b} : 1 \leq b \leq 13\} = \{13, 14 \cdots 25\}. \]

**Case(A):** \(x_{4,3} = 21\) (we have \(x_{4,0} = 24, x_{4,1} = 25, x_{4,2} = 23\) and \(x_{1,0} = 22\)).

Now, 22 is a label of either \(x_{a,0}\) for \(1 \leq a \leq 2\) or \(x_{a,b}\) for \(1 \leq a \leq 2; 1 \leq b \leq 3\). That is, 22 is a label of pendent or non-pendent node in \(K_{1,2}\) or \(K_{1,3}\) component of \(G\). Without loss of generality, let us assume that \(x_{1,0} = 22\).

**Case(A\(_1\)):** \(x_{1,0} = 22\) (we have \(x_{4,0} = 24, x_{4,1} = 25, x_{4,2} = 23\))

If \(x_{1,0} = 22\), then, \(x_{1,1}\) take the values one among 1,2 [As \(x_{1,1} \geq 3\) would imply that \(y_{1,1} \geq 13\). This is not possible]. Let \(x_{1,1} = 1\) and \(x_{4,13} = 2\), then, respective link labels are \(y_{1,1} = 12\) and \(y_{4,3} = 23\). Next, \(t_{4,4}\) is either 20 or 19.

**Case(A\(_2\)):** \(x_{4,4} = 19\) or 20

If \(x_{4,4} = 19\), then, let \(x_{1,2} = 20\), then, \(y_{1,2} = 21\), but, \(y_{4,5} = 21\) is already alloted. Hence, \(x_{1,2} = 20\) is not possible.

**Case(B):** \(x_{4,13} = 2\) or 1 we have \(x_{4,0} = 24, x_{4,1} = 25, x_{4,2} = 23\) and \(x_{1,0} = 1\).

Now, 2 is a label of either \(x_{a,0}\) for \(1 \leq a \leq 2\) or \(x_{a,b}\) for \(1 \leq a \leq 2; 1 \leq b \leq 3\). That is, 2 is a label of pendent or non-pendent node in \(K_{1,2}\) or \(K_{1,3}\) component.
of \( G \). Without loss of generality, let us assume that \( x_{1,0} = 2 \). Then, 2 is a label of non-pendent node in \( K_{1,2} \) component of \( G \).

**Case (B₁):** \( x_{4,3} = 22 \) or \( 21 \)

Let \( x_{4,3} = 21 \). That is, 22 is a label of pendent node in a \( K_{1,2} \) component of \( G \). Let us assume that \( x_{1,1} = 22 \). Let \( x_{4,0} = 24, x_{4,1} = 25, x_{4,2} = 23, x_{4,3} = 21, x_{4,13} = 1, x_{1,0} = 2, x_{1,1} = 22 \). Then, \( y_{4,13} = 13, y_{4,1} = 25, y_{4,2} = 24, y_{4,3} = 23 \) and \( y_{1,1} = 12 \).

**Case (B₂):** \( x_{4,4} = 19 \) or \( 20 \)

Let \( t_{4,4} = 19 \), then, 20 should be a label of another pendent or adjacent node in \( K_{1,2} \) component of \( G \). Then \( x_{1,2} = 20 \). Let \( x_{4,0} = 24, x_{4,1} = 25, x_{4,2} = 23, x_{4,3} = 21, x_{4,13} = 1 \) and \( x_{4,4} = 19, x_{1,0} = 2, x_{1,1} = 22, x_{1,2} = 20 \). Then, \( y_{4,13} = 13, y_{4,1} = 25, y_{4,2} = 24, y_{4,3} = 23 \), \( y_{1,1} = 12 \) and \( y_{1,2} = 11 \).

**Case (C):** \( x_{4,5} = 17 \) or \( 18 \)

Let \( x_{4,5} = 18 \), then, 17 should be a label of pendent or non-pendent node in \( K_{1,3} \) component of \( G \). Without loss of generality. Let \( x_{2,0} = 17 \), then 17 should be label of non-pendent node in \( K_{1,3} \) component of \( G \).

**Case (C₁):** \( x_{4,12} = 3 \) or \( 4 \)

If \( x_{4,12} = 4 \) and \( x_{1,2} \geq 4 \), then, 3 should be a label of pendent node of \( K_{1,3} \) component of \( G \), then, \( x_{2,1} \geq 11 \). This is not possible.

