A STUDY ON CONCEPTS OF BALLS IN A INTUITIONISTIC FUZZY
$D$–METRIC SPACES

S. YAHYA MOHAMED AND E. NAARGEES BEGUM

Abstract. Dhage [2] introduced the concept of open balls in a D-Metric space in two different ways and discussed at length the properties of the topologies generated by the family of all open balls of each kind. In this paper, a new concept of balls in a Intuitionistic Fuzzy D-Metric spaces are introduced.

1. INTRODUCTION

Fuzzy sets are sets whose elements have degrees of membership. Fuzzy sets are introduced by Lotti.A. Zadeh [3] (1965) as an extension of the classical notion of sets. The concept of an intuitionistic fuzzy set can be viewed as an alternative approach to define a fuzzy set in cases where available information is not sufficient for the definition of an imprecise concept by means of a conventional fuzzy set. In general, the theory of intuitionistic fuzzy sets is the generalization of fuzzy sets.

The idea of an intuitionistic fuzzy set was first published by Krassimir Atanassov [1]. In general, the theory of intuitionistic fuzzy sets is the generalization of fuzzy sets. Several researches have shown interest in the intuitionistic fuzzy set

1 Corresponding author

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theory and successfully applied in many other field. Fuzzy application in almost every direction of mathematics such as arithmetic, topology, graph theory, probability theory, logic etc.,

In 1992, B.C.Dhage [2] proposed the notion of a $D$–metric space in an attempt to obtain analogous results to those for metric spaces, but in a more general setting. This paper is organized as follows. The definition of intuitionistic fuzzy metric space, $D$–metric space and fuzzy $D$–metric space are introduced in section 2. In section 3, we introduce the new concept of intuitionistic fuzzy $D$–metric spaces and also we discuss about the theorems on balls in a intuitionistic fuzzy $D$–metric spaces.

2. Preliminaries

**Definition 2.1.** Let $A$ be a non-empty set. A function $\rho : A \times A \times A \to [0,\infty)$ is called a $D$–metric on $A$ if

1. $\rho(a, b, c) = 0$ if and only if $a = b = c$ (coincidence);
2. $\rho(a, b, c) = \rho(p(a, b, c))$ for all $a, b, c \in A$ and for any permutation $p(a, b, c)$ of $a, b, c$ (symmetry),
3. $\rho(a, b, c) \leq \rho(a, b, r) + \rho(a, r, c) + \rho(r, b, c)$ for all $a, b, c, r \in A$ (tetrahedral inequality).

If $A$ is a non-empty set and $\rho$ is a $D$–metric on $A$, then the ordered pair $(A, \rho)$ is called a $D$–metric space. When the $D$–metric $\rho$ is understood, $A$ itself is called a $D$–metric space.

**Definition 2.2.** The 3-tuple $(A, M, *)$ is said to be a fuzzy $D$–metric space, where $A$ is an arbitrary set, $*$ is continuous t-norm and $M$ is a fuzzy set on $A \times A \times A \to [0, \infty)$ satisfying the following conditions: For all $a, b, c, r \in A, s, t, u > 0$.

1. $M(a, b, c, t) = 0$
2. $M(a, b, c, t) = 1$ if and only if $a = b = c$
3. $M(a, b, c, t) = M(p(a, b, c, t))$ for all $a, b, c \in A$ and for any permutation $p(a, b, c, t)$ of $a, b, c, t$
4. $M(a, b, c, t + s + u) \geq M(a, b, r, t) * M(a, r, c, s) * M(r, b, c, u)$
5. $M(a, b, c, t) : [0, \infty) \to [0, 1]$ is continuous.
Definition 2.3. A 5-tuple \((A, M, N, *, \circ)\) is said to be an intuitionistic fuzzy metric space if \(A\) is an arbitrary set, \(*\) is a continuous t-norm, \(\circ\) is a continuous t-conorm and \(M, N\) are fuzzy sets on \(A^2 \times [0, \infty)\) satisfying the conditions:

