INTEGRAL INEQUALITIES OF HADAMARD TYPE FOR SUB $E$-FUNCTIONS

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ABSTRACT. In this paper, we show that the power function of sub $E$-function $f^n(x)$ is sub $E$-function. Furthermore, we establish some new integral inequalities of Hadamard type involving sub $E$-functions and concave $E$-functions.

1. INTRODUCTION

Let $f : I \to \mathbb{R}$ be a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. There are many generalizations of the notion of convex functions see [3,4,7,10]. One way to generalize the notion of convex function is to replace linear functions by another family of functions in the sense of Beckenbach [3]. In this paper, we deal with a family $\{E(x)\}$ of exponential functions

$$E(x) = A \exp Bx,$$

where $A, B$ arbitrary constants.

The Hermite-Hadamard integral inequality for convex functions $f : [a, b] \to \mathbb{R}$

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2},$$

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is well known in the literature and has many applications for special means, see for example [2, 6, 9]. The Hermite-Hadamard integral inequality (1.1) was established for sub-$E$-functions in [1] as

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq L(f(a), f(b)),$$

where, $L(f(a), f(b)) := \frac{f(b) - f(a)}{\ln f(b) - \ln f(a)}$, $f(a), f(b) \geq 0$, $f(a) \neq f(b)$.

In this work, we proved that the higher powers of sub-$E$-function is sub-$E$-function in addition to establish some new integral inequalities of Hadamard type involving sub-$E$-functions and concave $E$-functions.

2. Definitions and Preliminary Results

In this section, we introduce the basic definitions and results which will be used later. For more informations see [1], [5], [8].

**Definition 2.1.** A positive function $f : I \to (0, \infty)$ is called sub-$E$-function on $I$, if for any $a, b \in I$ with $a < b$ the graph of $f(x)$ for $a < x < b$ lies on or under the graph of a function

$$E(x) = Ae^{Bx},$$

where $A$ and $B$ are taken so that $E(a) = f(a)$, and $E(b) = f(b)$.

Equivalently, for all $x \in [a, b]$

$$f(x) \leq E(x) \leq \exp \left[ \frac{(b - x) \ln f(a) + (x - a) \ln f(b)}{b - a} \right].$$

(2.1)

If the inequality (2.1) holds with “$\geq$”, then the function will be called concave $E$-function on $I$.

Note the following: There is more than one formula for the function $E(x)$ other than that stated in (2.1); for example,

$$E(x) = f(a)e^{B(x - a)}; \quad B = \frac{\ln f(b) - \ln f(a)}{b - a},$$

or in a multiplicative form

$$E(x) = [f(a)]^{\frac{b - x}{b - a}}[f(b)]^{\frac{x - a}{b - a}}.$$
Remark 2.1. The sub $E$-functions possess a number of properties analogous to those of convex functions. For example: If $f : I \to (0, \infty)$ is sub $E$-function, then for any $a, b \in I$, the inequality $f(x) \geq E(x)$ holds outside the interval $[a, b]$.

Definition 2.2. Let a function $f : I \to (0, \infty)$ be sub $E$-function. A function
$$T_u(x) = Ae^{Bx},$$
is said to be supporting function for $f(x)$ at the point $u \in (a, b)$ if
1. $T_u(u) = f(u)$,
2. $T_u(x) \leq f(x) \quad \forall x \in I$.
That is, if $f(x)$ and $T_u(x)$ agree at $x = u$, the graph of $f(x)$ lies on or above the support curve.

Proposition 2.1. If $f : I \to \mathbb{R}$ is a differentiable sub $E$-function, then the supporting function for $f(x)$ at the point $u \in I$ has the form
$$T_u(x) = f(u) \exp \left[ (x - u) \frac{f'(u)}{f(u)} \right].$$

Remark 2.2. For a sub $E$-function $f : I \to (0, \infty)$, we write the supporting function at $u \in I$ in the following form
$$T_u(x) = f(u) \exp \left[ (x - u) \frac{M_{u,f}}{f(u)} \right].$$
The constant $M_{u,f}$ is equal to $f'(u)$ if $f$ is differentiable at the point $u \in I$; otherwise $f'_{-}(u) \leq M_{u,f} \leq f'_{+}(u)$.