**Case (C₂):** \( x_{4,12} = 3 \) or \( 4 \)

Now, let \( x_{4,5} = 18 \), so 17 should be a label of unlabeled node. To avoid the complication, let us allot 17 to a pendent node. Without loss of generality, let it be \( x_{2,1} = 17 \), i.e \( x_{2,1} = 17, y_{2,b}, 1 \leq b \leq 3 \).
Case (D): \( x_{4,12} = 3 \) or 4

Let \( x_{4,12} = 4 \), then, 3 should be a label of non-pendent node in \( K_{1,3} \) component of G. Then, \( t_{2,0} = 3 \).

Case (D_1): \( t_{4,6} = 15 \) or 16

If \( t_{4,6} = 16 \), so 15 should be a label of a pendent node in \( K_{1,3} \) of G. Then, \( x_{2,2} = 15 \).

Case (D_2): \( x_{4,7} = 13 \) or 14

Let \( x_{4,7} = 14 \), then, 13 should a be label of another pendent node in \( K_{1,3} \) component of G.

Without loss of generality, let \( x_{2,3} = 13 \). Let \( x_{4,0} = 24 \), \( x_{4,1} = 25 \), \( x_{4,2} = 23 \), \( x_{4,3} = 21 \), \( x_{4,13} = 1 \), \( x_{4,4} = 19 \), \( x_{4,5} = 18 \), \( x_{4,6} = 16 \), \( x_{4,7} = 14 \), \( x_{4,12} = 4 \), \( x_{1,0} = 2 \), \( x_{1,1} = 22 \), \( x_{1,2} = 20 \), \( x_{2,0} = 3 \), \( x_{2,1} = 17 \), \( x_{2,2} = 15 \) and \( x_{2,3} = 13 \). Then \( y_{4,13} = 13 \), \( y_{4,1} = 25 \), \( y_{4,2} = 24 \), \( y_{4,3} = 23 \), \( y_{1,1} = 12 \), \( y_{1,2} = 11 \), \( y_{2,1} = 10 \), \( y_{2,2} = 9 \) and \( y_{2,3} = 8 \).

Case (E): \( x_{4,8} = 11 \) or 12 (we have \( x_{4,0} = 24 \), \( x_{4,1} = 25 \), \( x_{4,2} = 23 \) and \( x_{1,0} = 1 \))

Now, 12 is a label of \( t_{b,0} \) for \( 1 \leq b \leq 3 \). That is, 12 is a label of pendent or non-pendent node in another \( K_{1,3} \) component of G. Without loss of generality, let us assume that \( t_{3,0} = 12 \). Then 12 is a label of non-pendent node in \( K_{1,3} \) component of G.

Case (E_1): \( x_{4,11} = 5 \) or 6

If \( x_{4,11} = 6 \), so 5 should be a label of a pendent node in \( K_{1,3} \) component of G. Then, \( x_{3,1} = 5 \). Let \( x_{3,0} = 11 \), \( x_{3,1} = 5 \), \( x_{4,8} = 12 \), \( x_{4,11} = 6 \). Then, \( y_{4,8} = 18 \), \( y_{4,11} = 15 \), \( y_{3,1} = 8 \). But, the link value 8 is already allotted \( y_{2,3} = 8 \). Hence,
$x_{3,0} = 11$ is not a non-pendent node in second $K_{1,3}$ component of $G$. Hence, $y_{3,1} = 8$ is not possible. Hence, $x_{3,0} \neq 11$. Similarly $x_{3,0} \neq 12$.

**Case(F):** $x_{4,11} = 5$ or 6

If $x_{4,11} = 6$, so 5 should be a label of a pendent or non-pendent node in another $K_{1,3}$ component of $G$. Without loss of generality, let $x_{3,0} = 5$, then, 5 should be a label of a non-pendent node in $K_{1,3}$ component of $G$.

**Case(F$_1$):** $x_{4,8} = 11$ or 12

If $x_{4,8} = 12$, so 11 should be a label of a pendent node in second $K_{1,3}$ component of $G$. Let us assume that $x_{3,1} = 11$. Then, $y_{3,1} = 8$, but, the link value 8 is already allotted $y_{2,3} = 8$. Then, $x_{3,1} \neq 11$ and similarly $x_{3,1} \neq 12$.