1. \(M(a, b, t) + N(a, b, t) \leq 1\) for all \(a, b \in A\) and \(t > 0\);
2. \(M(a, b, 0) = 0\) for all \(a, b \in A\);
3. \(M(a, b, t) = 1\) for all \(a, b \in A\) and \(t > 0\) if and only if \(a = b\);
4. \(M(a, b, t) = M(b, a, t)\) for all \(a, b \in A\) and \(t > 0\);
5. \(M(a, b, t) * M(b, c, s) \leq M(a, c, t + s)\), for all \(a, b, c \in A\) and \(s, t > 0\);
6. \(M(a, b, \cdot) : [0, \infty) \to [0, \infty]\) is left continuous, for all \(a, b \in A\);
7. \(\lim_{t \to \infty} M(a, b, t) = 1\) for all \(a, b \in A\) and \(t > 0\);
8. \(N(a, b, 0) = 1\) for all \(a, b \in A\);
9. \(N(a, b, t) = 0\), for all \(a, b \in A\) and \(t > 0\) if and only if \(a = b\);
10. \(N(a, b, t) = N(b, a, t)\) for all \(a, b \in A\) and \(t > 0\);
11. \(N(a, b, t) \circ N(b, c, s) \geq N(a, c, t + s)\) for all \(a, b, c \in A\) and \(s, t > 0\);
12. \(N(a, b, \cdot) : [0, \infty) \to [0, 1]\) is right continuous, for all \(a, b \in A\);
13. \(\lim_{t \to \infty} N(a, b, t) = 0\) for all \(a, b \in A\).

The functions \(M(a, b, t)\) and \(N(a, b, t)\) denote the degree of nearness and the degree of non-nearness between \(a\) and \(b\) w.r.t. \(t\) respectively.

3. Balls in an intuitionistic fuzzy \(D\)-metric spaces

Definition 3.1. A 5-tuple \((A, M, N, *, \circ)\) is said to be an intuitionistic fuzzy \(D\)-metric space if \(A\) is an arbitrary set, \(*\) is a continuous t-norm, \(\circ\) is a continuous \(t\)-conorm and \(M, N\) are fuzzy sets on \(A^3 \times [0, \infty)\) satisfying the conditions:

1. \(M(a, b, c, t) + N(a, b, c, t) \leq 1\) for all \(a, b, c \in A\) and \(t > 0\);
2. \(M(a, b, c, 0) = 0\) for all \(a, b, c \in A\);
3. \(M(a, b, c, t) = 1\) for all \(a, b, c \in A\) and \(t > 0\) if and only if \(a = b = c\);
4. \(M(a, b, c, t) = M(p(a, b, c, t))\) for all \(a, b, c \in A\) and for any permutation \(p(a, b, c)\) of \(a, b, c\) for \(t > 0\);
5. \((M(a, b, c, t + s + u) \geq M(a, b, r, t) * M(a, r, c, t) * M(r, b, c, u),\) for all \(a, b, c, r \in A\) and \(s, t, u > 0\);
6. \((M(a, b, c, \cdot) : [0, \infty) \to [0, \infty]\) is left continuous for all \(a, b, c \in A\);
7. \(\lim_{t \to \infty} M(a, b, c, t) = 1\) for all \(a, b, c \in A\) and \(t > 0\);
8. \(N(a, b, c, 0) = 1\) for all \(a, b, c \in A\);
\[ N(a, b, c, t) = 0 \text{ for all } a, b, c \in A \text{ and } t > 0 \text{ if and only if } a = b = c; \]

(10) \[ N(a, b, c, t) = N(p(a, b, c, t)) \text{ for all } a, b, c \in A \text{ and for any permutation } p(a, b, c) \text{ of } a, b, c, t > 0; \]

(11) \[ N(a, b, r, t) \circ N(a, r, c, s) \circ N(r, b, c, s) \geq N(a, b, c, t+s+u) \text{ for all } a, b, c, r \in A \text{ and } s, t, u > 0; \]

(12) \[ N(a, b, c, \cdot) : [0, \infty) \to [0, 1] \text{ is right continuous, for all } a, b, c \in A; \]

