SOLUTIONS OF EMDEN-FOWLER TYPE EQUATIONS OF DIFFERENT ORDER BY ADOMIAN DECOMPOSITION METHOD

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ABSTRACT. This study efficiently applies Adomian decomposition method to approximate solutions of linear and nonlinear Emden-Fowler type equations of higher order. A few illustrative problems are discussed to confirm the accuracy and applicability of the presented approach.

1. INTRODUCTION

The Adomian decomposition method (ADM) [1–3] was first introduced by Adomian in the years 1980s. After that many researchers offered various modifications on this method. ADM is powerful for solving linear and nonlinear problems and has been used as a highly effective technique for solving different types of differential equations with initial and boundary conditions such as Emden-Fowler Equation [5, 8–10] which is considered as one of the most equations that has been great interest by numerous mathematicians and physicists in many studies. Several approaches were investigated to find solutions for this type of equations and to overcome the difficulty of the singular point as [4, 6, 7]. This article is attempt to handle Emden-fowler type equations of higher order with boundary conditions by using ADM. We therefore introduce

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a novel differential operator to solve different orders of Emden-Fowler equations.

2. Structure of the Emden equations of higher order

To establish different order of Emden-Fowler type equations we will propose the generalized formula as follows

\[ x^{-1} \frac{d^{n-1}}{dx^{n-1}} x^{1-m} \frac{d^2}{dx^2} x^m y = f(x, y), \]  

(2.1)

where \( m \geq 0 \) and \( n \geq 1 \). To obtain the first type, we put \( n = 1 \) in equation (2.1) gives

\[ y'' + \frac{2m}{x} y' + \frac{m(m-1)}{x^2} y = f(x, y). \]

To obtain the second type, we set \( n = 2 \) in equation (2.1) to find

\[ y''' + \frac{1 + 2m}{x} y'' + \frac{m(m-1)}{x^2} y' - \frac{m(m-1)}{x^3} y = f(x, y). \]

To obtain the third type, we put \( n = 3 \) in equation (2.1) yields:

\[ y^{(4)} + \frac{2 + 2m}{x} y''' + \frac{m(m-1)}{x^2} y'' - \frac{2m(m-1)}{x^3} y' + \frac{2m(m-1)}{x^4} y = f(x, y). \]

To obtain the fourth type, we put \( n = 4 \) in equation (2.1) gives:

\[ y^{(5)} + \frac{3 + 2m}{x} y^{(4)} + \frac{m(m-1)}{x^2} y''' - \frac{3m(m-1)}{x^3} y'' + \frac{6m(m-1)}{x^4} y' - \frac{6m(m-1)}{x^5} y = f(x, y) \]

\[ \ldots \]

\[ y^{(n+1)} + \frac{(n-1) + 2m}{x} y^{(n)} + \frac{m(m-1)}{x^2} y^{(n-1)} + \]

\[ + \sum_{r=2}^{n} \frac{(-1)^{r+1}(n-1)!m(m-1)}{(n-r)!x^{1+r}} y^{(n-r)} = f(x, y). \]

(2.2)
3. Essentials of the Method

Consider a general Emden-Fowler equation in the form (2.2) with the following boundary conditions: \( y(b) = a_1, y'(b) = a_2, \ldots, y^{(n)}(b) = a_n \), where the function \( f(x, y) \) is known, \( a_1, a_2, \ldots, a_n \) are constants, if \( m \neq 1, b \neq 0 \), and if \( m = 1, b \) takes any value. Equation (2.2) can be rewritten as

\[
Ly = f(x, y),
\]

where

\[
L(.) = x^{-m} \frac{d^{n-1}}{dx^{n-1}} x^{1-m} \frac{d^2}{dx^2} x^m(\cdot),
\]

then the inverse operator \( L^{-1} \) is given by

\[
L^{-1}(\cdot) = x^{-m} \int_0^x \int_0^x \int_0^x \cdots \int_0^x f(x, y)dx_1dx_2dx_3\ldots dx_n.
\]

By using \( L^{-1} \) to both sides of (3.1) to obtain

\[
y = \gamma(x) + L^{-1}f(x, y),
\]

such that

\[
L\gamma(x) = 0.
\]