Theorem 2.1. Let $f : I \to (0, \infty)$ be a two times continuously differentiable function. The function $f$ is sub $E$-function on $I$ if and only if $f(x)f''(x) - (f'(x))^2 \geq 0$ for all $x$ in $I$.

Theorem 2.2. A function $f : I \to (0, \infty)$ is sub $E$-function on $I$ if and only if there exist a supporting function for $f(x)$ at each point $x \in I$.

Theorem 2.3. If a function $f : [a, b] \to \mathbb{R}$ is continuous and $g$ is an integrable function that does not change sign on $[a, b]$, then there exists $c$ in $(a, b)$ such that
$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx.$$
3. Main Results

**Theorem 3.1.** Let \( f : I \to (0, \infty) \) be a sub \( E \)-function and two times continuously differentiable then the higher powers of \( f(x) \) is sub \( E \)-function.

**Proof.** Since \( f(x) \) is non-negative and sub \( E \)-function then,

\[
(3.1) \quad f(x) \geq 0, \quad f(x)f''(x) - (f'(x))^2 \geq 0 \quad \forall x \in I.
\]

Hence,

\[
(f^n(x))' = nf^{n-1}(x)f'(x),
\]

\[
(f^n(x))'' = n(n - 1)f^{n-2}(x)(f'(x))^2 + n f^{n-1}(x)f''(x).
\]

Now using (3.1), we conclude that:

\[
f^n(x)(f^n(x))'' - ((f^n(x))')^2 \geq 0.
\]

Hence, \( f^n(x) \) is sub \( E \)-function.

\[\square\]

**Theorem 3.2.** Let \( f : I \to (0, \infty) \) be sub \( E \)-function, \( n \in \mathbb{N} \) and \( a, b \in I \) with \( a < b \), then

\[
\frac{f^n(a)}{nB} \left[ e^{nB(b-a)} - 1 \right] \leq \int_a^b f^n(x)dx \leq \frac{f^{n+1}(u)}{nf'(u)} \left[ \exp \left( n(b-u) \frac{f'(u)}{f(u)} \right) - \exp \left( n(a-u) \frac{f'(u)}{f(u)} \right) \right],
\]

where, \( B = \frac{\ln f(b) - \ln f(a)}{b-a} \).

**Proof.** Let \( u \) an arbitrary point in \((a, b)\). As \( f(x) \) is a sub \( E \)-function, then from Definition 2.1 we observe that the graph of \( f(x) \) lies nowhere above the function

\[
E(x) = f(a)e^{B(x-a)}; \quad B = \frac{\ln f(b) - \ln f(a)}{b-a},
\]
and nowhere below any supporting function.

\[ T_u(x) = f(u) \exp \left[ (x - u) \frac{f'(u)}{f(u)} \right] \]  

(3.3)

at the point \( u \in (a, b) \). Thus,

\[ T_u(x) \leq f(x) \leq E(x), \quad x \in [a, b]. \]

As \( f(x) \), \( T_u(x) \) are non-negative functions, then

\[ T_u^n(x) \leq f^n(x) \leq E^n(x) \quad \forall n \in \mathbb{N}. \]  

(3.4)

\[ \int_a^b T_u^n(x) dx \leq \int_a^b f^n(x) dx \leq \int_a^b E^n(x) dx. \]

Using (3.2), one has

\[ \int_a^b f^n(x) dx = \int_a^b E^n(x) dx \]

\[ = \int_a^b f^n(a) e^{nB(x-a)} dx \]

\[ = \left. \frac{f^n(a)}{nB} e^{nB(x-a)} \right|_a^b \]

\[ = \frac{f^n(a)}{nB} \left( e^{nB(b-a)} - 1 \right). \]  

(3.5)

Using (3.3), (3.4), one obtains:

\[ \int_a^b f^n(x) dx \geq \int_a^b T_u^n(x) dx \]

\[ = \int_a^b f^n(u) \exp \left[ n(x - u) \frac{f'(u)}{f(u)} \right] dx \]

\[ = \left. f^n(u) \frac{f(u)}{nf'(u)} \exp \left[ n(x - u) \frac{f'(u)}{f(u)} \right] \right|_a^b \]

\[ = \frac{f^{n+1}(u)}{nf'(u)} \left[ \exp \left[ n(b - u) \frac{f'(u)}{f(u)} \right] - \exp \left[ n(a - u) \frac{f'(u)}{f(u)} \right] \right]. \]  