(13) \[ \lim_{n \to \infty} N(a, b, c, t) = 0 \text{ for all } a, b, c \in A. \text{ The functions } M(a, b, c, t) \text{ and } \]

\[ N(a, b, c, t) \text{ denote the degree of nearness and the degree of non-nearness between } a, b \text{ and } c \text{ w.r.t. } t \text{ respectively.} \]

**Definition 3.2.** Let \((A, M, N, *, \circ)\) be an intuitionistic fuzzy \(D\)-metric space. Then

1. A sequence \(\{a_n\}\) in \(A\) is said to be Cauchy sequence if for all and \(t > 0\) and \(p, q > 0, \lim_{n \to \infty} M(a_{n+p+q}, a_{n+p}, a_n, t) = 1\) and \(\lim_{n \to \infty} N(a_{n+p+q}, a_{n+p}, a_n, t) = 1.\)

2. A sequence \(\{a_n\}\) in \(X\) is said to be convergent to a point \(a \in A\) if for all \(t > 0\) and \(p > 0, \lim_{n \to \infty} M(a_{n+p}, a_n, a, t) = 1\) and \(\lim_{n \to \infty} N(a_{n+p}, a_n, a, t) = 0.\)

**Remark 3.1.** Let \((A, M, N, *, \circ)\) be an intuitionistic fuzzy \(D\)-metric space, \(a \in A\) and \(\alpha \in (0, \infty).\)

Let

\[ \bar{B}(a, \alpha, t) = \{b \in A : M(a, b, b, t) > 1 - \alpha, N(a, b, c, t) < \alpha\}, \]

\[ B(a, \alpha, t) = \{b \in \bar{B}(a, \alpha, t) : M(a, b, c, t) > 1 - \alpha, N(a, b, c, t) < \alpha, \forall c \in \bar{B}(a, \alpha, t)\} \]

\[ \bar{\bar{B}}(a, \alpha, t) = \{\{a\} \cup b \in A : \sup_{c \in A} M(a, b, c, t) > 1 - \alpha, \sup_{c \in A} N(a, b, c, t) < \alpha\}. \]

**Remark 3.2.**

(i) It is clear that \(B(a, \alpha, t) \subseteq \bar{B}(a, \alpha, t),\)

(ii) If \(0 < \alpha_1 < \alpha_2\) then \(\bar{B}(a, \alpha_1, t) \subseteq \bar{B}(a, \alpha_2, t), B(a, \alpha_1, t) \subseteq \bar{B}(a, \alpha_2, t).\)

By \(\bar{B}(a, \alpha, t)\) we mean a set in \(A\) is given by \(\bar{B}(a, \alpha, t) = \{b \in \bar{B}(a, \alpha, t) : \)

\(\text{if } b, c \in \bar{B}(a, \alpha, t)\}. \text{ Then}

\[ M(a, b, c, t) > 1 - \alpha, N(a, b, c, t) < \alpha \]

\[ = \{b, c \in A : M(a, b, c, t) > 1 - \alpha, N(a, b, c, t) < \alpha\}. \]

It is clear that \(B(a, \alpha, t) \subseteq \bar{B}(a, \alpha, t).\)
Theorem 3.1. Let \((A, M, N, *, \circ)\) be an intuitionistic fuzzy \(D\)-metric space. Then for a fixed \(a \in A\), the balls \(\tilde{B}(a, \alpha, t)\) and \(B(a, \alpha, t)\) are the sets in \(A\) given by,
\[
\tilde{B}(a, \alpha, t) = \left(\frac{a - \alpha t}{1 - \alpha}, \frac{a + \alpha t}{1 - \alpha}\right), \quad B(a, \alpha, t) = \left(\frac{a - \alpha t}{2(1 - \alpha)}, \frac{a + \alpha t}{2(1 - \alpha)}\right).
\]

Proof. Let \(a, b, c \in A\) be arbitrary. Let \(\alpha > 0\) be fixed. Then
\[
\tilde{B}(a, \alpha, t) = \{b \in A : (M(a, b, b, t) > 1 - \alpha, N(a, b, b, t) < \alpha)\} = \{b \in A : |a - b| < \frac{\alpha t}{1 - \alpha}\}
\]