The Adomian decomposition method assumes that solution \( y(x) \) by an infinite series

\[
y(x) = \sum_{n=0}^{\infty} y_n(x),
\]

and the nonlinear term \( f(x, y) \) by an infinite series of polynomials

\[
f(x, y) = \sum_{n=0}^{\infty} A_n,
\]

where the components \( y_n(x) \) of the solution \( y(x) \) will be determined recurrently, and the \( A_n \) are the Adomian polynomials, specific algorithms were seen.
in [3] to formulate Adomian polynomials. The following algorithm:

\begin{align*}
A_0 &= Z(y_0), \\
A_1 &= Z'(y_0)y_1, \\
A_2 &= Z'(y_0)y_2 + \frac{1}{2}Z''(y_0)y_1^2, \\
A_3 &= Z'(y_0)y_3 + Z''(y_0)y_1y_2 + \frac{1}{3!}Z'''(y_0)y_1^3, \\
&\vdots
\end{align*}

(3.5)

From (3.2), (3.3) and (3.4) we get

\begin{equation}
\sum_{n=0}^{\infty} y(n) = \gamma(x) + L^{-1} \sum_{n=0}^{\infty} A_n.
\end{equation}

(3.6)

To determine the components $y_n(x)$, we use Adomian decomposition method by using the relation

\begin{align*}
y_0 &= \gamma(x) + L^{-1} f(x), \\
y_{n+1} &= -L^{-1} A_n, n \geq 0,
\end{align*}

therefore

\begin{align*}
y_0 &= \gamma(x) + L^{-1} f(x), \\
y_1 &= -L^{-1} A_0, \\
y_2 &= -L^{-1} A_1, \\
y_3 &= -L^{-1} A_3, \\
&\vdots
\end{align*}

(3.7)

Using the equation (3.5) and (3.7) we can determine the components $y_n(x)$, and therefore, we can directly obtain series solution of $y(x)$ in (3.6). In addition, and for numerical reasons, we can be the n-term approximate

\[ \Psi_n = \sum_{n=0}^{n-1} y_n(x), \]

in order to approximate the exact solution.
4. Applications of ADM

In this part, we present some examples and apply ADM to obtain approximate and exact solutions of the Emden-Fowler type equations of different order.

Example 1. First, we consider the linear Emden-Fowler type equation

\[
y'' + \frac{5}{x} y' + \frac{2}{x^2} y - \frac{2}{x^3} y = \frac{e^x}{x^3} \left( -2 + 42 x^3 + 99 x^6 + 27 x^9 \right),
\]

\[
y\left(\frac{1}{2}\right) = 1.13315, \quad y'\left(\frac{1}{2}\right) = 0.849861, \quad y''\left(\frac{1}{2}\right) = 4.03684,
\]

where \( L(\cdot) = x^{-1} \frac{d}{dx} x^{-1} \frac{d^2}{dx^2} x^2 (\cdot) \).

Then the inverse operator \( L^{-1} \) is given by

\[
L^{-1}(\cdot) = x^{-2} \int_0^x \int_0^x x \cdot x(\cdot) dx dx dx.
\]

We write equation (4.1) by ADM operator form as

\[
Ly = \frac{e^x}{x^3} \left( -2 + 42 x^3 + 99 x^6 + 27 x^9 \right).
\]

Applying \( L^{-1} \) to (4.2) we get the exact solution

\[
y(x) = e^x.
\]

Example 2. Next, assume the nonlinear Emden-Fowler type equation

\[
y^{(4)} + \frac{14}{x} y'' + \frac{30}{x^2} y' - \frac{60}{x^3} y + \frac{60}{x^4} y =
\]

\[
e^x \left( 540 + 780 x + 270 x^2 + 30 x^3 + x^4 \right) - x^{16} e^{4x} + y^4,
\]

\[
y(-1) = 0.367879, \quad y'(-1) = -1.10364, \quad y''(-1) = 1.8394, \quad y'''(-1) = 0.367879,
\]

the exact solution is \( y(x) = x^4 e^x \), where

\[
L(\cdot) = x^{-1} \frac{d^2}{dx^2} x^{-5} \frac{d^2}{dx^2} x^6 (\cdot).
\]

Then the inverse operator \( L^{-1} \) is given by

\[
L^{-1}(\cdot) = x^{-6} \int_0^x \int_0^x x^5 \int_{-1}^x x(\cdot) dx dx dx dx.
\]
equation (4.3) in ADM operator form becomes

\begin{equation}
Ly = e^x (540 + 780 x + 270 x^2 + 30 x^3 + x^4) - x^6 e^{4x} + y^4.
\end{equation}