**Case(F$_2$):** $x_{4,9} = 9$ or 10

If $x_{4,9} = 10$, then, 9 should be a label of a pendent node in second $K_{1,3}$ component of $G$. Let us assume that the node is $x_{3,1} = 9$, then we get the link value $y_{3,1} = 7$.

**Case(F$_3$):** $x_{4,10} = 7$ or 8

If $x_{4,9} = 8$, then, 7 should be a label of a pendent node in second $K_{1,3}$ component of $G$. Let us assume that $x_{3,2} = 7$. Then, $x_{3,1} = 6$. But, $x_{3,2} = 8$ is not possible. Suppose $x_{3,2} = 8$, then, we get the link value $y_{3,2} = 7$, but the link value 7 is already allotted $y_{3,1} = 7$. Hence, $x_{3,2} \neq 8$.

**Case(G):** $x_{4,8} = 11$ or 12

If $x_{4,8} = 12$, then, 11 should be a label of a pendent node in second $K_{1,3}$ component of $G$. Let us assume that $x_{3,3} = 11$. Then, $y_{3,3} = 8$, but, the link value 8 is already allotted $y_{2,3} = 8$. Then, $x_{3,3} \neq 11$ and similarly $x_{3,3} \neq 12$.

![Figure 5](image-url)
Let $x_{4,0} = 24$, $x_{4,1} = 25$, $x_{4,2} = 23$, $x_{4,3} = 21$, $x_{4,13} = 1$, $x_{4,4} = 19$, $x_{4,5} = 18$, $x_{4,6} = 16$, $x_{4,7} = 14$, $x_{4,12} = 4$, $x_{4,8} = 12$, $x_{4,9} = 10$, $x_{4,10} = 8$, $x_{4,11} = 6$, $x_{1,0} = 2$, $x_{1,1} = 22$, $x_{1,2} = 20$, $x_{2,0} = 3$, $x_{2,1} = 17$, $x_{2,2} = 15$, $x_{2,3} = 13$, $x_{3,0} = 5$, $x_{3,1} = 9$, $x_{3,2} = 7$, $x_{3,3} = 11$.

Then, $y_{4,13} = 13$, $y_{4,1} = 25$, $y_{4,2} = 24$, $y_{4,3} = 23$, $y_{1,1} = 12$, $y_{1,2} = 11$, $y_{2,1} = 10$, $y_{2,2} = 9$, $y_{2,3} = 8$, $y_{3,0} = 7$, $y_{3,2} = 6$, $y_{3,3} = 8$.

Suppose that $x_{4,8} = 11$ and one of the unlabeled node should be 12, we know that all the node labels smaller than 5 are already allotted to the nodes. So, giving label greater than 5 to the adjacent node of the unknown node, labeled 12 will induce a link label 9, but, 9 is already the link label of $y_{2,2}$, which fails the bijection of the labeling defined. Obviously, $G = 2K_{1,3} \cup K_{1,2} \cup K_{1,13}$ is not a skolem mean graph for $x_{4,0} = 24$.

A similar argument can prove that $G$ is not a skolem mean graph, when, $x_{4,0}$ takes other values as such the edges $y_{4,3}$ get the higher values.

Hence, we failed to generate a skolem mean labeling for $G = 2K_{1,3} \cup K_{1,2} \cup K_{1,13}$, even, when the $K_{1,13}$ component of $G$ takes the smaller of the values. Hence, $G = 2K_{1,3} \cup K_{1,2} \cup K_{1,13}$ is not a skolem mean graph, when, $G$ assumes smaller as well as greater values. Hence, $G = 2K_{1,3} \cup K_{1,2} \cup K_{1,13}$ is not a skolem mean graph. That is, $G$ is not a skolem mean graph, when, $|s - t| = 5 + 2r$. In a similar way, we shall prove that $G = 2K_{1,4} \cup K_{1,2} \cup K_{1,15}$ is also not a skolem mean graph. Arguably, we may assert that graph with bigger difference between $s$ and $t$ will never make a skolem mean graph.

Hence, the four star $G = K_{1,r} \cup K_{1,r} \cup K_{1,s} \cup K_{1,t}$ is not a skolem mean graph, if $|s - t| > 4 + 2r$ for $r = 2, 3 \cdots ; s = 1, 2 \cdots \cdot$.

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