(3.6)

Hence, from (3.4), (3.5), (3.6) we get the required inequality. \( \square \)

**Theorem 3.3.** If \( f : I \to (0, \infty) \) is sub \( E \)-function on \( I \) then,

\[ f \left( \frac{x + y}{2} \right) \leq \sqrt{f(x)f(y)}, \quad \forall x, y \in I. \]
Hence, the theorem follows. □

\[ f \left( \frac{a+b}{2} \right) \leq \exp \left[ \frac{b-a}{2} \ln f(a) + \frac{b-a}{2} \ln f(b) \right] \]

= \exp \left[ \ln f(a) + \ln f(b) \right] \\
= \exp \left[ \frac{\ln f(a)f(b)}{2} \right] \\
= \sqrt{f(a)f(b)}. \\

\[ \Box \]

**Theorem 3.4.** Let \( f, g : I \to (0, \infty) \) be continuous, sub E-functions on \( I, a, b \in I \) with \( a < b, c_1, c_2 \in (a, b) \) and \( \alpha, \beta > 0 \) with \( \alpha + \beta = 1 \). Then the following inequality holds

\[
\int_a^b f(x)g(x)dx \leq \alpha f(a)g(c_1) \left[ \frac{1}{B_1} (e^{B_1(b-a)} - 1) \right] + \beta g(a)f(c_2) \left[ \frac{1}{B_2} (e^{B_2(b-a)} - 1) \right].
\]

**Proof.** Since \( f, g \) are sub E-functions, we have

\[
f(x) \leq f(a)e^{B_1(x-a)}, \quad B_1 = \frac{\ln f(b) - \ln f(a)}{b-a},
\]

\[
g(x) \leq g(a)e^{B_2(x-a)}, \quad B_2 = \frac{\ln g(b) - \ln g(a)}{b-a},
\]

multiplying both sides of (3.7) and (3.8) by \( \alpha g(x) \) and \( \beta f(x) \) respectively and adding the resulting inequalities we get

\[
f(x)g(x) \leq \alpha f(a)g(x)e^{B_1(x-a)} + \beta g(a)f(x)e^{B_2(x-a)}.
\]

Integrating both sides of (3.9) with respect to \( x \) from \( a \) to \( b \), we get

\[
\int_a^b f(x)g(x)dx \leq \alpha f(a) \int_a^b g(x)e^{B_1(x-a)}dx + \beta g(a) \int_a^b f(x)e^{B_2(x-a)}dx.
\]

Let \( c_1, c_2 \in (a, b) \), by using integral form of mean value theorem, we get

\[
\int_a^b f(x)g(x)dx \leq \alpha f(a)g(c_1) \int_a^b e^{B_1(x-a)}dx + \beta g(a)f(c_2) \int_a^b e^{B_2(x-a)}dx,
\]

\[
= \alpha f(a)g(c_1) \left[ \frac{1}{B_1} (e^{B_1(b-a)} - 1) \right] + \beta g(a)f(c_2) \left[ \frac{1}{B_2} (e^{B_2(b-a)} - 1) \right].
\]

Hence, the theorem follows. □
**Theorem 3.5.** Let \( f, g : I \to (0, \infty) \) be continuous, sub \( E \)-functions on \( I \), \( a, b \in I \) with \( a < b \) and \( \alpha, \beta > 0 \) with \( \alpha + \beta = 1 \). Then the following inequality holds:

\[
\int_a^b f(x)g(x)dx \geq \alpha \frac{f^2(a)}{M_{a,f}} g(c_1) \left[ \exp \frac{M_{a,f}(b-a)}{f(a)} - 1 \right] + \beta \frac{g^2(a)}{M_{a,g}} f(c_2) \left[ \exp \frac{M_{a,g}(b-a)}{g(a)} - 1 \right].
\]