Applying $L^{-1}$ to (4.4) we get

\begin{equation}
y = 0.464229 x + 0.81459 x^2 + L^{-1}(e^x (540 + 780 x + 270 x^2 + 30 x^3 + x^4))
- L^{-1}(x^6 e^{4x}) + L^{-1} y^4,
\end{equation}

therefore

\begin{equation}
y = 0.0000286088 x + 0.0000229384 x^2 + e^x x^4 + L^{-1} y^4.
\end{equation}

We use the recursive relation

\begin{equation}
y_0 = 0.0000286088 x + 0.0000229384 x^2 + e^x x^4,
y_{n+1} = L^{-1} A_n, \quad n \geq 0,
\end{equation}

where the nonlinear term $y^4$ has Adomian polynomials $A_n$ as the following

\begin{equation}
A_0 = y_0^4, \quad A_1 = 4y_0^3 y_1,
\end{equation}

so the solution components are

\begin{align*}
y_0 &= 0.0000286088 x + 0.0000229384 x^2 + e^x x^4, \\
y_1 &= -0.0000152908 x - 0.0000229363 x^2 - 8.91966 \times 10^{-6} x^3 - 1.64375 \times 10^{-18} x^4 - \ldots \\
y_2 &= -3.38524 \times 10^{-10} x - 5.07786 \times 10^{-10} x^2 - 1.97472 \times 10^{-10} x^3 - 8.29769 \times 10^{-23} x^6 + \ldots
\end{align*}

this yields the solution in a series form

\begin{equation}
y(x) = y_0 + y_1 + y_2 = x^4 e^x + 0.0000133176 x + 1.63957 \times 10^{-9} x^2 - 8.91985 \times 10^{-6} x^3 - 1.64365 \times 10^{-18} x^4 - \ldots
\end{equation}

Example 3. Consider the following equation

\begin{equation}
y^{(5)} + \frac{23}{x} y^{(4)} + \frac{90}{x^2} y''' - \frac{270}{x^3} y'' + \frac{540}{x^4} y - \frac{270}{x^5} y = 5040 + e^x - e^y,
\end{equation}

\begin{equation}
y(1) = 1, y'(1) = 5, y''(1) = 20, y'''(1) = 60, y^{(4)}(1) = 120,
\end{equation}
the exact solution is \( y(x) = x^5 \). Here:

\[
L(\cdot) = x^{-1} \frac{d^2}{dx^2} x^{-9} \frac{d^2}{dx^2} x^{10}(\cdot),
\]
and the inverse operator \( L^{-1} \) is given by

\[
L^{-1}(\cdot) = x^{-10} \int_0^x \int_0^x \int_1^x \int_1^x x(\cdot) dx dx dx dx.
\]

In ADM operator form equation (4.5) becomes

(4.6) \[ Ly = 5040 + e^{x^2} - e^y. \]

Applying \( L^{-1} \) to (4.6) we get

\[
y = -\frac{63}{11} x + \frac{140 x^2}{11} - \frac{105 x^3}{13} + x^5 + L^{-1}(e^{x^5}) - L^{-1}e^y.
\]

We use the recursive relation

\[
y_0 = -0.00180375 x + 0.0037636 x^2 - 0.00219399 x^3 + 1.0002 x^5 + \ldots
\]

\[
y_{n+1} = -L^{-1}A_n, \quad n \geq 0,
\]

where the nonlinear term \( e^y \) has Adomian polynomials \( A_n \) as the following

\[
A_0 = e^{y_0},
\]
\[
A_1 = y_1 e^{y_0},
\]
\[
A_2 = (y_2 + \frac{1}{2}y_1^2) e^{y_0},
\]

and so on. So the solution components are given by

\[
y_0 = -0.00180375 x + 0.0037636 x^2 - 0.00219399 x^3 + 1.0002 x^5 + \ldots
\]
\[
y_1 = -0.00180361 x + 0.00376324 x^2 - 0.00219373 x^3 + 0.000198413 x^5 + \ldots
\]
\[
y_2 = 1.48956 \times 10^{-7} x - 3.67325 \times 10^{-7} x^2 + 2.63754 \times 10^{-7} x^3 - 1.25251 \times 10^{-7} x^6 + \ldots
\]
\[
y_3 = -1.72379 \times 10^{-11} x + 4.73759 \times 10^{-11} x^2 - 3.82888 \times 10^{-11} x^3 + 1.03442 \times 10^{-11} x^6 + \ldots
\]

and so on. Thus, the approximate solution is

\[
y(x) = y_0 + y_1 + y_2 + y_3 = -2.92299 \times 10^{-7} x + 7.24058 \times 10^{-7} x^2
\]
\[
-5.22375 \times 10^{-7} x^3 + x^5 + 2.50501 \times 10^{-7} x^6 + \ldots
\]
Table 1. Comparison of numerical errors between the exact solution \( y = x^5 \) and the ADM solution