Again,
\[
B(a, \alpha, t) = \{b \in \tilde{B}(a, \alpha, t) : (M(a, b, c, t) > 1 - \alpha, N(a, b, c, t) < \alpha, \forall c \in \tilde{B}(a, \alpha, t))\} \quad (3.1)
\]

This relation (3.1) implies that the set \(B(a, \alpha, t)\) contain all the points \(b, c \in A\) for which one has
\[
|a - b| < \frac{\alpha t}{1 - \alpha}, \quad |a - c| < \frac{\alpha t}{(1 - \alpha)} \quad \text{with} \quad |b - c| < \frac{\alpha t}{1 - \alpha}. \quad (3.2)
\]

In order to hold inequalities in (3.2) we must have,
\[
|a - b| + |a - c| < \frac{\alpha t}{1 - \alpha},
\]

since
\[
|b - c| \leq |b - a| + |a - c|.
\]

Therefore if we take
\[
|a - b| < \frac{\alpha t}{2(1 - \alpha)},
\]

and
\[
|a - c| < \frac{\alpha t}{2(1 - \alpha)},
\]
then the inequalities in (3.2) are satisfied. Thus we have

\[ B(a, \alpha, t) = \{ b \in \tilde{B}(a, \alpha, t) : |a - b| < \frac{\alpha t}{2(1 - \alpha)} \} \]

\[ B(a, \alpha, t) = (a - \frac{\alpha t}{2(1 - \alpha)}, a + \frac{\alpha t}{2(1 - \alpha)}) \].

\[ \square \]

**Theorem 3.2.** Every ball \( B(a, \alpha, t), a \in A, \alpha > 0 \), is an open set in \( A \). (ie.) it contains a ball of each of its points

**Proof.** Let \( a \in A \) be arbitrary and \( \alpha > 0 \). Consider the ball \( B(a, \alpha, t) \) in \( A \) and supposed that \( x \in B(a, \alpha, t) \). We will show that there is an \( \tilde{\alpha} > 0, \tilde{\alpha} > \alpha \) such that \( B(x, \tilde{\alpha}, t) \subseteq B(a, \alpha, t) \). Since \( x \in B(a, \alpha, t) \) there is number \( \alpha_1 > 0 \) such that \( M(a, x, x, t) > 1 - \alpha \) and \( N(a, x, x, t) < \alpha_1 \) and \( \alpha_1 < \alpha \). We may choose an arbitrary \( \epsilon > 0 \) such that \( \tilde{B}(a, \alpha_1 + \epsilon, t) \subseteq B(a, \alpha, t) \) which is possible in view of \( \alpha_1 < \alpha \). Since \( \tilde{B}(a, \alpha_1 + \epsilon, t) \) is open ,there is an open ball \( \tilde{B}(x, \tilde{\alpha}, t), \tilde{\alpha} < 0 \) such that \( \tilde{B}(x, \tilde{\alpha}, t) \subseteq B(a, \alpha_1 + \epsilon, t) \subseteq B(a, \alpha, t) \). Again \( B(x, \tilde{\alpha}, t) \subseteq \tilde{B}(x, \tilde{\alpha}, t) \). Hence \( B(x, \tilde{\alpha}, t) \subseteq B(x, \tilde{\alpha}, t) \).

Thus proves that \( B(a, \alpha, t) \) is an open in \( A \). \[ \square \]

4. **Conclusion**

Last three decades were very productive for the fuzzy mathematics and the recent literature has observed the fuzzy application in almost every direction of mathematics. In this paper a general analysis has been done open balls in intuitionistic fuzzy \( D \)-metric space. In future from this concept can be to extend in various spaces.

**References**


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PG AND RESEARCH DEPARTMENT OF MATHEMATICS
GOVERNMENT ARTS COLLEGE, TRICHY-22
AFFILIATED TO BHARATHIDASAN UNIVERSITY
E-mail address: yahya_md@yahoo.com

DEPARTMENT OF MATHEMATICS
SRI KAILASH WOMENS COLLEGE
THALAIVASAL, TAMILNADU, INDIA
E-mail address: mathmb@gmail.com