**Proof.** Since \( f, g \) are sub \( E \)-functions on \( I \), from Definition 2.2, we have that \( \forall x, y \in I \)

\[
f(x) \geq f(y) \exp \left( (x-y) \frac{M_{y,f}}{f(y)} \right),
\]

\[
g(x) \geq g(y) \exp \left( (x-y) \frac{M_{y,g}}{g(y)} \right),
\]

where \( M_{y,f} \) is a fixed real number depending on \( y, f \). Multiplying both sides of (3.10) and (3.11) by \( \alpha f(x) \) and \( \beta g(x) \) respectively and adding the resulting inequalities, we get

\[
f(x)g(x) \geq \alpha g(x)f(y) \exp \left[ (x-y) \frac{M_{y,f}}{f(y)} \right] + \beta f(x)g(y) \exp \left[ (x-y) \frac{M_{y,g}}{g(y)} \right],
\]

by taking \( y = a \) in (3.12), we get

\[
f(x)g(x) \geq \alpha g(x)f(a) \exp \left[ (x-a) \frac{M_{a,f}}{f(a)} \right] + \beta f(x)g(a) \exp \left[ (x-a) \frac{M_{a,g}}{g(a)} \right].
\]

Integrating both sides of (3.13) with respect to \( x \) from \( a \) to \( b \), we get

\[
\int_a^b f(x)g(x)dx \geq \alpha f(a) \int_a^b g(x) \exp \left[ (x-a) \frac{M_{a,f}}{f(a)} \right] dx + \beta g(a) \int_a^b f(x) \exp \left[ (x-a) \frac{M_{a,g}}{g(a)} \right] dx.
\]

Let \( c_1, c_2 \in (a, b) \), by using integral form of mean value theorem, we get

\[
\int_a^b f(x)g(x)dx \geq \alpha f(a)g(c_1) \int_a^b \exp \left[ (x-a) \frac{M_{a,f}}{f(a)} \right] dx + \beta g(a)f(c_2) \int_a^b \exp \left[ (x-a) \frac{M_{a,g}}{g(a)} \right] dx,
\]

\[
= \frac{\alpha}{M_{a,f}} f^2(a)g(c_1) \left[ \exp \frac{M_{a,f}(b-a)}{f(a)} - 1 \right] + \frac{\beta}{M_{a,g}} g^2(a)f(c_2) \left[ \exp \frac{M_{a,g}(b-a)}{g(a)} - 1 \right].
\]

Hence, the theorem follows. \( \square \)

**Theorem 3.6.** Let \( f : I \to (0, \infty) \) be sub \( E \)-function on \( I \), \( g : I \to (0, \infty) \) be concave \( E \)-function on \( I \), \( a, b \in I \) with \( a < b \) and \( \alpha > 1 \) with \( \alpha + \beta = 1 \). Then the
following inequality holds

\[
\int_a^b f(x)g(x)dx \geq \alpha f\left(\frac{a+b}{2}\right) \int_a^b g(x) \exp \left[ (x - \frac{a+b}{2}) \frac{M_{\frac{a+b,f}}}{f\left(\frac{a+b}{2}\right)} \right] dx \\
+ \beta g\left(\frac{a+b}{2}\right) \int_a^b f(x) \exp \left[ (x - \frac{a+b}{2}) \frac{M_{\frac{a+b,g}}}{g\left(\frac{a+b}{2}\right)} \right] dx.
\]

**Proof.** Since \(f\) is sub-\(E\)-function on \(I\) and \(g\) is concave \(E\)-function on \(I\), we have that \(\forall x, y \in I\)

\[
f(x) \geq f(y) \exp \left[ (x - y) \frac{M_{y,f}}{f(y)} \right],
\]

(3.14)

\[
g(x) \leq g(y) \exp \left[ (x - y) \frac{M_{y,g}}{g(y)} \right],
\]

(3.15)

where \(M_{y,f}\) is a fixed real number depending on \(y, f\). Multiplying both sides of (3.14) and (3.15) by \(\alpha g(x)\) and \(\beta f(x)\) respectively and adding the resulting inequalities, we get

\[
f(x)g(x) \geq \alpha g(x)f(y) \exp \left[ (x - y) \frac{M_{y,f}}{f(y)} \right] + \beta f(x)g(y) \exp \left[ (x - y) \frac{M_{y,g}}{g(y)} \right].
\]

(3.16)