<table>
<thead>
<tr>
<th>( x )</th>
<th>Exact solution</th>
<th>ADM solution</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.000</td>
<td>0.0000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.00001</td>
<td>0.0000097</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.2</td>
<td>0.00032</td>
<td>0.0003199</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.3</td>
<td>0.00243</td>
<td>0.0024299</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.4</td>
<td>0.01024</td>
<td>0.0102399</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.5</td>
<td>0.03125</td>
<td>0.0312499</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.6</td>
<td>0.07776</td>
<td>0.0777599</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.7</td>
<td>0.16807</td>
<td>0.1680699</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.8</td>
<td>0.32768</td>
<td>0.3276999</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.9</td>
<td>0.59049</td>
<td>0.5904899</td>
<td>0.000000</td>
</tr>
<tr>
<td>1.0</td>
<td>1.000</td>
<td>0.9999999</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

Example 4. We finally study the nonlinear Emden-Fowler type equation when we set \( m = 1 \) and \( n = 5 \) in (2.2), we obtain

\[
y^{(6)} + \frac{6}{x} y^{(5)} = 144 \left( 35 - 4080 x^6 + 20005 x^{12} - 13705 x^{18} + 1056 x^{24} + x^{30} \right) e^{-6y}, \tag{4.7}
\]

\( y(0) = 0, y'(0) = 0, y''(0) = 0, y'''(0) = 0, y^{(4)}(0) = 0, y^{(5)}(0) = 0, \)

the exact solution is \( y(x) = \log(1 + x^6) \).

Here

\[
L(.) = x^{-1} \frac{d^6}{dx^6} x(.),
\]

and the inverse operator \( L^{-1} \) is given by

\[
L^{-1}(.) = x^{-1} \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x x(.) dx dx dx dx dx.
\]

equation (4.7) in ADM operator form becomes:

\[
Ly = 144 \left( 35 - 4080 x^6 + 20005 x^{12} - 13705 x^{18} + 1056 x^{24} + x^{30} \right) e^{-6y}. \tag{4.8}
\]

Applying \( L^{-1} \) to (4.8) we have

\[
y = L^{-1} \left( 144 \left( 35 - 4080 x^6 + 20005 x^{12} - 13705 x^{18} + 1056 x^{24} + x^{30} \right) e^{-6y} \right).
\]
We use the recursive relation
\[ y_0 = 0, \]
\[ y_{n+1} = L^{-1}(144 \left( 35 - 4080 x^6 + 20005 x^{12} - 13705 x^{18} + 1056 x^{24} + x^{30} \right) ) A_n \]
\[ n \geq 0, \]
where the nonlinear term \( e^{-6y} \) has Adomian polynomials \( A_n \) as the following
\[ A_0 = e^{-6y_0}, \]
\[ A_1 = -6y_1 e^{-6y_0}, \]
\[ A_2 = (-6y_2 + 18y_1^2) e^{-6y_0}, \]
and so on. So the solution components are
\[ y_0 = 0, \]
\[ y_1 = x^6 - 0.475524 x^{12} + 0.147464 x^{18} - 0.0154771 x^{24} + 0.000286847 x^{30} + \ldots \]
\[ y_2 = -0.0244755 x^{12} + 0.181187 x^{18} - 0.148732 x^{24} + 0.0388222 x^{30} - \ldots \]
\[ y_3 = 0.00468185 x^{18} - 0.0843324 x^{24} + 0.118886 x^{30} - \ldots \]
and so on. This yields the solution in a series form
\[ y(x) = y_0 + y_1 + y_2 + y_3 = x^6 - 0.5 x^{12} + 0.333333 x^{18} - 0.248541 x^{24} + 0.157995 x^{30}. \]

Note that, the solution by ADM converges to the exact solution
\[ y(x) = \log(1 + x^6) = x^6 - 0.5 x^{12} + 0.333333 x^{18} - 0.25 x^{24} + 0.2 x^{30} + \ldots \]

5. Conclusion

The Emden-fowler type equations of higher order are investigated by suggested modification of ADM. Numerical examples showed that the method is successful for finding solutions of these equations with some iterations approximate the exact solutions very well. The results demonstrated that the proposed strategy is powerful and valid.
REFERENCES


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