By taking \(y = \frac{a+b}{2}\) in (3.16), hence

\[
f(x)g(x) \geq \alpha g(x)f\left(\frac{a+b}{2}\right) \exp \left[ (x - \frac{a+b}{2}) \frac{M_{\frac{a+b,f}}}{f\left(\frac{a+b}{2}\right)} \right] \\
+ \beta f(x)g\left(\frac{a+b}{2}\right) \exp \left[ (x - \frac{a+b}{2}) \frac{M_{\frac{a+b,g}}}{g\left(\frac{a+b}{2}\right)} \right],
\]

(3.17)

integrating both sides of (3.17) with respect to \(x\) from \(a\) to \(b\), we get the desired inequality

\[
\int_a^b f(x)g(x)dx \geq \alpha f\left(\frac{a+b}{2}\right) \int_a^b g(x) \exp \left[ (x - \frac{a+b}{2}) \frac{M_{\frac{a+b,f}}}{f\left(\frac{a+b}{2}\right)} \right] dx \\
+ \beta g\left(\frac{a+b}{2}\right) \int_a^b f(x) \exp \left[ (x - \frac{a+b}{2}) \frac{M_{\frac{a+b,g}}}{g\left(\frac{a+b}{2}\right)} \right] dx.
\]

□
Theorem 3.7. Let $f, g : I \to (0, \infty)$ be sub $E$-functions on $I$, $a, b \in I$ with $a < b$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$. Then the following inequality holds

$$\int_a^b f(x)g(x)dx \geq \alpha f\left(\frac{a+b}{2}\right) \int_a^b g(x) \exp \left[\left(x - \frac{a+b}{2}\right) \frac{M\left(\frac{a+b}{2}, f\right)}{f\left(\frac{a+b}{2}\right)}\right] dx$$

$$+ \beta g\left(\frac{a+b}{2}\right) \int_a^b f(x) \exp \left[\left(x - \frac{a+b}{2}\right) \frac{M\left(\frac{a+b}{2}, g\right)}{g\left(\frac{a+b}{2}\right)}\right] dx.$$ 

Proof. Since $f, g$ are sub $E$-functions on $I$, from Definition 2.2, we have that for all $x, y \in I$

$$f(x) \geq f(y) \exp \left[\left(x - y\right) \frac{M_{y,f}}{f(y)}\right], \quad (3.18)$$

$$g(x) \geq g(y) \exp \left[\left(x - y\right) \frac{M_{y,g}}{g(y)}\right], \quad (3.19)$$

where $M_{y,f}$ is a fixed real number depending on $y, f$. Multiplying both sides of (3.18) and (3.19) by $\alpha g(x)$ and $\beta f(x)$ respectively and adding the resulting inequalities, we get

$$f(x)g(x) \geq \alpha g(x) f\left(\frac{a+b}{2}\right) \exp \left[\left(x - \frac{a+b}{2}\right) \frac{M\left(\frac{a+b}{2}, f\right)}{f\left(\frac{a+b}{2}\right)}\right]$$

$$+ \beta f(x) g\left(\frac{a+b}{2}\right) \exp \left[\left(x - \frac{a+b}{2}\right) \frac{M\left(\frac{a+b}{2}, g\right)}{g\left(\frac{a+b}{2}\right)}\right]. \quad (3.20)$$

By taking $y = \frac{a+b}{2}$ in (3.20), hence

$$f(x)g(x) \geq \alpha g(x) f\left(\frac{a+b}{2}\right) \exp \left[\left(x - \frac{a+b}{2}\right) \frac{M\left(\frac{a+b}{2}, f\right)}{f\left(\frac{a+b}{2}\right)}\right]$$

$$+ \beta f(x) g\left(\frac{a+b}{2}\right) \exp \left[\left(x - \frac{a+b}{2}\right) \frac{M\left(\frac{a+b}{2}, g\right)}{g\left(\frac{a+b}{2}\right)}\right], \quad (3.21)$$

integrating both sides of (3.21) with respect to $x$ from $a$ to $b$, we get

$$\int_a^b f(x)g(x)dx \geq \alpha f\left(\frac{a+b}{2}\right) \int_a^b g(x) \exp \left[\left(x - \frac{a+b}{2}\right) \frac{M\left(\frac{a+b}{2}, f\right)}{f\left(\frac{a+b}{2}\right)}\right] dx$$

$$+ \beta g\left(\frac{a+b}{2}\right) \int_a^b f(x) \exp \left[\left(x - \frac{a+b}{2}\right) \frac{M\left(\frac{a+b}{2}, g\right)}{g\left(\frac{a+b}{2}\right)}\right] dx.$$
Theorem 3.8. Let \( f, g \) and \( h : I \to (0, \infty) \) be sub \( E \)-functions on \( I \) and \( a, b \in I \) with \( a < b \). Then the following inequality holds

\[
3 \int_a^b f(x)g(x)h(x) \, dx \geq f\left(\frac{a+b}{2}\right) \int_a^b g(x)h(x) \exp \left[ (x - \frac{a+b}{2}) \frac{M(\frac{a+b}{2}, f)}{f(\frac{a+b}{2})} \right] \, dx \\
+ g\left(\frac{a+b}{2}\right) \int_a^b f(x)h(x) \exp \left[ (x - \frac{a+b}{2}) \frac{M(\frac{a+b}{2}, g)}{g(\frac{a+b}{2})} \right] \, dx \\
+ h\left(\frac{a+b}{2}\right) \int_a^b f(x)g(x) \exp \left[ (x - \frac{a+b}{2}) \frac{M(\frac{a+b}{2}, h)}{h(\frac{a+b}{2})} \right] \, dx.
\]

Proof. Since \( f, g \) and \( h \) are sub \( E \)-functions on \( I \), from Definition 2.2, we have \( \forall x, y \in I \)

\[
(3.22) \quad f(x) \geq f(y) \exp \left[ (x - y) \frac{M_y,f}{f(y)} \right],
\]

\[
(3.23) \quad g(x) \geq g(y) \exp \left[ (x - y) \frac{M_y,g}{g(y)} \right],
\]

\[
(3.24) \quad h(x) \geq h(y) \exp \left[ (x - y) \frac{M_y,h}{h(y)} \right],
\]

multiplying both sides of (3.22), (3.23) and (3.24) by \( g(x)h(x) \), \( f(x)h(x) \) and \( f(x)g(x) \) respectively and adding the resulting inequalities

\[
3 f(x)g(x)h(x) \geq g(x)h(x)f(y) \exp \left[ (x - y) \frac{M_y,f}{f(y)} \right] \\
+ f(x)h(x)g(y) \exp \left[ (x - y) \frac{M_y,g}{g(y)} \right] \\
+ f(x)g(x)h(y) \exp \left[ (x - y) \frac{M_y,h}{h(y)} \right].
\]

(3.25)
Now, if we choose \( y = \frac{a + b}{2} \) in (3.25), we obtain

\[
3f(x)g(x)h(x) \geq g(x)h(x)f\left(\frac{a + b}{2}\right)\exp \left( x - \frac{a + b}{2} \right) \frac{M_{\alpha b}}{f\left(\frac{a + b}{2}\right)} \\
+ f(x)h(x)g\left(\frac{a + b}{2}\right)\exp \left( x - \frac{a + b}{2} \right) \frac{M_{\alpha b}}{g\left(\frac{a + b}{2}\right)} \\
+ f(x)g(x)h\left(\frac{a + b}{2}\right)\exp \left( x - \frac{a + b}{2} \right) \frac{M_{\alpha b}}{h\left(\frac{a + b}{2}\right)}.
\]

(3.26)

Integrating both sides of (3.26) with respect to \( x \) from \( a \) to \( b \), we get

\[
3 \int_a^b f(x)g(x)h(x)dx \geq f\left(\frac{a + b}{2}\right) \int_a^b g(x)h(x) \exp \left( x - \frac{a + b}{2} \right) \frac{M_{\alpha b}}{f\left(\frac{a + b}{2}\right)} dx \\
+ g\left(\frac{a + b}{2}\right) \int_a^b f(x)h(x) \exp \left( x - \frac{a + b}{2} \right) \frac{M_{\alpha b}}{g\left(\frac{a + b}{2}\right)} dx \\
+ h\left(\frac{a + b}{2}\right) \int_a^b f(x)g(x) \exp \left( x - \frac{a + b}{2} \right) \frac{M_{\alpha b}}{h\left(\frac{a + b}{2}\right)} dx.
\]

Hence, the theorem follows. \( \square \)

**Theorem 3.9.** Let \( f_1, f_2, \ldots, f_n \) and \( h : I \to (0, \infty) \) be sub \( E \)-functions on \( I \) and \( a, b \in I \) with \( a < b \). Further, let \( \alpha_1, \alpha_2, \ldots, \alpha_n > 0 \) with \( \sum_{i=1}^n \alpha_i = 1 \). Then the following inequality holds

\[
\int_a^b \prod_{i=1}^n f_i(x)dx \geq \alpha_1 f_1\left(\frac{a + b}{2}\right) \int_a^b f_2(x)f_3(x)\ldots f_n(x) \exp \left( x - \frac{a + b}{2} \right) \frac{M_{\alpha b}}{f_1\left(\frac{a + b}{2}\right)} dx \\
+ \alpha_2 f_2\left(\frac{a + b}{2}\right) \int_a^b f_1(x)f_3(x)\ldots f_n(x) \exp \left( x - \frac{a + b}{2} \right) \frac{M_{\alpha b}}{f_2\left(\frac{a + b}{2}\right)} dx \\
+ \cdots \\
+ \alpha_n f_n\left(\frac{a + b}{2}\right) \int_a^b f_1(x)f_2(x)\ldots f_{n-1}(x) \exp \left( x - \frac{a + b}{2} \right) \frac{M_{\alpha b}}{f_n\left(\frac{a + b}{2}\right)} dx.
\]
Proof. Since \( f_1, f_2, \ldots, f_n \) are sub E-functions on \( I \), we have \( \forall x, y \in I \)

\[
\begin{align*}
(3.27) \quad f_1(x) & \geq f_1(y) \exp \left( (x - y) \frac{M_{y,f_1}}{f_1(y)} \right) \\
(3.28) \quad f_2(x) & \geq f_2(y) \exp \left( (x - y) \frac{M_{y,f_2}}{f_2(y)} \right), \\
& \vdots \\
(3.29) \quad f_n(x) & \geq f_n(y) \exp \left( (x - y) \frac{M_{y,f_n}}{f_n(y)} \right).
\end{align*}
\]

Multiplying both sides of (3.27), (3.28),... and (3.29) by \( \alpha_1 f_2(x) f_3(x) \ldots f_n(x), \alpha_2 f_1(x) f_3(x) \ldots f_n(x), \ldots, \) and \( \alpha_n f_1(x) f_2(x) \ldots f_{n-1}(x) \) respectively and adding the resulting inequalities

\[
\prod_{i=1}^{n} f_i(x) \geq \alpha_1 f_2(x) f_3(x) \ldots f_n(x) f_1(y) \exp \left( (x - y) \frac{M_{y,f_1}}{f_1(y)} \right) \\
+ \alpha_2 f_1(x) f_3(x) \ldots f_n(x) f_2(y) \exp \left( (x - y) \frac{M_{y,f_2}}{f_2(y)} \right) \\
\vdots \\
+ \alpha_n f_1(x) f_2(x) \ldots f_{n-1}(x) f_n(y) \exp \left( (x - y) \frac{M_{y,f_n}}{f_n(y)} \right).
\]

Now, if we choose \( y = \frac{a + b}{2} \) in (3.30), we obtain

\[
\prod_{i=1}^{n} f_i(x) \geq \alpha_1 f_2(x) f_3(x) \ldots f_n(x) f_1 \left( \frac{a + b}{2} \right) \exp \left( x - \frac{a + b}{2} \right) \frac{M_{\frac{a+b}{2},f_1}}{f_1 \left( \frac{a + b}{2} \right)} \\
+ \alpha_2 f_1(x) f_3(x) \ldots f_n(x) f_2 \left( \frac{a + b}{2} \right) \exp \left( x - \frac{a + b}{2} \right) \frac{M_{\frac{a+b}{2},f_2}}{f_2 \left( \frac{a + b}{2} \right)} \\
\vdots \\
+ \alpha_n f_1(x) f_2(x) \ldots f_{n-1}(x) f_n \left( \frac{a + b}{2} \right) \exp \left( x - \frac{a + b}{2} \right) \frac{M_{\frac{a+b}{2},f_n}}{f_n \left( \frac{a + b}{2} \right)}.
\]

Integrating both sides of (3.31) with respect to \( x \) from \( a \) to \( b \), we get the desired inequality. \( \square \)
References


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